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Engineering Mathematics - II

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1

Differential Equations

1.1 Differential Equations

Differential equations are important in engineering because while solving the problems in engineering it is obvious that we deal with derivatives and the relations appear in the form of equations, which involves derivatives.

1.2 Definition

An equation containing dependent variable, independent variables, differentials or differential co-efficient is called a differential equation.

Ordinary differential equation : A differential equation is said to be **ordinary** if the differential co-efficients are ordinary and containing only one independent variable, one or more dependent variable.

1.3 Order of a Differential Equation

The order of the highest derivative occuring in the differential equation is called it's order.

1.4 Degree of a Differential Equation

The degree **or** power **or** index of highest ordered derivative appearing in the differential equation (Provided the differential equation should be free from radicals and rational terms, fractions), is known as degree of differential equation.

Example 1.1 : *Determine order and degree of the differential equation*

$$\frac{dy}{dx} = 4x+y$$

Solution : The order of this D.E. is 1 and degree 1.

Example 1.2 : *Determine order and degree of the differential equation*

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^4 + y = 0$$

Solution : The order of this D.E. is 2 and degree 3.

Example 1.3 : *Determine order and degree of the differential equation*

$$\left[1+\frac{dy}{dx}\right]^{3/2} = \frac{d^3y}{dx^3}$$

Solution : On squaring both sides we get,

$$\left[1+\frac{d^2y}{dx^2}\right]^3 = \left(\frac{d^3y}{dx^3}\right)^2$$

The order of this D.E. is 3 and degree 2.

Example 1.4 : *Determine order and degree of the differential equation*

$$\frac{dy}{dx}+\frac{k}{\frac{dy}{dx}} = 6$$

Solution : Rewriting given equation we get,

$$\left(\frac{dy}{dx}\right)^2 + k = 6\,\frac{dy}{dx}$$

The order of this D.E. is 1 and degree 2.

Example 1.5 : *Determine the order and degree of the differential equation*

$$\frac{\left(x+\frac{dy}{dx}\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = c$$

Solution : Rewriting the equation and squaring both sides we get

$$\left(x+\frac{dy}{dx}\right)^2 = c^2\left[1+\left(\frac{dy}{dx}\right)^2\right]$$

The order of this D.E. is 1 and the degree is 2.

Example 1.6 : *Determine the order and degree of the differential equation*

$$\left[1+\left(\frac{dy}{dx}\right)^3\right]^{3/2} = \left(\frac{d^2y}{dx^2}\right)$$

Solution : Rewriting the equation and squaring both sides we get, $\left[1+\left(\frac{dy}{dx}\right)^3\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$.

Therefore order = 2 and degree = 2.

Solution of a Differential Equation

Definition : Any relation between dependent and independent variables which is free from derivatives and satisfies the given differential equation is called it's solution.

Example :

1. Consider the differential equation $\frac{dy}{dx} = k$, the relation $y = kx$ ... (1)

   satisfies the differential equation.

$\therefore$ It is solution of the differential equation.

2. Consider the differential equation :

$$\frac{d^2y}{dx^2} + y = 0 \quad \text{... (2)}$$

Now the relation $y = C_1 \sin x$ is the solution of the differential equation (2).

Since $\qquad \frac{dy}{dx} = C_1 \cos x$

and $\qquad \frac{d^2y}{dx^2} = -C_1 \sin x$

$\qquad\qquad = -y$

or $\qquad \frac{d^2y}{dx^2} + y = 0$

Similarly, $y = C_2 \cos x$ also solution of the differential equation. ... (3)

Also, $y = C_1 \sin x + C_2 \cos x$ is the solution of the differential equation (2) ... (4)

Definition

A solution or primitive of a differential equation is any relation between the dependent and independent variables which is free from derivatives and which satisfies the differential equation.

Definition of General Solution

General solution : A relation between dependent variable, independent variable and free from derivative, contains number of arbitrary constants equal to the order of the differential equation which satisfies the differential equation is called it's **General solution.**

For example $y = C_1 \cos x$, $y = C_2 \sin x$ are simply solutions of D.E. $\frac{d^2y}{dx^2} + y = 0$ and $y = C_1 \cos x + C_2 \sin x$ is the general solution of the D.E. $\frac{d^2y}{dx^2} + y = 0$.

Example :

The solutions (1), (3), (4) are called as general solutions.

Definition of Particular Solution

Particular solution : A solution which is obtained from general solution by assigning values to the arbitrary constants is called particular solution.

For example $y = 2 \sin x - \cos x$ is a particular solution of the D.E. $\frac{d^2y}{dx^2} + y = 0$.

In $y = kx$ if $k = 2$ then $y = 2x$ is called as particular solution.

Also, if $C_1 = 1$, $C_2 = -1$ in (4), then $y = \cos x - \sin x$ also particular solution.

Formation of Ordinary Differential Equation

An ordinary differential equation can be formed by eliminating number of arbitrary constants in it's general solution.

If there is one arbitrary constant, the relation or solution to be differentiated once with respect to independent variable. Also, if there are 'n' arbitrary constants in the relation, then it is to be differentiated 'n' times and eliminating those n arbitrary constants, order of the differential equation for used would be 'n'.

For example $\quad y = C_1 x + C_2 \quad$ ... (1)

(where C_1 and C_2 are arbitrary constants).

i.e. there are two arbitrary constants.

For the elimination of two arbitrary constants we need two or more relations.

Differentiate equation (1) twice

$$\frac{dy}{dx} = C_1 \quad \text{... (2)}$$

$$\frac{d^2y}{dx^2} = 0 \quad \text{... (3)}$$

Equation (3) is independent of arbitrary constants.

Therefore equation (3) is the D.E. whose G.S. is equation (1).

Thus we begin with a relation and find a differential equation for which this relation serves as a solution.

Consider a relation between dependent and independent variables involving n arbitrary constants. By differentiating successively these relation n times, we get (n + 1) relations, eliminating the n arbitrary constants from (n + 1) relations, we obtain the required n^{th} order differential equation.

Example 1.7 : *Find the D.E. whose G.S. is $y = C_1 x + C_2 e^x$.*

Solution : Consider

$$y = C_1 x + C_2 e^x \quad \text{... (1)}$$

As there are two arbitrary constants differentiating twice we get,

$$\frac{dy}{dx} = C_1 + C_2 e^x \quad \text{... (2)}$$

$$\frac{d^2y}{dx^2} = C_2 e^x \quad \text{... (3)}$$

From equation (2) and (3)

$$\frac{dy}{dx} = C_1 + \frac{d^2y}{dx^2}$$

$$\therefore \quad C_1 = \frac{dy}{dx} - \frac{d^2y}{dx^2}$$

Substitute in (1)

$$\therefore \quad y = \left(\frac{dy}{dx} - \frac{d^2y}{dx^2}\right)x + \frac{d^2y}{dx^2}$$

i.e. $(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ is the required differential equation.

Example 1.8 : *Find the D.E. of all circles touching Y-axis at the origin and centres on X-axis.* **[May-99, Dec.-2003]**

Solution : The equation of such a circle is

$$(x-a)^2 + (y-0)^2 = a^2$$

i.e. $$x^2 + y^2 = 2ax \quad \text{... (1)}$$

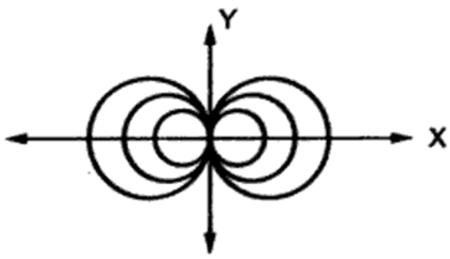


Fig. 1.1

There is only one arbitrary constant 'a'. Differentiating once

$$2x + 2y\frac{dy}{dx} = 2a$$

i.e. $$x + y\frac{dy}{dx} = a \quad \text{... (2)}$$

Substituting in (1) we get

$$x^2 + y^2 = 2x\left[x + y\frac{dy}{dx}\right]$$ is the required D.E.

Example 1.9 : *Form the differential equation of all circles of radius 'a'.*
[May-2000, Dec.-2000, May-2005]

Solution : The equation of any circle of radius 'a' is

$$(x-h)^2+(y-k)^2 = a^2 \quad \text{... (1)}$$

Here h and k are arbitrary constants and a is a particular constant.

$\therefore$ Differentiating we get $2(x-h)+2(y-k)\dfrac{dy}{dx}=0$

i.e.
$$(x-h)+(y-k)\frac{dy}{dx} = 0 \quad \text{... (2)}$$

Differentiating again

$$1+(y-k)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 = 0 \quad \text{... (3)}$$

From (3)
$$(y-k) = \frac{1+\left(\frac{dy}{dx}\right)^2}{-\left(\frac{d^2y}{dx^2}\right)} \quad \text{... (4)}$$

From (2)
$$(x-h) = -(y-k)\frac{dy}{dx}$$

$\therefore$
$$(x-h) = \left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]\frac{dy}{dx}$$

Substituting $(x-h)$ and $(y-k)$ in (1)

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2}\left(\frac{dy}{dx}\right)^2+\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2} = a^2$$

Simplify
$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2 \text{ is the required D.E.}$$

Example 1.10 : *Form the differential equation whose G.S. is $y = A\sin(\omega t+B)$, where A and B are arbitrary constants.* **[Dec.-98]**

Solution : Consider

$$y = A\sin(\omega t+B) \quad \text{... (1)}$$

Given G.S. involves two arbitrary constants A and B hence we will differentiate twice. (with respect to t)

$$\frac{dy}{dt} = A\omega\cos(\omega t + B) \quad \text{... (2)}$$

$$\frac{d^2y}{dt^2} = -A\omega^2\sin(\omega t + B) \quad \text{... (3)}$$

From (1) $$\frac{d^2y}{dt^2} = -\omega^2 y$$

i.e. $$\frac{d^2y}{dt^2} + \omega^2 y = 0 \text{ is the required D.E.}$$

Example 1.11 : *Form a D.E. whose G.S. is* **[Dec.-2002, May-2005]**

$$y = \log\cos(x - a) + b \quad \text{... (1)}$$

Solution : G.S. involves two arbitrary constants, therefore, differentiating two times.

$$\frac{dy}{dx} = \frac{-\sin(x-a)}{\cos(x-a)}$$

i.e. $$\frac{dy}{dx} = -\tan(x-a) \quad \text{... (2)}$$

$$\frac{d^2y}{dx^2} = -\sec^2(x-a)$$

$$= -[1 + \tan^2(x-a)]$$

$$\frac{d^2y}{dx^2} = -\left[1 + \left(\frac{dy}{dx}\right)^2\right] \quad \text{... by using (2)}$$

i.e. $$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0 \text{ is the required D.E.}$$

Example 1.12 : *Find the D.E. of which $Ax^2 + By^2 = 1$ is the G.S.* **[May-98, May-2002, May-2004]**

Solution : Given

$$Ax^2 + By^2 = 1 \quad \text{... (1)}$$

Differentiating w.r.t. x, we get

$$2Ax + 2By\,y_1 = 0 \quad \text{... (2)}$$

or $$Ax + By\,y_1 = 0$$

Differentiating again, we get

$$A + B[y_1^2 + yy_2] = 0 \quad \text{... (3)}$$

If it is not easy to eliminate constants from obtained equations then use method of determinent.

Eliminating A, B from (1), (2) and (3) we get

$$\begin{vmatrix} x^2 & y^2 & -1 \\ x & yy_1 & 0 \\ 1 & y_1^2 + yy_2 & 0 \end{vmatrix} = 0$$

$$-1[xy_1^2 + xyy_2 - yy_1] = 0$$

i.e. $$x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} - y\frac{dy}{dx} = 0 \text{ is the required D.E.}$$

Example 1.13 : *Eliminate the constants from the equation* **[Dec.-99]**

$$y = e^x [a\cos x + b\sin x] \quad \text{... (1)}$$

Solution : Differentiating w.r.t x we get

$$\frac{dy}{dx} = e^x [a\cos x + b\sin x] + [-a\sin x + b\cos x]\, e^x$$

$$\frac{dy}{dx} = y + e^x (-a\sin x + b\cos x)$$

Differentiating again $$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-a\sin x + b\cos x) + (-a\cos x - b\sin x)\, e^x$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-a\sin x + b\cos x) - y$$

i.e. $$\frac{d^2y}{dx^2} = \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - y$$

$$\therefore \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \text{ is the required D.E.}$$

Example 1.14 : *Form the D.E. whose primitive is $y = a e^{2x} + b e^{3x}$* **[Dec.-2003, Dec.-2004]**

Solution : We have $$y = a e^{2x} + b e^{3x} \quad \text{... (1)}$$

$$\therefore \quad y_1 = 2a e^{2x} + 3b e^{3x} \quad \text{... (2)}$$

and $$y_2 = 4a e^{2x} + 9b e^{3x} \quad \text{... (3)}$$

Eliminating a, b using (1), (2) and (3)

$$\begin{vmatrix} e^{2x} & e^{3x} & -y \\ 2e^{2x} & 3e^{3x} & -y_1 \\ 4e^{2x} & 9e^{3x} & -y_2 \end{vmatrix} = 0$$

$$\Rightarrow \quad -e^{2x} \cdot e^{3x} \begin{vmatrix} 1 & 1 & y \\ 2 & 3 & y_1 \\ 4 & 9 & y_2 \end{vmatrix} = 0$$

Solving we get,

$$-e^{5x}\ [1\,(3y_2-9y_1)-1\,(2y_2-4y_1)+y\,(18-12)]=0$$

or $$y_2 - 5y_1 + 6y = 0$$

i.e. $$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Example 1.15 : *Form the D.E. whose G.S. is $y = (C_1 + C_2x)\,e^x$* **[Dec.-2005]**

Solution : Given $$y = (C_1 + C_2x)\,e^x \quad \text{... (1)}$$

$\therefore$ $$y_1 = (C_1 + C_2x)\,e^x + e^x C_2$$

or $$y_1 = y + e^x \cdot C_2 \quad \text{... (2)}$$

Now, $$y_2 = y_1 + e^x \cdot C_2 \quad \text{... (3)}$$

(3) – (2) gives $$y_2 - y_1 = y_1 - y$$

i.e. $$y_2 - 2y_1 + y = 0$$

i.e. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0$ is the required D.E.

Exercise 1.1

Form the differential equations by eliminating the arbitrary constants from the following equations.

1) $y = Ae^x + Be^{-x}$ **[Ans : $\frac{d^2y}{dx^2} - y = 0$]**

2) $y = A\cos(\log x) + B\sin(\log x)$ **[Dec.-2004, May-2005] [Ans : $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$]**

3) $xy = Ae^x + Be^{-x}$ **[Ans : $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy$]**

4) $y = Ae^{-9t}\cos(3t + B)$ **[May-2004] [Ans : $\frac{d^2y}{dx^2} + 18\frac{dy}{dx} + 90y = 0$]**

5) $y = ae^{2x} + be^{-3x} + ce^x$ **[Ans : $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$]**

6) $y = \log(Ax)$ **[Ans : $x\frac{dy}{dx} = 1$]**

7) $y = ae^{2x} + be^{-x}$ **[Ans : $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$]**

8) $y = ax + bx^{-1}$ **[Ans : $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$]**

9) $\frac{x}{A} + \frac{y}{B} = 1$ **[Ans : $\frac{d^2y}{dx^2} = 0$]**

10) $y = Ax^2 + Bx + C$ **[Ans : $\frac{d^3y}{dx^3} = 0$]**

11) $y = \alpha e^{Bx}$ **[Ans : $y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$]**

Solution of Ordinary D.E. of 1st Order and 1st Degree

Type 1

Variable separable form

If the given differential equation can be put in the form :

$$f_1(x)\,dx + f_2(y)\,dy = 0$$

(i.e. the variables are separated)

then its general solution is $\int f_1(x)\,dx + \int f_2(y)\,dy = C$

Example 1.16 : *Solve* $\dfrac{dy}{dx} + \dfrac{1+y^2}{1+x^2} = 0$

Solution : The given equation can be written as

$$\frac{dy}{1+y^2} + \frac{dx}{1+x^2} = 0$$

Integrating both sides

$$\int \frac{dy}{1+y^2} + \int \frac{dx}{1+x^2} = C$$

$$\therefore \quad \tan^{-1} y + \tan^{-1} x = \tan^{-1} C$$

$$\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1} C$$

$$\Rightarrow \quad \frac{x+y}{1-xy} = C \text{ is the G.S.}$$

Example 1.17 : *Solve* $\dfrac{dy}{dx} + \dfrac{y^2+y+1}{x^2+x+1} = 0$

Solution : The given equation can be written as

$$\frac{dy}{y^2+y+1} + \frac{dx}{x^2+x+1} = 0$$

Making the perfect square,

$$\frac{dy}{\left(y+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 0$$

Integrating both sides,

$$\int \frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = C$$

$$\int \frac{dy}{\left(\frac{2y+1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\int \frac{dx}{\left(\frac{2x+1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = C \qquad \left(\text{Use} \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right)$$

$$\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right)+\frac{2}{\sqrt{3}}\tan^{-1}\frac{(2x+1)}{\sqrt{3}} = C$$

$$\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right)+\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) = C_1$$

$$\tan^{-1}\left\{\frac{\frac{2y+1}{\sqrt{3}}+\frac{2x+1}{\sqrt{3}}}{1-\left(\frac{2y+1}{\sqrt{3}}\right)\left(\frac{2x+1}{\sqrt{3}}\right)}\right\} = C_1$$

$$\frac{\left(\frac{2(x+y+1)}{\sqrt{3}}\right)}{\left(\frac{3-(2y+1)(2x+1)}{3}\right)} = \tan C_1 = C_2 \text{ (say)}$$

$$2(x+y+1)\sqrt{3} = C_2(2-4xy-2x-2y)$$

$$\sqrt{3}(x+y+1) = C_2(1-2xy-x-y)$$

Example 1.18 : *Solve* $y\frac{dy}{dx} = \sqrt{1+x^2+y^2+x^2y^2}$

Solution : The given equation can be written as

$$y\frac{dy}{dx} = \sqrt{1+x^2+y^2(1+x^2)} = \sqrt{(1+x^2)(1+y^2)}$$

or $$\frac{1}{2}\times\frac{2y\,dy}{\sqrt{1+y^2}} = \sqrt{1+x^2}\,dx$$

Integrating both sides,

$$\frac{1}{2}\int \frac{2y\,dy}{\sqrt{1+y^2}} = \int \sqrt{1+x^2}\,dx + C$$

Use $\left[\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log\left[x + \sqrt{x^2 + a^2}\right]\right]$

Use $\boxed{\int \frac{f'(x)}{\sqrt{(f(x))}} dx = 2\sqrt{f(x)}}$

$$\therefore \quad \sqrt{1 + y^2} = \frac{x}{2}\sqrt{1 + x^2} + \frac{1}{2}\log\left(x + \sqrt{1 + x^2}\right) + C \text{ is the G.S.}$$

Example 1.19 : *Solve* $\frac{dy}{dx} = \frac{x \cos x}{2e^y \sinh y}$

Solution : The given equation can be written as

$$2e^y \sinh y\, dy = x \cos x\, dx \quad \text{(V.S. form)}$$

On integrating both sides, we get

$$2\int e^y \sinh y\, dy = \int x \cos x\, dx + C$$

$$2\int e^y\left(\frac{e^y - e^{-y}}{2}\right) dy = \int x \cos x\, dx + C$$

$$\frac{e^{2y}}{2} - y = \int x \cos x\, dx + C$$

Integrating by parts,

$$\frac{e^{2y}}{2} - y = x \sin x - \int \sin x\, dx + C$$

$$\frac{e^{2y}}{2} - y = x \sin x + \cos x + C \text{ is G.S.}$$

Example 1.20 : *Solve* $y^2 \cos\sqrt{x}\, dx - 2\sqrt{x}\, e^{1/y}\, dy = 0.$

Solution : The given equation can be written as

$$\frac{\cos\sqrt{x}}{2\sqrt{x}} dx - \frac{e^{1/y}}{y^2} dy = 0$$

On integrating we get,

$$\int \frac{\cos\sqrt{x}}{2\sqrt{x}} dx - \int \frac{e^{1/y}}{y^2} dy = C \Rightarrow \sin\sqrt{x} + e^{1/y} = C$$

Note

$$\boxed{\int \cos f(x)\, f'(x)\, dx = \sin f(x) \text{ and } \int e^{f(y)} f'(y)\, dy = e^{f(y)}}$$

Example 1.21 : *Solve* $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$

Solution : The given equation can be written as,

$$y - ay^2 = \frac{dy}{dx}(a + x)$$

$$\frac{dx}{a + x} = \frac{dy}{1 - ay} = \left(\frac{1}{y} + \frac{a}{1-ay}\right)dy \qquad \text{(by using partial fractions)}$$

Integrating both sides,

$$\int \frac{dx}{a + x} = \int\left(\frac{1}{y} + \frac{a}{1 - ay}\right) dy + \log C$$

$$\log (a + x) = \log y - a \log (1 - ay) + \log C = \log [Cy (1-ay)^{-a}]$$

$$\therefore \quad a + x = Cy (1-ay)^{-a}$$

$$\therefore \quad (a+x)(1-ay)^{-a} = Cy \text{ which is required G.S.}$$

Example 1.22 : *Solve* $\dfrac{dy}{dx} = \dfrac{x(2 \log x + 1)}{\sin y + y \cos y}$

Solution : The equation can be written as,

$$(\sin y + y \cos y)\, dy = x (2 \log x + 1)\, dx$$

Integrating both sides,

$$\therefore \quad \int(\sin y + y \cos y)\, dy = \int x (2 \log x + 1)\, dx + C$$

Integrating by parts,

$$-\cos y + y \sin y + \cos y = 2\left[(\log x)\cdot\frac{x^2}{2} - \int\frac{1}{x}\cdot\frac{x^2}{2}\cdot dx\right] + \frac{x^2}{2} + C$$

$$\therefore \quad y \sin y = x^2 \log x - \frac{x^2}{2} + \frac{x^2}{2} + C$$

$$\therefore \quad y \sin y = x^2 \log x + C$$

Example 1.23 : *Solve* $\dfrac{dy}{dx} = e^{x-y} + 3x^2e^{-y}$.

Solution : Multiplying the given equation by e^y, we get

$$e^y \frac{dy}{dx} = e^x + 3x^2$$

$$e^y\, dy = (e^x + 3x^2)\, dx$$

Integrating both sides we get,

$$\therefore \quad e^y = e^x + x^3 + C \text{ is the G.S.}$$

Example 1.24 : *Solve $3e^x \tan y \cdot dx + \left(1+e^x\right)\sec^2 y \cdot dy = 0$. Given $y = \frac{\pi}{4}$ when $x = 0$.*

Solution : The given differential equation can be written as,

$$\frac{3e^x}{1+e^x}\,dx + \frac{\sec^2 y}{\tan y}\,dy = 0$$

On integrating, we get

$$\int \frac{3e^x}{1+e^x}\,dx + \int \frac{\sec^2 y}{\tan y}\,dy = C \qquad \left(\because \int \frac{f'}{f}\,dx = \log f\right)$$

$$3\log(1+e^x) + \log\tan y = \log C$$

$$\therefore \left(1+e^x\right)^3 \tan y = C \qquad ...(1)$$

Given, $x = 0$ when $y = \frac{\pi}{4}$

$$\therefore (1+1)^3 \tan\frac{\pi}{4} = C$$

$$\therefore \quad C = 8$$

Particular solution is $(1+e^x)^3 \cdot \tan y = 8$

Example 1.25 : *Solve $\frac{dy}{dx} + x^2 = x^2 e^{3y}$*

Solution : The equation can be written as,

$$\frac{dy}{dx} = x^2(e^{3y} - 1)$$

or

$$\frac{dy}{e^{3y}-1} = x^2\,dx$$

$$\therefore \int \frac{dy}{e^{3y}-1} = \int x^2 dx + C$$

Multiply and divide by $3e^{-3y}$

$$\frac{1}{3}\int \frac{3e^{-3y}dy}{1-e^{-3y}} = \frac{x^3}{3} + C$$

$$\log(1 - e^{-3y}) - x^3 = C_1 \text{ where } C_1 = 3C$$

which is the G.S.

Exercise 1.2

Solve the following differential equations

1) $xy\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}\left(1+x+x^2\right)$ **[Ans. : $\log\left(\dfrac{1+y^2}{x^2}\right) - 2\tan^{-1}x = C$]**

2) $\log\dfrac{dy}{dx} = ax + by$ **[Ans. : $ae^{-by} + be^{ax} + C = 0$]**

3) $\dfrac{y\,dy}{x\,dx} = \sqrt{1+x^2+y^2+x^2y^2}$ **[Ans. : $\sqrt{1+y^2} = \dfrac{1}{3}\left(1+x^2\right)^{3/2} + C$]**

4) $x(1-y)\,dx + (1+y^2)(x-1)\,dy = 0$ **[Ans. : $x + \log(x-1) - \dfrac{y^2}{2} - y - \log(1-y) + C = 0$]**

5) $\dfrac{dy}{dx} = e^{x+y} + x^2e^y$ **[Ans. : $e^{-y} + e^x + \dfrac{x^3}{3} + C = 0$]**

6) $3e^x\tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$ **[Ans. : $\dfrac{\tan y}{\left(1-e^x\right)^3} = C$]**

7) $(1+x)\dfrac{dy}{dx} + 1 = 2e^{-y}$ **[Ans. : $\left(e^y - 2\right)(x+1) = C$]**

8) $y - x\dfrac{dy}{dx} = y^2 + \dfrac{dy}{dx}$ *and* $y(1) = \dfrac{1}{2}$

Note $y(1) = \dfrac{1}{2} \Rightarrow$ *for* $x = 1;\ y = \dfrac{1}{2}$ **[Ans. : $(x+1)(y+1) + 2y = 0$]**

9) $xe^{x^2+y^2}\,dx = y\,dy$ **[Ans. : $e^{x^2} + e^{-y^2} = C$]**

10) $\left(x^2 + yx^2\right)\dfrac{dy}{dx} + y^2 + xy^2 = 0$ **[Ans. : $\log\left(\dfrac{x}{y}\right) - \dfrac{1}{x} - \dfrac{1}{y} = C$]**

11) $y\sqrt{1-x^2}\,dy = x\sqrt{1-y^2}\,dx$ **[Ans. : $\sqrt{1-x^2} + \sqrt{1-y^2} = C$]**

12) $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$ **[Ans. : $y\sqrt{1-x^2} + x\sqrt{1-y^2} = C$]**

13) $x(1-y)\,dx + (1+y^2)(x-1)\,dy = 0$ **[Ans. : $x + \log(x-1) - \dfrac{y^2}{2} - y - \log(1-y) + C = 0$]**

14) $x\cos x\cos y + \sin y\dfrac{dy}{dx} = 0$ **[Ans. : $x\sin x + \cos x + \log\sec y = C$]**

15) $\left(xy^2 - x\right)dx = \left(y + x^2y\right)dy$ **[Ans. : $\left(1+x^2\right) = C\left(1-y^2\right)$]**

16) $a\left(x\dfrac{dy}{dx} + 2y\right) = 2xy\dfrac{dy}{dx}$ **[Ans. : $\log x + \dfrac{1}{2}\log y - \dfrac{y}{a} = C$]**

17) $xy^3\dfrac{dy}{dx} = \left(1-x^2\right)\left(1+y^2\right)$ **[Ans. : $\log x - \dfrac{1}{2}\left(x^2+y^2\right) + \dfrac{1}{2}\log\left(1+y^2\right) = C$]**

18) $y - x\dfrac{dy}{dx} = 3\left(1 + x^2\dfrac{dy}{dx}\right)$ **[Ans. : $(y-3)(3x+1) = Cx$]**

19) $\left(1-x^2\right)(1+y)\,dx = xy(1-y)\,dy$ **[Ans. : $\log x - \dfrac{x^2-y^2}{2} = 2y - 2\log(1+y) + C$]**

20) $\left(4+e^{2x}\right)\dfrac{dy}{dx} = ye^{2x}$ **[Ans. : $y^2 = C\left(4+e^{2x}\right)$]**

21) $\dfrac{dy}{dx} = \dfrac{1+y^2}{\left(1+x^2\right)xy}$ **[Ans. : $(1+x^2)(1+y^2) = Cx^2$]**

22) $(3 + 2\sin x + \cos x)\,dy = (1 + 2\sin y + \cos y)\,dy$

Hint : Use half angle formulae **[Ans. : $\dfrac{1}{2}\log(1 + 2\tan y/2) = \tan^{-1}(\tan x/2 + 1) + C$]**

Differential Equations of Reducible to Variables Separable Form by Substitution

The differential equation of form $\frac{dy}{dx} = f(ax + by + c)$ can be reduced to V.S. form by putting $ax + by + c = V$, then find $\frac{dy}{dx}$ and separate the variables in a x and V.

Example 1.26 : *Solve* $(x + y)^2 \left(x\frac{dy}{dx} + y\right) = xy\left(1 + \frac{dy}{dx}\right)$

Solution : Let

$$x + y = u,\ xy = v$$

$$\therefore \quad 1 + \frac{dy}{dx} = \frac{du}{dx} \text{ and } x\frac{dy}{dx} + y \cdot 1 = \frac{dv}{dx}$$

With this equation becomes

$$u^2 \frac{dv}{dx} = v\frac{du}{dx}$$

$$\therefore \quad \frac{dv}{v} = \frac{du}{u^2}$$

V.S. form, integrating,

$$\log v = -\frac{1}{u} + C$$

i.e. $\log xy + \frac{1}{x + y} = C$ is the G.S.

Example 1.27 : *Solve* $\left(ye^{xy} - \tan x\right)dx + \left(xe^{xy} - \sec y\right)dy = 0$

Solution : We rewrite the given differential equation as

$$e^{xy}(y\,dx + x\,dy) + (-\tan x\,dx - \sec y\,dy) = 0$$

(Note this step)

V.S. form integrating,

$$\int e^{xy}d(xy) - \int \tan x\,dx - \int \sec y\,dy = C$$

$$e^{xy} - \log \sec x - \log[\sec y + \tan y] = C$$

Example 1.28 : *Solve* $\frac{dy}{dx} + \frac{x\left(x^2 + y^2 - 1\right)}{y\left(x^2 + y^2 + 1\right)} = 0$

Solution : The given differential equation can be written as

$$x(x^2 + y^2 - 1)\,dx + y(x^2 + y^2 + 1)\,dy = 0$$

$$(x^2 + y^2)(xdx + ydy) - xdx + ydy = 0 \qquad \text{... (1)}$$

Put $\quad x^2 + y^2 = v$

$\therefore \quad 2xdx + 2ydy = dv$

Thus from (1)

$$v\left(\frac{dv}{2}\right) - xdx + ydy = 0$$

Integrating we get

$$\frac{1}{2}\int vdv - \int xdx + \int ydy = C$$

$$\frac{v^2}{4} - \frac{x^2}{2} + \frac{y^2}{2} = C$$

Multiply by 4

$$(x^2 + y^2)^2 - 2x^2 + 2y^2 = 2C = C_1 \text{ (say)}$$

$$\Rightarrow \quad (x^2 + y^2)^2 + 2(y^2 - x^2) = C_1$$

Example 1.29 : *Solve* $x\,dx + y\,dy = y\left(x^2 + y^2\right)dy$

Solution : The equation is

$$x\,dx + y\,dy = y\,(x^2 + y^2)\,dy$$

Put $\quad x^2 + y^2 = u$

$\therefore \quad 2\,(xdx + ydy) = du$

Equation becomes,

$$\frac{du}{2} = y\,u\,dy$$

Or $$\frac{du}{u} = 2y\,dy \quad \text{(V.S. form)}$$

Integrating we get,

$$\int \frac{1}{u}\,du = 2\int ydy + C$$

$$\log u = y^2 + C$$

Example 1.30 : *Solve* $(4x + y)^2 \dfrac{dx}{dy} = 1$

Solution : The equation can be written as

$$\frac{dy}{dx} = (4x + y)^2 \quad \text{... (1)}$$

Put $4x + y = u$

$$\therefore \quad 4 + \frac{dy}{dx} = \frac{du}{dx}$$

From (1)

$$\frac{du}{dx} - 4 = u^2$$

$$\frac{du}{u^2 + 4} = dx \quad \text{(V.S. form)}$$

On integrating, we get

$$\int \frac{du}{u^2+4} = \int \cdot dx + C$$

Or $\dfrac{1}{2} \tan^{-1} \dfrac{u}{2} = x + C$

i.e. $\tan^{-1}\left(\dfrac{4x + y}{2}\right) = 2x + C_1$

Where $C_1 = 2C$

Example 1.31 : *Solve* $\left(\dfrac{x + y - a}{x + y + b}\right) \dfrac{dy}{dx} = \left(\dfrac{x + y + a}{x + y - b}\right)$

Solution : The given equation can be written as,

$$\frac{dy}{dx} = \frac{(x + y + a)\,(x + y + b)}{(x + y - a)\,(x + y - b)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x + y)^2 + (a + b)\,(x + y) + ab}{(x + y)^2 - (a + b)\,(x + y) + ab} \quad \text{...(1)}$$

Put $x + y = V \quad \therefore \quad 1 + \dfrac{dy}{dx} = \dfrac{dV}{dx}$

From (1), we get

$$\frac{dV}{dx} - 1 = \frac{V^2 + (a+b)V + ab}{V^2 - (a+b)V + ab}$$

$$\Rightarrow \quad \frac{dV}{dx} = \frac{V^2 + (a+b)V + ab}{V^2 - (a+b)V + ab} + 1$$

$$= \frac{V^2 + (a+b)\,V + ab + V^2 - (a+b)\,V + ab}{V^2 - (a+b)\,V + ab}$$

$$\frac{V^2 - (a+b)\,V + ab}{(V^2 + ab)}\,dV = 2dx$$

Integrating it,

$$\int \frac{V^2 + ab}{V^2 + ab}\,dV - (a+b)\int \frac{VdV}{V^2 + ab} - 2\int dx = C$$

$$\int dV - \frac{(a+b)}{2}\int \frac{2VdV}{V^2 + ab} - 2\int dx = C$$

Note : $\left[\int \frac{f'(x)}{f(x)}\,dx = \log\, f(x)\right]$

$$\therefore \quad V - \frac{(a+b)}{2}\log(V^2 + ab) - 2x = C$$

$$2V - (a+b)\log(V^2 + ab) - 4x = 2C = C_1$$

$$2x + 2y - (a+b)\log(V^2 + ab) - 4x = C_1$$

$$\Rightarrow \quad 2\,(y - x) - (a+b)\log[(x+y)^2 + ab] = C_1$$

Example 1.32 : *Solve $(x - 2\sin y + 3)\,dx + (2x - 4\sin y - 3)\cos y\,dy = 0$*

Solution : The given differential equation can be written as

$$(x - 2\sin y + 3)\,dx + [2(x - 2\sin y) - 3]\cos y\,dy = 0$$

Put $\quad x - 2\sin y = u$

$$\therefore \quad 1 - 2\cos y\cdot\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{1}{2}\left(1 - \frac{du}{dx}\right) = \cos y\frac{dy}{dx}$$

Given D.E. reduces to,

$$(u+3) + (2u-3)\frac{1}{2}\left(1-\frac{du}{dx}\right) = 0$$

$$2u + 6 + 2u - 3 - (2u - 3)\frac{du}{dx} = 0$$

$$4u + 3 - (2u - 3)\frac{du}{dx} = 0$$

$$dx - \frac{2u - 3}{4u + 3}du = 0$$

$$\Rightarrow \quad 2\int dx - \int \frac{4u + 3 - 9}{4u + 3}du = C \quad \text{... (Multiplying by 2)}$$

$$2\int dx - \int du + 9\int \frac{1}{4u+3}du = C$$

$$2x - u + \frac{9}{4}\log(4u + 3) = C$$

$$x + 2\sin y + \frac{9}{4}\log(4x - 8\sin y + 3) = C \text{ is G.S.}$$

Example 1.33 : *Solve* $\left(2 + 2x^2y^{1/2}\right)y\,dx + \left(x^2y^{1/2} + 2\right)x\,dy = 0$

Solution : Here the term $x^2y^{1/2}$ is the repeated term in the equation.

Thus we substitute

$$x^2y^{1/2} = V \Rightarrow y = \frac{V^2}{x^4}$$

$$\Rightarrow \quad dy = \frac{2V}{x^4}dV - \frac{4V^2}{x^5}dx$$

Substitute these in given equation we have

$$(2 + 2V)\frac{V^2}{x^4}dx + x(V + 2)\left(\frac{2V}{x^4}dV - \frac{4V^2}{x^5}dx\right) = 0$$

$$2(V + 1)\frac{V^2}{x^4}dx + \frac{x}{x^5}(V + 2)\,2V\,[xdV - 2Vdx] = 0$$

$$[V(V + 1) - 2V(V + 2)]\,dx + (V + 2)\,xdV = 0$$

$$V(3 + V)\,dx - x(V + 2)\,dV = 0$$

$$\frac{dx}{x} - \frac{2}{3}\frac{dV}{V} - \frac{1}{3}\frac{dV}{V + 3} = 0$$

Solution : We divide by x^2,

$$\frac{y}{x}\left(\cos\frac{y}{x} + \frac{y}{x}\sin\frac{y}{x}\right) - \left(\frac{y}{x}\sin\frac{y}{x} - \cos\frac{y}{x}\right)\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{y}{x}\left(\cos\frac{y}{x} + \frac{y}{x}\sin\frac{y}{x}\right)}{\left(\frac{y}{x}\sin\frac{y}{x} - \cos\frac{y}{x}\right)}$$

Put $\quad \frac{y}{x} = v,\ y = xv,\ \frac{dy}{dx} = x\frac{dv}{dx} + v$

$$\therefore \quad x\frac{dv}{dx} + v = \frac{v(\cos v + v\sin v)}{(v\sin v - \cos v)}$$

$$x\frac{dv}{dx} = \frac{v(\cos v + v\sin v)}{v\sin v - \cos v} - v$$

$$x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\therefore \quad \left(\frac{v\sin v - \cos v}{v\cos v}\right)dv = \frac{2}{x}dx$$

V.S. form integrating using formula

$$\int\frac{f'(x)}{f(x)}dx = \log f(x)$$

$$2\int\frac{1}{x}dx + \int\frac{\cos v - v\sin v}{v\cos v}dv = \log C$$

$$\Rightarrow \quad 2\log x + \log(v\cos v) = \log C$$

$$\log(x^2 v\cos v) = \log C$$

$$\therefore \quad x^2\frac{y}{x}\cos\frac{y}{x} = C$$

$$xy\cos\frac{y}{x} = C \text{ is the required G.S.}$$

Example 1.41 : *Solve* $xy\log\left(\frac{x}{y}\right)dx + \left(y^2 - x^2\log\left(\frac{x}{y}\right)\right)dy = 0.$

Solution : Given equation can be written as

$$\frac{dx}{dy} = \frac{x^2\log\frac{x}{y} - y^2}{xy\log\frac{x}{y}}$$

$$\therefore \qquad e^v \cdot y \cdot \frac{dv}{dy} = y$$

$$e^v \, dv = dy \qquad \text{(V.S. form)}$$

Integrating,

$$\int e^v \, dv = \int dy + C \Rightarrow e^v - y = C$$

$$\Rightarrow \qquad e^{x/y} - y = C \text{ is the general solution}$$

Example 1.43 : ***Solve*** $\frac{dy}{dx} = \sqrt{y - x}$

Solution : Put $y - x = t^2,$

$$\frac{dy}{dx} - 1 = 2t\frac{dt}{dx}$$

$$\Rightarrow \qquad \frac{dy}{dx} = 2t\frac{dt}{dx} + 1$$

Equation reduces to

$$2t\frac{dt}{dx} + 1 = t$$

$$\therefore \qquad 2t\frac{dt}{dx} = t - 1$$

$$\frac{2t}{t-1} dt = dx$$

Integrating, i.e.

$$\int \frac{2(t-1)+2}{t-1} dt = \int dx + C$$

$$2t + 2 \log (t - 1) - x = C$$

$$2\sqrt{y-x} + 2\log(\sqrt{y-x} - 1) - x = C$$

is the required G.S.

Exercise 1.3

Solve the following differential equations.

1) $e^{x+y}\left(x\frac{dy}{dx} + y\right) - e^{xy}\left(1 + \frac{dy}{dx}\right) = 0$

Hint $x + y = u$, $xy = V$, $1 + \frac{dy}{dx} = \frac{du}{dx}$, $x\frac{dy}{dx} + y = \frac{dV}{dx}$ **[Ans. :** $e^{-xy} = e^{-(x+y)} + C$**]**

2) $\frac{dy}{dx} = \left(\frac{x+y+1}{x+y+3}\right)^2$

Hint : $x + y = u$ **[Ans. : $(x+y)^2 + 4(x+y) + 5 = Ce^{x-y}$]**

3) $\frac{2dy}{dx} + \cos^2(x-2y) = 1$ **[Ans. : $\tan(x-2y) = x + C$]**

4) $(x-y)^2 \frac{dy}{dx} = a^2$ **[Ans. : $\frac{a}{2}\log\frac{x-y-a}{x-y+a} = y + a$]**

5) $(x-2y)\,dy = (3x+y)(3x^2-5xy-2y^2)\,dx$

Hint : $3x^2-5xy-2y^2 = (3x+y)(x-2y)$, put $3x+y = u$. **[Ans. : $x = \frac{1}{\sqrt{3}}\tan^{-1}\frac{3x+y}{\sqrt{3}} + C$]**

6) $(x-2y)\,dy = (2x+y)(2x^2-2y^2-3xy)\,dx$ **[Ans. : $\frac{1}{\sqrt{2}}\tan^{-1}\frac{2x+y}{\sqrt{2}} = x + C$]**

7) $y(x^2+y^2+a^2)\frac{dy}{dx} + x(x^2+y^2-a^2) = 0.$ **[Ans. : $x^4 + 2x^2y^2 - 2a^2x^2 + 2a^2y^2 + y^4 = C$]**

8) $(2x-y)\,e^{y/x} + (y + x\,e^{y/x})\frac{dy}{dx} = 0$ **[Ans. : $y^2 + 2x^2e^{y/x} = C$]**

9) $\frac{dy}{dx} = \tan^2(x+y)$ **[Ans. : $y - x + \sin(x+y)\cos(x+y) = C$]**

10) $(x-y)^2\frac{dy}{dx} + a^2 = 0$ **[Ans. : $y - x = a\tan\left(\frac{C+y}{a}\right)$]**

11) $\frac{dy}{dx} = (x-y+1)^2 + (x-y)$, put $x - y + 1 = u$ **[Ans. : $\frac{x-y+3}{x-y} = Ce^{3x}$]**

12) $\frac{dy}{dx} = \cos x\cos y + \sin x\sin y$ **[Ans. : $x + \cot\left(\frac{x-y}{2}\right) = C$]**

13) $\left[\tan\frac{y}{x} - \frac{y}{x}\sec^2\frac{y}{x}\right]dx + \sec^2\frac{y}{x}\,dy = 0$ **[Ans. : $x\tan\left(\frac{y}{x}\right) = C$]**

14) Solve $\frac{dy}{dx} + e^{y/x} = \frac{y}{x}$ **[Ans. : $\log Cx = e^{-y/x}$]**

15) Solve $\cos(x+y)\,dy = dx$ **[Ans. : $y - \tan\left(\frac{x+y}{2}\right) = C$]**

16) Solve $\frac{dy}{dx} = \sin(y-x)$ **[Ans. : $\tan(y-x) + \sec(y-x) + x = C$]**

17) Solve $x\frac{dy}{dx} - y = x\sqrt{x^2+y^2}$ **[Ans. : $\sinh^{-1}\left(\frac{y}{x}\right) = x + C$]**

18) Solve $(x+y+1)\frac{dy}{dx} = 1$ **[Ans. : $x + y + 2 = C_1 e^y$]**

19) Solve $\frac{dy}{dx} = \frac{2}{x+2y-3}$ **[Ans. : $2y - 3 - 4\log(x+2y+1) = C$]**

20) Solve $\frac{dy}{dx} = \frac{x+2y-3}{3x+6y-1}$ **[Ans. : $\frac{3(x+2y)}{5} + \frac{16}{25}\log\{5(x+2y)-7 = x + C\}$]**

Definition of Homogeneous Functions

A differential equation M dx + N dy = 0 is said to be homogeneous if M and N are homogeneous functions in x and y of same degree.

This equation can be reduced to V.S. form by the substitution.

$$y = vx$$

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Example 1.44 : *Solve* $\left(y^4 - 2x^3y\right) dx + \left(x^4 - 2xy^3\right) dy = 0$

Solution : As M and N are homogeneous functions in x and y of degree 4. ∴ Given D.E. is homogeneous. It can be written as

$$\frac{dy}{dx} = \frac{2x^3y - y^4}{x^4 - 2xy^3} \qquad \dots (1)$$

Put y = vx

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting values of y and $\frac{dy}{dx}$ in equation (1), we get,

$$v + x\frac{dv}{dx} = \frac{2x^4v - x^4v^4}{x^4 - 2x^4v^3}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{2v - v^4}{1 - 2v^3} - v$$

$$= \frac{v + v^4}{1 - 2v^3}$$

$$\therefore \quad \frac{1 - 2v^3}{v + v^4}dv = \frac{dx}{x} \quad \text{which is in variable separable form.}$$

$$\frac{dx}{x} = \left(\frac{1}{v} - \frac{3v^2}{1 + v^3}\right)dv$$

Integrating,

$$\therefore \quad \int\frac{dx}{x} = \int\frac{1}{v}dv - \int\frac{3v^2dv}{1 + v^3} + \log C$$

$$\log x = \log v - \log(1 + v^3) + \log C$$

$$\Rightarrow \quad \frac{x(1 + v^3)}{v} = C$$

$$\therefore \quad x^3 + y^3 = C\,xy \quad \text{is the general solution.}$$

Example 1.45 : *Solve* $y\sqrt{1+\frac{x^2}{y^2}}\,dx - \left(y^3 + x\sqrt{1+\frac{x^2}{y^2}}\right)dy = 0$

Solution : The given D.E. can be written as,

$$\frac{dx}{dy} = \frac{y^3 + x\sqrt{1+\frac{x^2}{y^2}}}{y\sqrt{1+\frac{x^2}{y^2}}} \qquad \text{...(1)}$$

Put $\quad \frac{x}{y} = v \quad$ i.e. $\quad x = vy$

$$\therefore \quad y\frac{dv}{dx} = v + y\frac{dv}{dy}$$

From (1), we get,

$$v + y\frac{dv}{dy} = \frac{y^3 + vy\sqrt{1+v^2}}{y\sqrt{1+v^2}}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{y^2 + v\sqrt{1+v^2}}{\sqrt{1+v^2}} - v$$

$$= \frac{y^2 + v\sqrt{1+v^2} - v\sqrt{1+v^2}}{\sqrt{1+v^2}}$$

$$\Rightarrow \quad \sqrt{1+v^2}\,dv = y\,dy$$

Integrating,

$$\int\sqrt{1+v^2}\,dv = \int y\,dy + C$$

$$\Rightarrow \quad \frac{v}{2}\sqrt{1+v^2} + \frac{1}{2}\log\{v + \sqrt{1+v^2}\} = \frac{y^2}{2} + C$$

$$\frac{x}{y}\sqrt{1+\frac{x^2}{y^2}} + \log\left\{\frac{x}{y} + \sqrt{1+\frac{x^2}{y^2}}\right\} = y^2 + 2C = y^2 + C_1$$

Example 1.46 : *Solve* $x^3\frac{dy}{dx} = y^3 + y^2\sqrt{y^2 - x^2}$

Solution :

$$\frac{dy}{dx} = \frac{y^3}{x^3} + \frac{y^2}{x^2}\sqrt{\frac{y^2}{x^2} - 1}$$

Put $\quad y = xv,\ \frac{dy}{dx} = x\frac{dv}{dx} + v$

$$x\frac{dv}{dx}+v = v^3+v^2\sqrt{v^2-1}$$

$$x\frac{dv}{dx} = v^3+v^2\sqrt{v^2-1}-v$$

$$\Rightarrow \qquad x\frac{dv}{dx} = v\left(v^2-1\right)+v^2\sqrt{v^2-1}$$

$$x\frac{dv}{dx} = v\sqrt{v^2-1}\,(\sqrt{v^2-1}+v)$$

Rationalizing,

$$= v\sqrt{v^2-1}\,\frac{(\sqrt{v^2-1}+v)\,(\sqrt{v^2-1}-v)}{(\sqrt{v^2-1}-v)}$$

$$= \frac{v\sqrt{v^2-1}\left(v^2-1-v^2\right)}{\sqrt{v^2-1}-v}$$

$\therefore \quad \dfrac{v-\sqrt{v^2-1}}{v\sqrt{v^2-1}}\,dv = \dfrac{dx}{x}$ which is in variable separable form.

Integrating,

$$\int\frac{1}{\sqrt{v^2-1}}\,dv-\int\frac{1}{v}\,dv = \int\frac{dx}{x}+\log C$$

$$\left[\because\int\frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left(x+\sqrt{x^2-a^2}\right)\right]$$

$$\log\left(v+\sqrt{v^2-1}\right)-\log v-\log x = \log C$$

$$\frac{(v+\sqrt{v^2-1})}{xv} = C \quad\Rightarrow\quad y+\sqrt{y^2-x^2} = C\,xy$$

Example 1.47 : *Solve* $x^3dx-y^3dy = 3xy\left(y\,dx-x\,dy\right)$

Solution : As given D.E. can be written as

$$(x^3-3\,xy^2)\,dx-(y^3-3\,x^2y)\,dy = 0$$

As degree of each term is same.

$\therefore$ Given D.E. is homogeneous D.E. Above equation can be written as

$$\frac{dy}{dx} = \frac{x^3-3xy^2}{y^3-3x^2y}$$

Let $\qquad y = vx, \quad \dfrac{dy}{dx} = v+x\dfrac{dv}{dx}$

$$\therefore \qquad \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$x\frac{dv}{dx} + v = \frac{1 - 3v^2}{v^3 - 3v}$$

$$x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - v = \frac{1 - v^4}{v^3 - 3v}$$

$$\frac{v^3 - 3v}{1 - v^4} dv = \frac{dx}{x}$$

Integrating,

$$\int \frac{v^3 - 3v}{1 - v^4} dv = \int \frac{dx}{x} + C$$

$$-\frac{1}{4}\int \frac{-4v^3}{1 - v^4} dv - 3\int \frac{vdv}{1 - v^4} - \log x = C$$

$$-\frac{1}{4} \log(1 - v^4) - \frac{3}{2}\int \frac{dt}{1 - t^2} - \log x = C \qquad \left[\because \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left[\frac{a + x}{a - x}\right]\right]$$

Put $\qquad v^2 = t \therefore 2vdv = dt$

$$-\frac{1}{4} \log(1 - v^4) - \frac{3}{4} \log\left(\frac{1 + t}{1 - t}\right) - \log x = C$$

Multiply by (– 4)

$$\log(1 - v^4) + 3\log\left(\frac{1 + t}{1 - t}\right) + 4\log x = -4C = C_1$$

$$\log(1 - v^4) + \log\left(\frac{1 + t}{1 - t}\right)^3 + \log x^4 = \log C_1$$

$$\frac{(1 - v^4)(1 + v^2)^3 x^4}{(1 - v^2)^3} = C_1$$

$$\frac{(1 - v^2)(1 + v^2)(1 + v^2)^3 x^4}{(1 - v^2)^3} = C_1$$

$$\frac{(1 + v^2)^4 x^4}{(1 - v^2)^2} = C_1$$

i.e. $\qquad (1 + v^2)^4 x^4 = C_1 (1 - v^2)^2$

i.e. $\qquad (x^2 + y^2)^4 = C_1 (x^2 - y^2)^2$

Example 1.48 : *Solve* $\left(x + y\cot\frac{x}{y}\right)dy - y\,dx = 0$

Solution : x/y indicates that we use the substitution x = vy.

The equation can be written as

$$x + y\cot\frac{x}{y} - y\frac{dx}{dy} = 0$$

Put $\quad x = vy$

$$\therefore \quad \frac{dx}{dy} = v + y\frac{dv}{dy}, \quad \text{Equation becomes}$$

$$\therefore \quad vy + y\cot v - y\left(v + y\frac{dv}{dy}\right) = 0$$

$$\therefore \quad y\cot v - y^2\frac{dv}{dy} = 0$$

$$\therefore \quad \cot v - y\frac{dv}{dy} = 0$$

$$\therefore \quad \frac{dy}{y} = \tan v\,dv \text{ which is in variable separable form.}$$

Integrating

$$\int\frac{dy}{y} = \int\tan v\,dv + C$$

$$\log y = \log\sec v + \log C$$

$$\therefore \quad y = C\sec v$$

i.e. $\quad y = C\sec\frac{x}{y}$ is G.S.

Example 1.49 : *Solve* $\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx + x\cdot\sec^2\frac{y}{x}\,dy = 0$

Solution : The equation can be written as

$$\frac{dy}{dx} = \frac{y\sec^2\frac{y}{x} - x\tan\frac{y}{x}}{x\sec^2\frac{y}{x}} = \frac{y}{x} - \frac{\tan\frac{y}{x}}{\sec^2\frac{y}{x}}$$

is a homogeneous equation.

Put $\quad y = vx$

$$\therefore \quad v + x\frac{dv}{dx} = v - \frac{\tan v}{\sec^2 v}$$

$$\therefore \quad \frac{\sec^2 v}{\tan v}\,dv + \frac{dx}{x} = 0$$

V.S. form, integrate

$$\int \frac{\sec^2 v}{\tan v}\,dv + \int \frac{dx}{x} = \log C$$

$$\therefore \quad \log \tan v + \log x = \log C$$

$$\therefore \quad x \tan v = C$$

$$\text{i.e.} \quad x \tan \frac{y}{x} = C$$

Example 1.50 : *Solve* $x\dfrac{dy}{dx} = y(\log y - \log x + 1)$

Solution : The given equation can be written as,

$$\frac{dy}{dx} = \frac{y}{x}\left[\log \frac{y}{x} + 1\right] \qquad \ldots (1)$$

$$\text{Put} \quad \frac{y}{x} = V \text{ or } y = Vx$$

$$\therefore \quad \frac{dy}{dx} = V + x\frac{dV}{dx}$$

$$\text{From (1),} \quad V + x\frac{dV}{dx} = V(\log V + 1) = V\log V + V$$

$$\Rightarrow \quad x\frac{dV}{dx} = V\log V$$

$$\frac{dV}{V\log V} = \frac{dx}{x} \quad \text{(V.S. form)}$$

$$\text{Integrating,} \quad \int \frac{dV}{V\log V} = \int \frac{dx}{x} + C_1$$

$$\Rightarrow \quad \log(\log V) = \log x + \log C$$

$$\log\left(\log \frac{y}{x}\right) = \log(Cx)$$

$$\Rightarrow \quad \log\left(\frac{y}{x}\right) = Cx \text{ i.e. } \frac{y}{x} = e^{Cx}$$

$$\Rightarrow \quad y = xe^{Cx}$$

Example 1.51 : *Solve* $x\cos\left(\frac{y}{x}\right)(y\,dx + x\,dy) = y\sin\left(\frac{y}{x}\right)(x\,dy - y\,dx)$

Solution : The given equation can be written as,

$$\cos\left(\frac{y}{x}\right)\left(\frac{y}{x}+\frac{dy}{dx}\right) = \frac{y}{x}\sin\frac{y}{x}\left(\frac{dy}{dx}-\frac{y}{x}\right) \quad ...(1)$$

Put $\frac{y}{x} = V$ or $y = Vx$

$$\therefore \quad \frac{dy}{dx} = V + x\frac{dV}{dx}$$

Substitute in (1)

$$\cos V\left[V + V + x\frac{dV}{dx}\right] = V\sin V\left[V + x\frac{dV}{dx} - V\right]$$

$$\frac{dV}{dx}[x\cos V - xV\sin V] = -2V\cos V$$

$$\frac{(V\sin V - \cos V)}{V\cos V}\,dV = 2\frac{dx}{x}$$

Integrating,

$$\int\left(\frac{V\sin V - \cos V}{V\cos V}\right)dV = 2\int\frac{dx}{x} + C_1$$

$$\Rightarrow \quad -\log(V\cos V) = 2\log x + \log C$$

$$\log\frac{1}{V\cos V} = \log x^2 + \log C$$

$$\log\left\{\frac{1}{\frac{y}{x}\cos\frac{y}{x}}\right\} = \log(x^2C)$$

$$\Rightarrow \quad \frac{x}{y}\sec\frac{y}{x} = x^2\,C$$

$$\sec\frac{y}{x} = Cxy \quad \text{is the G.S.}$$

Exercise 1.4

Solve the following differential equations.

1) $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$ **[Ans. : $\sin\frac{y}{x} = Cx$]**

2) $(xy - x^2)\frac{dy}{dx} = y^2$ **[Ans. : $Cy = e^{y/x}$]**

3) $2\,xy\,dy = (3y^2 + x^2)\,dx$ **[Ans. : $x^3 = C(x^2 + y^2)$]**

4) $x^2\,ydx - (x^3 + y^3)\,dy = 0$ **[Ans. : $\log y - \frac{x^3}{3y^3} = C$]**

5) $\frac{dy}{dx} \frac{x^2 - 3xy + 2y^2}{2xy - x^2}$ **[Ans. : $Cx = e^{y/x}$]**

6) $2xy\,dx + (y^2 + x^2)\,dy = 0$ **[Ans. : $x^2 + y^2 = Cy$]**

7) $(x + y)\frac{dy}{dx} + (x - y) = 0$ **[Ans. : $\log\sqrt{x^2 + y^2} + \tan^{-1}\frac{y}{x} = C$]**

8) $x\sqrt{x^2 + y^2}\,dx - x(x + \sqrt{x^2 + y^2})\,dy = 0$ **[Ans. : $\frac{y}{2x^2}\sqrt{x^2 + y^2} + \frac{1}{2}\log\left(\frac{y + \sqrt{x^2 + y^2}}{x}\right) + y = C$]**

9) $xdy - ydx = \sqrt{x^2 + y^2}\,dx$ **[Ans. : $y + \sqrt{x^2 + y^2} = Cx^2$]**

10) $(x^4 + y^4)dx - 2x^3y\,dy = 0$ **[Ans. : $\log x + \frac{x^2}{y^2 - x^2} = C$]**

11) $x(x - y)\,dy = y(x + y)\,dx$ **[Ans. : $\frac{x}{y} + \log(xy) = C$]**

12) $(2x - y)\,e^{y/x} + (y + xe^{y/x})\,dy = 0$ **[Ans. : $y^2 + 2x^2 + e^{y/x} = C$]**

13) $(x^2y - 2xy^2)\,dx = (x^3 - 3x^2y)\,dy$ **[Ans. : $Cy^3 = x^2\,e^{x/y}$]**

14) $x\frac{dy}{dx} = y + x\cos^2\frac{y}{x}$ **[Ans. : $\log x = \tan\frac{y}{x} + C$]**

15) $yx^2dx = (x^3 - y^3)\,dy$ **[Ans. : $Cy = e^{-\frac{x^3}{3y^3}}$]**

16) $\frac{dy}{dx} + e^{y/x} = \frac{y}{x}$ **[Ans. : $\log Cx = e^{-y/x}$]**

17) $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$ **[Ans. : $Cy = e^{y/x}$]**

18) $(x^2 + y^2)\,dx = 8xy\,dy = 0$ **[Ans. : $x(x^2 + 9y^2)^4 = C$]**

19) $x\frac{dy}{dx} = y(\log y - \log x)$ (*put* $y = vx$) **[Ans. : $y = x\,e^{1 + Cx}$]**

20) $(x^2 - y^2)\,dx = 2xy \cdot dy$ **[Ans. : $x(x^2 - 3y^2) = C$]**

21) $x\frac{dy}{dx} = y + \sqrt{x^2 + y^2}\ y(4) = 3$ *put* $y = vx$ **[Ans. : $y + \sqrt{x^2 + y^2} = 2xe^{x-4}$]**

22) $y^2 + x^2\frac{dy}{dx} = xy \cdot \frac{dy}{dx}$ **[Ans. : $Cy = e^{y/x}$]**

23) $\frac{dy}{dx} = \sec\frac{y}{x} + \frac{y}{x}$ $\quad y(2) = \pi$ [Ans. : $\sin\frac{y}{x} = \log\left(\frac{ex}{2}\right)$]

24) $x\frac{dy}{dx} = y + \sqrt{x^2 + y^2}$ $\quad y(4) = 3$ [Ans. : $y + \sqrt{x^2 + y^2} = 2x\,e^{x-4}$, $x\tan\frac{y}{x} = C$]

25) $\frac{dy}{dx} = \frac{y^3 + 3x^3y}{x^3 + 3xy^2}$ [Ans. : $xy = C(x^2 - y^2)^2$]

26) Solve $y^2\,dx + (x^2 - xy - y^2)\,dy = 0$ [Ans. : $\log y - \tan^{-1}\frac{y}{x} = C$]

27) Solve $x\frac{dy}{dx} + \frac{y^2}{x} = y$ [Ans. : $\log x - \frac{x}{y} = C$]

Non-homogeneous Equations of First Order in x and y

Non-homogeneous equations of first order in x and y can be written as

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \qquad \ldots (A)$$

Case (1)

When $\frac{a_1}{a_2} = \frac{b_1}{b_2}$; we proceed to solve the equation as follows.

If the $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ then in this case $a_2 = ka_1$ and $b_2 = kb_1$

Therefore equation (A) becomes

$$\frac{dy}{dx} = \frac{(a_1x + b_1y) + C_1}{K(a_1x + b_1y) + C_2}$$

As $a_1x + b_1y$ is the repeated factor in this D.E. substituting $a_1x + b_1y = V$. We get variable separable form in v and x.

Example 1.52 : *Solve* $\frac{dy}{dx} = \frac{x - y + 5}{2x - 2y + 5}$

Solution : The given equation is

$$\frac{dy}{dx} = \frac{x - y + 5}{2x - 2y + 5} = \frac{(x-y)+5}{2(x-y)+5} \qquad \ldots (1)$$

Put $\quad x - y = V$

$\Rightarrow \quad 1 - \frac{dy}{dx} = \frac{dV}{dx}$ or $\frac{dy}{dx} = 1 - \frac{dV}{dx}$

$$\therefore \text{ From (1), } 1 - \frac{dV}{dx} = \frac{V+5}{2V+5}$$

$$\Rightarrow \qquad \frac{dV}{dx} = \frac{V}{2V+5}$$

$$\frac{2V+5}{V}\, dV = dx$$

Integrating,

$$\Rightarrow \quad 2\int dV + 5\int \frac{dV}{V} = dx + C$$

$$\Rightarrow \quad 2V + 5 \log V = x + C$$

$$2(x-y) + 5 \log (x-y) = x + C$$

$$(x-y)^5 = C_1 e^{2y-x} \text{ is the G.S.}$$

Example 1.53 : *Solve* $(3y + 2x + 4)\, dx - (4x + 6y + 5)\, dy = 0$

Solution : The given equation can be written as

$$\frac{dy}{dx} = \frac{(2x+3y)+4}{2(2x+3y)+5} \qquad \left(\text{Here } \frac{a_1}{a_2} = \frac{b_1}{b_2}\right)$$

Put $\quad 2x + 3y = u$

$$\therefore \quad 2 + 3\frac{dy}{dx} = \frac{du}{dx}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{3}\left(\frac{du}{dx} - 2\right)$$

The given equation becomes

$$\frac{1}{3}\left(\frac{du}{dx} - 2\right) = \frac{u+4}{2u+5}$$

$$\therefore \quad \frac{du}{dx} - 2 = \frac{3u+12}{2u+5}$$

$$\therefore \quad \frac{du}{dx} = \frac{3u+12}{2u+5} + 2 = \frac{3u+12+4u+10}{2u+5}$$

$$\therefore \quad \frac{(2u+5)}{7u+22}\, du = dx$$

Dividing we get,

i.e. $$\left(\frac{2}{7} - \frac{9}{7}\frac{1}{7u+22}\right) du = dx$$

Integrating,

$$\int \frac{2}{7}\,du - \frac{9}{7}\int \frac{1}{7u+22}\,du = \int dx + C$$

i.e.
$$\frac{2}{7}u - \frac{9}{7}\cdot\frac{1}{7}\log(7u+22) = x + C$$

$$\therefore \frac{2}{7}(2x+3y) - \frac{9}{49}\log(14x+21y+22) = x + C \text{ is G.S.}$$

Example 1.54 : *Solve* $\dfrac{dy}{dx} = \dfrac{x+2y-3}{3x+6y-1}$

Solution : We have, $\dfrac{dy}{dx} = \dfrac{x+2y-3}{3(x+2y)-1}$ ...(1)

Put $x + 2y = V$

Differentiate w.r.t. x

$$1 + 2\frac{dy}{dx} = \frac{dV}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{dV}{dx} - 1\right)$$

Substituting in (1)

$$\frac{1}{2}\left(\frac{dV}{dx} - 1\right) = \frac{V-3}{3V-1}$$

$$\frac{dV}{dx} = \frac{2V-6}{3V-1} + 1$$

$$\frac{3V-1}{5V-7}\,dV = dx$$

Integrating,

$$\int \frac{3}{5}\,dV + \frac{16}{5}\int \frac{dV}{5V-7} = \int dx + C$$

$$\frac{3V}{5} + \frac{16}{5}\left(\frac{1}{5}\right)\log(5V-7) = x + C$$

$$\frac{3(x+2y)}{5} + \frac{16}{25}\log\{5(x+2y)-7\} = x + C \text{ is G.S.}$$

Exercise 1.5

Solve the following

1) $(2x+y+1)dx + (4x+2y-1)dy = 0$ **[Ans. : $x + 2y + \log(2x + y - 1) = C$]**

2) $(6x+9y+6)\dfrac{dy}{dx} = 2x+3y-1$ **[Ans. : $2x + 3y + 1 = Ce^{(x-3y)}$]**

3) $(6x-4y+1)\dfrac{dy}{dx} = 3x+2y-1$ **[Ans. : $x - 2y + \frac{1}{4}\log\left(3x - 2y + \frac{1}{4}\right) = C$]**

4) $\frac{dy}{dx} = \frac{6x - 4y + 3}{3x - 2y + 1}$ **[Ans. : $2x - y = \log(3x - 2y + 3) + C$]**

5) $(4x + 6y + 5)dy = (3y + 2x + 4)\,dx$ **[Ans. : $7x - 14y + 3\log(14x + 21y + 22) = C$]**

6) $\frac{dy}{dx} = \frac{3x - 6y + 1}{6x - 12y + 5}$ **[Ans. : $2(5y - x) = 3(x - 2y)^2 + C$]**

7) $\frac{dy}{dx} = \frac{4x - 6y + 3}{6x - 9y - 1}$ **[Ans. : $\frac{3}{2}(2x - 3y)^2 - (2x - 3y) + 11x = C$]**

8) $\frac{dy}{dx} = \frac{x - y + 3}{2x - 2y + 5}$ **[Ans. : $x - 2y + \log(x - y + 2) = C$]**

9) $\frac{dy}{dx} = \frac{8x + 6y + 12}{4x + 3y + 2}$ **[Ans. : $5(3y - 6x) - 12\log(20x + 15y + 22) = C$]**

10) $\frac{dy}{dx} = \frac{x + 2y - 3}{3x + 6y - 1}$ **[Ans. : $5(x - 3y) = 8\log(5x + 10y - 7) + C$]**

11) $\frac{dy}{dx} = \frac{x + y + 1}{2x + 2y + 3}$ **[Ans. : $6y - 3x + \log(3x + 3y + 4) = C$]**

12) $\frac{dy}{dx} = \frac{3x - 4y - 2}{6x - 8y - 5}$ **[Ans. : $\log(6x - 8y - 7) = C - 2x + 4y$]**

13) Solve $\frac{dy}{dx} = \frac{x + y + 1}{2x + 2y + 1}$ **[Ans. : $6y - 3x - \log(3x + 3y + 2) = C_1$]**

14) Solve $(x + 2y + 1)\,dx - (2x + 4y + 3)\,dy = 0$ **[Ans. : $2y - x\frac{1}{4}\log(4x + 8y + 5) = C_1$]**

Case (2)

$$\frac{dy}{dx} = \frac{a_1x + b_1y + C_1}{a_2x + b_2y + C_2} \qquad \text{...(A)}$$

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

Put $\quad x = X + h, \; y = Y + k \qquad$ ...(1)

Where h, k are constants to be determined.

Using these in equation (A), we have

$$\frac{dY}{dX} = \frac{a_1(X + h) + b_1(Y + k) + C_1}{a_2(X + h) + b_2(Y + k) + C_2} \qquad \text{...(2)}$$

Now choose h and k such that

$$\left.\begin{aligned} a_1h + b_1k + C_1 &= 0 \\ a_2h + b_2k + C_2 &= 0 \end{aligned}\right\} \qquad \text{...(3)}$$

Then equation (2) reduces to

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y} \qquad \text{...(4)}$$

Which is homogeneous differential equation, can be solved by substitution Y = VX.

Example 1.55 : *Solve* $(3y - 7x + 7)\,dx + (7y - 3x + 3)\,dy = 0$

Solution : The given equation can be written as

$$\frac{dy}{dx} = \frac{-(3y - 7x + 7)}{7y - 3x + 3} \quad \text{... (1)}$$

Here $a_1 = +7, \; b_1 = -3$

$a_2 = -3, \; b_2 = 7$

$$\frac{a_1}{a_2} = \frac{7}{-3} \neq \frac{-3}{7} = \frac{b_1}{b_2}$$

Given D.E. is non-homogeneous D.E. of case (2). $\therefore$ Put $x = X + h$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{dY}{dX} \qquad y = Y + k$

Equation (1) becomes,

$$\frac{dY}{dX} = -\frac{3(Y+k) - 7(X+h) + 7}{7(Y+k) - 3(X+h) + 3}$$

$$= -\frac{3Y - 7X + (-7h + 3k + 7)}{7Y - 3X + (-3h + 7k + 3)} \quad \text{... (2)}$$

Now, h and k are so chosen that

$-7h + 3k + 7 = 0$

and $-3h + 7k + 3 = 0$

On solving $h = 1$, $k = 0$ and (2) becomes

$$\frac{dY}{dX} = -\frac{3Y - 7X}{7Y - 3X}$$ a homogeneous equation in X, Y.

Putting $Y = V.X$; we get

$$V + X\frac{dV}{dX} = -\frac{3VX - 7X}{7VX - 3X}$$

$$\therefore \quad X\frac{dV}{dX} = -\frac{7 - 7V^2}{7V - 3}$$

Separating the variables,

$$\frac{7V - 3}{1 - V^2}\,dV = -7\frac{dX}{X}$$

$$\frac{7V - 3}{(1 - V)(1 + V)}\,dV = -7\frac{dX}{X}$$

By partial fractions,

$$\left(\frac{2}{1-V} - \frac{5}{1+V}\right) dV = -7\frac{dX}{X}$$

Integrating,

$$-2\log(1 - V) - 5\log(1 + V) = -7\log X + C$$

$$\therefore\ 7 \log X - 2 \log (1 - V) - 5 \log (1 + V) = \log C$$

$$\therefore \quad \frac{X^7}{(1 - V)^2 (1 + V)^5} = C$$

$$\frac{X^7 (X^7)}{(X-Y)^2 \ (X+Y)^5} = C$$

Substituting $\quad X = x - 1$ and $Y = y$

$$(x - y - 1)^2 \ (x + y - 1)^5 = C_1 (x-1)^{14} \text{ is the general solution.}$$

Example 1.56 : *Solve* $(2x + y - 3)\, dy = (x + 2y - 3)\, dx$

Solution : The given equation is

$$\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3} \quad \therefore \text{ As } \frac{1}{2} \neq \frac{2}{1} \qquad \ldots (1)$$

Put, $\quad x = X + h, y = Y + k$

Substituting in (1)

$$\frac{dY}{dX} = \frac{(X + 2Y) + (h + 2k - 3)}{(2X + Y) + (2h + k - 3)} \qquad \ldots (2)$$

Choose h, k such that $h + 2k - 3 = 0$

And $\quad 2h + k - 3 = 0 \Rightarrow h = k = 1$

From (2), we get

$$\therefore \quad \frac{dY}{dX} = \frac{X + 2Y}{2X + Y}$$

Which is homogeneous.

$$\therefore \quad \text{Put} \quad Y = VX \Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}$$

$$V + X\frac{dV}{dX} = \frac{X + 2VX}{2X + VX} = \frac{1 + 2V}{2 + V}$$

$$X\frac{dV}{dX} = \frac{1 + 2V}{1 - V^2} - V$$

$$\frac{2 + V}{1 - V^2} = \frac{dX}{X}$$

$$\left(\frac{1/2}{1 + V} + \frac{3/2}{1 - V}\right) dV = \frac{dX}{X}$$

Integrating,

$$\frac{1}{2}\int \frac{dV}{1+V} + \frac{3}{2}\int \frac{dV}{1-V} = \int \frac{dX}{X} + C_1$$

$$\log(1+V) - 3\log(1-V) = 2\log X + C$$

$$\log \frac{1+V}{(1-V)^3 X^2} = \log C_1$$

$$\Rightarrow \qquad \frac{1+V}{(1-V)^3 X^2} = C_1$$

$$\frac{1+\frac{Y}{X}}{\left(1-\frac{Y}{X}\right)^3 X^2} = C_1$$

$$\Rightarrow \qquad \frac{X+Y}{(X-Y)^3} = C_1 \qquad \dots (3)$$

$$X + Y = C_1 (X - Y)^3$$

Now $\qquad X = x - 1$

And $\qquad Y = y - 1 \qquad$ Substituting in (3)

$$x + y - 2 = C_1 (x - y)^3$$

Example 1.57 : *Solve* $\left(2x^2 + 3y^2 - 7\right) x\,dx - \left(3x^2 + 2y^2 - 8\right) y\,dy = 0$

Solution : Let $\qquad x^2 = u \,;\, y^2 = v \;\therefore\; 2x\,dx = du \,;\, 2y\,dy = dv$

The equation becomes.

$$(2u + 3v - 7)\,du - (3u + 2v - 8)\,dv = 0.$$

Which is non-homogeneous differential equation.

The further transformation,

$u = p + 2,\ v = q + 1$; yields the homogeneous equation

$$(2p + 3q)\,dp - (3p + 2q)\,dq = 0$$

$$\Rightarrow \qquad \frac{dp}{dq} = \frac{3p + 2q}{2p + 3q} \qquad \text{(Homogeneous)}$$

Let $\qquad p = rq$

$$\Rightarrow \qquad \frac{dp}{dq} = r + q\frac{dr}{dq}$$

Substituting in above equation we have,

$$r + q\frac{dr}{dq} = \frac{3rq + 2q}{2rq + 3q} = \frac{3r + 2}{2r + 3}$$

$$q\frac{dr}{dq} = \frac{3r + 2}{2r + 3} - r = \frac{3r + 2 - 2r^2 - 3r}{2r + 3}$$

$$\frac{(2r + 3)\, dr}{r^2 - 1} + 2\frac{dq}{q} = 0$$

$$\frac{(2r + 3)}{(r - 1)(r + 1)}\, dr + 2\frac{dq}{q} = 0$$

By partial fractions,

$$-\frac{1}{2}\frac{dr}{r + 1} + \frac{5}{2}\frac{dr}{r - 1} + 2\,\frac{dq}{q} = 0$$

Integrating it,

$$-\frac{1}{2}\int\frac{dr}{r + 1} + \frac{5}{2}\int\frac{dr}{r - 1} + 2\int\frac{dq}{q} = 0$$

$$\Rightarrow -\frac{1}{2}\log\,(r + 1) + \frac{5}{2}\log\,(r - 1) + 2\log q = C$$

$$-\log\,(r + 1) + 5\log\,(r - 1) + 4\log q = \log C_1$$

$$\Rightarrow \log\frac{q^4\,(r - 1)^5}{(r + 1)} = \log C_1 \Rightarrow \frac{q^5\,(r - 1)^5}{q\,(r + 1)} = C_1$$

$$\frac{(p - q)^5}{(p + q)} = C_1$$

Substituting $p = u^{-2}$, $q = v^{-1}$

$$\frac{(u - v - 1)^5}{(u + v^2 - 3)} = C_1$$

$$(x^2 - y^2 - 1)^5 = C_1\,(x^2 + y^2 - 3) \quad \text{is G.S.}$$

Example 1.58 : *Solve* $\frac{dy}{dx} + \frac{2x+3y}{y+2} = 0$

Solution : Given differential equation can be written as,

$$\frac{dy}{dx} = \frac{-2x - 3y}{y + 2} \qquad \text{... (1)}$$

Here $\quad \dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$

Put
$$x = X + h;\ y = Y + k.$$
$$dx = dX\ ;\ dy = dY$$
$$\frac{dy}{dx} = \frac{dY}{dX}$$

From (1), we get
$$\therefore \quad \frac{dY}{dX} = \frac{-2X - 3Y - 2h - 3k}{Y + k + 2} \qquad \text{... (2)}$$

We choose h, k such that
$$-2h - 3k = 0,\ k + 2 = 0.\ \text{ Solving we get } h = 3,\ k = -2.$$

From (2), we get
$$\frac{dY}{dX} = \frac{-2X - 3Y}{Y} = \frac{-2 - 3\,Y/X}{Y/X}$$

Put
$$Y = XV,$$
$$\frac{dY}{dX} = V + X\frac{dV}{dX}$$
$$X\frac{dV}{dX} + V = \frac{-2 - 3V}{V}$$
$$X\frac{dV}{dX} = \frac{-2 - 3V}{V} - V = \frac{-2 - 3V - V^2}{V}$$
$$\frac{V\,dV}{V^2 + 3V + 2} = -\frac{dX}{X} \quad \text{(V.S. form)}$$

Integrating,
$$\therefore \quad \int \frac{V}{(V + 2)(V + 1)}\,dV + \int \frac{dX}{x} = \log C$$
$$\int \frac{2}{V + 2}\,dV - \int \frac{1}{V + 1}\,dV + \log X = \log C$$

(using partial fraction)
$$2\log(V + 2) - \log(V + 1) + \log X = \log C$$
$$\log \frac{(V + 2)^2}{(V + 1)}\,X = \log C$$
$$\log \frac{(Y + 2X)^2}{(Y + X)} = \log C$$

Substituting $X = x - 3;\ Y = y + 2$

$$\left[\frac{(y + 2 + 2x - 6)^2}{(x + y - 1)}\right] = C$$

$$(2x + y - 4)^2 = C(x + y - 1)$$

Exercise 1.6

1) $\frac{dy}{dx} = \frac{y - x + 1}{y + x - 5}$ **[Ans. : $\tan^{-1}\left(\frac{y-2}{x-3}\right) + \frac{1}{2}\log[(x-3)^2 + (y-2)^2] = C$]**

2) $\frac{dy}{dx} = \frac{2x + 2y + 1}{3x + y - 2}$ **[Ans. : $(y - x + 3)^4 = C\left(y + 2x - \frac{3}{4}\right)$]**

3) $(4x + 3y + 1)\,dx + (3x + 2y + 1)\,dy = 0$ **[Ans. : $2x^2 + 3xy + y^2 + x + y = C$]**

4) $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$ **[Ans. : $(x - y)^3 = C(x + y - 2)$]**

5) $(x + 2y + 1)\,dx - (2x - 3)\,dy = 0$ **[Ans. : $4y + 5 = (2x - 3)\log[C(2x - 3)]$]**

6) $\frac{dy}{dx} = \frac{y + 2}{x + y - 1}$ **[Ans. : $\log(y + 2) - \frac{x - 1}{y + 2} = C$]**

7) $\frac{dy}{dx} = \frac{2x + y + 3}{2y + x + 1}$ **[Ans. : $\left(x + y + \frac{4}{3}\right)(x - y + 2)^3 = C$]**

8) $(3x - 2y + 1)\,dx + (2x - 3y + 4)\,dy = 0$ **[Ans. : $(x - y + 1)(x + y - 3)^5 = C$]**

9) $(y + 2)\,dy + (2x + 3y)\,dx = 0$ **[Ans. : $(2x + y - 3)^2 = C(x + y - 1)$]**

10) $\frac{dy}{dx} = \frac{x - y + 5}{x + y - 1}$ **[Ans. : $y^2 - x^2 + 2xy - 10x - 2y - 8 = C$]**

11) Solve $(y - 2x)dx + (2y - 3x + 1)dy = 0$ **[Ans. : $(y + 2)^2 - (x + 1)(y + 2) - (x + 1)^2 \left(\frac{2y + 4 - x - 1 + \sqrt{5x} + \sqrt{5}}{2y + 4 - x - 1 - \sqrt{5x} - \sqrt{5}}\right)^{\frac{2}{\sqrt{5}}} = C$]**

12) Solve $(2x - y + 1)dy - (x + 2y + 3)dx = 0$ **[Ans. : $2\tan^{-1}\left(\frac{y+1}{x+1}\right) - \frac{1}{2}\log[(y + 1)^2 + (x + 1)^2] = C$]**

Exact Differential Equations :

A differential equation of the form $M(x, y)\,dx + N(x, y)\,dy = 0$ is said to be exact if there exists a function $u(x, y)$ such that $M\,dx + N\,dy = du$.

where $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Condition of Exactness :

The necessary and sufficient condition that

$$M\,dx + N\,dy = 0 \text{ be exact is } \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

When the condition of exactness is satisfied, the general solution can be obtained by the following formulae.

1) $$\int\limits_{y\,=\,\text{constant}} M\,dx + \int [\text{terms of N free from x}]\,dy = C$$

2) Some times we may write the G.S. by using the following rule.

$$\int\limits_{x\,=\,\text{constant}} N\,dy + \int \big[\text{terms of M free from y}\big]\,dx = C$$

Example 1.59 : *Solve* $\dfrac{dy}{dx} = \dfrac{y+1}{(y+2)\,e^{y} - x}$

Solution : The D.E. can be written as

$$(y+1)\,dx + [x - (y+2)e^{y}]\,dy = 0$$

$$M = y+1,\quad N = x - (y+2)\,e^{y}$$

$$\frac{\partial M}{\partial y} = 1,\quad \frac{\partial N}{\partial x} = 1$$

The given equation is exact. Its G.S. is given as

$$\int\limits_{y\text{ const}} M\,dx + \int\limits_{\text{free } x} N\,dy = C$$

$$\int (y+1)\,dx + \int -(y+2)\,e^{y}\,dy = C$$

$$x\,(y+1) - [(y+2)\,e^{y} - (1)\,e^{y}] = C$$

$$x\,(y+1) - e^{y}\,(y+1) = C$$

$$(y+1)\,(x - e^{y}) = C$$

Note : The above problem can be solved by linear D.E.

Example 1.60 : *Solve* $(ye^{xy} - \tan x)\,dx + (xe^{xy} - \sec y)\,dy = 0$

Solution : Here $M = ye^{xy} - \tan x$, $N = xe^{xy} - \sec y$

$$\therefore \quad \frac{\partial M}{\partial y} = e^{xy} + xye^{xy} = \frac{\partial N}{\partial x}$$

The given equation is exact. Its G.S. is given as

$$\int\limits_{y\text{ const}} M\,dx + \int\limits_{\text{free } x} N\,dy = C$$

$$\int (ye^{xy} - \tan x)\,dx + \int -\sec y\,dy = C$$

$$e^{xy} - \log \sec x - \log(\sec y + \tan y) = C$$

Note : The above problem can be solved by reducible to V.S. by substitution $e^{xy} = V$.

Example 1.61 : $[1 + \log xy]\,dx + \left(1 + \frac{x}{y}\right) dy = 0$

Solution : Here, $M = 1 + \log xy,\ N = 1 + \frac{x}{y}$

$$\frac{\partial M}{\partial y} = \frac{1}{xy} \cdot x = \frac{1}{y}, \frac{\partial N}{\partial x} = \frac{1}{y}$$

The given equation is exact. Its G.S. is given as

$$\int\limits_{y\ \text{const}} M\,dx + \int\limits_{\text{free } x} N\,dy = C$$

$$\int (1 + \log xy)\,dx + \int 1\,dy = C$$

$$\int 1\,dx + \int \log x\,y dx + \int 1\,dy = C$$

$$x + \frac{1}{y}[(xy)\log(xy) - (xy)] + y = C$$

$$\therefore \quad x + x \log xy - x + y = C$$

$$\therefore \quad x \log xy + y = C \text{ is G.S.}$$

Example 1.62 : $\left(x\sqrt{1 - x^2y^2} - y\right) dy + \left(x + y\sqrt{1 - x^2y^2}\right) dx = 0$

Solution : Here $M = x + y\sqrt{1 - x^2y^2},\ N = x\sqrt{1 - x^2y^2} - y$

$$\frac{\partial M}{\partial y} = \sqrt{1 - x^2y^2} - \frac{x^2y^2}{\sqrt{1 - x^2y^2}} = \frac{\partial N}{\partial x}$$

The given equation is exact. Its G.S. is given as

$$\int\limits_{y\ \text{const}} M\,dx + \int\limits_{\text{free } x} N\,dy = C$$

$$\int (x + y\sqrt{1 - x^2y^2})\,dx + \int - y\,dy = C$$

Let $\quad xy = u,$

$\therefore \quad y dx = du$

$$\therefore \quad \int x dx + \int \sqrt{1 - u^2}\,du - \int y dy = C$$

$$\frac{x^2}{2} + \frac{u}{2}\sqrt{1 - u^2} + \frac{1}{2}\sin^{-1}u - \frac{y^2}{2} = C$$

$$\left[\because \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C\right]$$

$$\frac{x^2}{2} + \frac{xy}{2}\sqrt{1 - x^2y^2} + \frac{1}{2}\sin^{-1}(xy) - \frac{y^2}{2} = C$$

$$x^2 - y^2 + xy\sqrt{1 - x^2y^2} + \sin^{-1}xy = C$$

Example 1.63 : *Solve* $\left(y^2 \cdot e^{xy^2} + 4x^3\right)dx + \left(2xy\ e^{xy^2} - 3y^2\right)dy = 0$

Solution : Here

$$M = y^2e^{xy^2} + 4x^3$$

$$\therefore \quad \frac{\partial M}{\partial y} = 2ye^{xy^2} + y^2e^{xy^2}\ (2xy)$$

and

$$N = 2xye^{xy^2} - 3y^2$$

$$\therefore \quad \frac{\partial N}{\partial x} = 2y\, e^{xy^2} + 2xy\, e^{xy^2}\,(y^2)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Given differential equation is exact. General solution is,

$$\int\limits_{y\ \text{const}} M\,dx + \int\limits_{\text{free } x} N\,dy = C$$

$$\int [y^2\, e^{xy^2} + 4x^3]\,dx - \int 3y^2dy = C$$

$$y^2\,\frac{e^{xy^2}}{y^2} + x^4 - y^3 = C$$

$$e^{xy^2} + x^4 - y^3 = C$$

Note : The given problem can be solved by reducible to V.S. substitution.

Example 1.64 : *Solve* $\left(1 + e^{x/y}\right)dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

Solution : Here $\quad M = 1 + e^{x/y};\ N = e^{x/y}\left(1 - \frac{x}{y}\right)$

$$\therefore \quad \frac{\partial M}{\partial y} = e^{x/y}\left(-\frac{x}{y^2}\right)$$

and $$\frac{\partial N}{\partial x} = e^{x/y}\frac{1}{y}\left(1-\frac{x}{y}\right) - e^{x/y}\frac{1}{y} = e^{x/y}\left(-\frac{x}{y^2}\right)$$

$\Rightarrow \quad \therefore$ $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Equation is exact.

$\therefore$ General solution is

$$\int_{y\text{ const}} M\,dx + \int_{\text{free } x} N\,dy = C$$

$$\int\left(1+e^{x/y}\right)dx + \int 0\,dy = C$$

$\Rightarrow$ $$x + y\,e^{x/y} = C$$

Example 1.65 : *Solve* $\left(\dfrac{y^2}{(y-x)^2} - \dfrac{1}{x}\right)dx + \left(\dfrac{1}{y} - \dfrac{x^2}{(x-y)^2}\right)dy = 0$

Solution : Here $$M = \frac{y^2}{(y-x)^2} - \frac{1}{x}$$

$\therefore$ $$\frac{\partial M}{\partial y} = \frac{2xy}{(y-x)^3}$$

and $$N = \frac{1}{y} - \frac{x^2}{(x-y)^2}$$

$\therefore$ $$\frac{\partial N}{\partial x} = \frac{2xy}{(y-x)^3}$$

Given equation is exact as $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

General solution is

$$\int_{y\text{ const}} M\,dx + \int_{\text{free } x} N\,dy = C$$

$$\int\left(\frac{y^2}{(y-x)^2} - \frac{1}{x}\right)dx + \int\frac{1}{y}\,dy = C$$

$$y^2\,\frac{1}{y-x} - \log x + \log y = C$$

$$\frac{y^2}{y-x} + \log\frac{y}{x} = C$$

Example 1.66 : *Solve* $\left(\frac{2x}{y^3}\right)dx + \left(\frac{y^2 - 3x^2}{y^4}\right) dy = 0$

Solution : Here, $M = \frac{2x}{y^3}$; $N = \frac{y^2 - 3x^2}{y^4}$

$$\therefore \quad \frac{\partial M}{\partial y} = -\frac{6x}{y^4} = \frac{\partial N}{\partial x}$$

Given differential equation is exact. Its G. S. is

$$\int_{y\ \text{const}} M dx + \int_{\text{free } x} N dy = C$$

$$\frac{2}{y^3}\int x\, dx + \int \frac{1}{y^2}\, dy = C$$

$$\frac{x^2}{y^3} - \frac{1}{y} = C$$

Example 1.67 : *Solve* $\left(\frac{y}{(x-y)^2} - \frac{1}{2\sqrt{1-x^2}}\right) dx - \frac{x}{(x-y)^2}\, dy = 0$

Solution : Here, $M = \frac{y}{(x-y)^2} - \frac{1}{2\sqrt{1-x^2}}$, $N = \frac{-x}{(x-y)^2}$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{1}{(x-y)^2} + \frac{2y}{(x-y)^3} = \frac{x+y}{(x-y)^3}$$

and

$$\frac{\partial N}{\partial x} = -\left[\frac{1}{(x-y)^2} - \frac{2x}{(x-y)^3}\right]$$

$$= -\left[\frac{x-y-2x}{(x-y)^3}\right]$$

$$= \frac{x+y}{(x-y)^3}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Given differential equation is exact and the G. S. is given by,

$$\int_{y\ \text{const}} M dx + \int_{\text{free } x} N dy = C$$

$$y\int \frac{1}{(x-y)^2}\, dx - \frac{1}{2}\int \frac{dx}{\sqrt{1-x^2}} = C$$

$$\frac{-y}{x-y} - \frac{1}{2}\sin^{-1} x = C$$

$$\frac{2y}{x-y} + \sin^{-1} x = C_1$$

Example 1.68 : $\dfrac{dy}{dx} = -\left(\dfrac{4x^3y^2 + y\cos xy}{2x^4y + x\cos xy}\right)$

Solution : The given equation can be written as

$$(4x^3y^2 + y\cos xy)\,dx + (2x^4y + x\cos xy)\,dy = 0$$

Comparing with M dx + N dy = 0; we have

$$M = 4x^3y^2 + y\cos xy$$

$$N = 2x^4y + x\cos xy$$

$$\therefore \quad \frac{\partial M}{\partial y} = 8x^3y + \cos xy - xy\sin xy$$

$$\frac{\partial N}{\partial x} = 8x^3y + \cos xy - xy\sin xy$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The equation is exact.

And its general solution is

$$\int_{y\,=\,\text{constant}} M dx + \int [\text{terms in N which are free from x}]\, dy = C$$

$$\int (4y^2x^3 + y\cos xy)\,dx + \int 0\,dy = C$$

i.e. $$x^4y^2 + y\cdot\frac{\sin xy}{y} = C$$

i.e. $x^4y^2 + \sin xy = C$

Example 1.69 : *Solve* $[\cos x \cdot \tan y + \cos(x+y)]\,dx + [\sin x\sec^2 y + \cos(x+y)]\,dy = 0$

Solution : Here $M = \cos x \cdot \tan y + \cos(x+y)$

and $N = \sin x \cdot \sec^2 y + \cos(x+y)$

$$\because \quad \frac{\partial M}{\partial y} = \cos x \cdot \sec^2 y - \sin(x+y) = \frac{\partial N}{\partial x}$$

Equation is exact and its general solution.

$$\int_{y\text{ const}} Mdx + \int_{\text{Free x}} Ndy = C$$

$$\int [\cos x \cdot \tan y + \cos(x+y)]\,dx + \int 0 \cdot dy = C$$

i.e. $$\sin x \cdot \tan y + \sin(x+y) = C$$

Example 1.70 : *Solve* $\left(2x^2y + 4x^3 - 12xy^2 + 3y^2 - xe^y + e^{2x}\right)dy + \left(12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y\right)dx = 0$

Solution : Here

$$M = 12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y$$

And

$$N = 2x^2y + 4x^3 - 12xy^2 + 3y^2 - xe^y + e^{2x}$$

$$\therefore \quad \frac{\partial M}{\partial y} = 12x^2 + 4xy - 12y^2 + 2e^{2x} - e^y$$

And

$$\frac{\partial N}{\partial x} = 4xy + 12x^2 - 12y^2 - e^y + 2e^{2x}$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Equation is exact. Its G.S. is

$$\int_{y\ \text{const}} M\,dx + \int_{\text{Free } x} N\,dy = C$$

$$\int\left(12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y\right)dx + \int\left(3y^2\right)dy = C$$

$$\Rightarrow \frac{12x^3y}{3} + \frac{2x^2}{2}y^2 + \frac{4x^4}{4} - 4xy^3 + \frac{2ye^{2x}}{2} - xe^y + \frac{3y^3}{3} = C$$

$$4x^3y + x^2y^2 + x^4 - 4xy^3 + ye^{2x} - xe^y + y^3 = C$$

Example 1.71 : *Solve* $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + (x + \log x - x\sin y)\,dy = 0$

Solution : Here

$$M = y\left(1 + \frac{1}{x}\right) + \cos y$$

$$N = x + \log x - x\sin y$$

$$\therefore \quad \frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y = \frac{\partial N}{\partial x}$$

Equation is exact and its general solution is

$$\int_{y\ \text{const}} M\,dx + \int_{\text{Free } x} N\,dy = C$$

$$\int\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + \int 0\,dy = C$$

i.e.

$$y(x + \log x) + x\cos y = C$$

Example 1.72 : *Solve* $x\,dx + y\,dy = \dfrac{a^2(x\,dy - y\,dx)}{x^2 + y^2}$

Solution :

$$\left(x + \frac{a^2 y}{x^2 + y^2}\right)dx + \left(y - \frac{a^2 x}{x^2 + y^2}\right)dy = 0$$

Here $$M = x + \frac{a^2 y}{x^2 + y^2};$$

$$N = y - \frac{a^2 x}{x^2 + y^2}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{a^2(x^2 - y^2)}{(x^2 + y^2)} = \frac{\partial N}{\partial x}$$

Thus equation is exact. Here G.S. is

$$\int_{y\text{ const}} M\,dx + \int_{\text{Free }x} N\,dy = C$$

$$\int\left(x + \frac{a^2 y}{x^2 + y^2}\right)dx + \int y\,dy = C$$

$$\Rightarrow \quad x^2 + y^2 - 2a\tan^{-1}\frac{y}{x} = C$$

Example 1.73 : *Solve* $\dfrac{dy}{dx} = \dfrac{\tan y - 2xy - y}{x^2 - x\tan^2 y + \sec^2 y}$

Solution : $(\tan y - 2xy - y)\,dx + (x\tan^2 y - x^2 - \sec^2 y)\,dy = 0$

$$\frac{\partial M}{\partial y} = \sec^2 y - 2x - 1 = \tan^2 y - 2x;$$

$$\frac{\partial N}{\partial x} = \tan^2 y - 2x$$

Equation is exact. Its G.S. is

$$\int_{y\text{ const}} M\,dx + \int_{\text{free }x} N\,dy = C$$

$$\int (\tan y - 2xy - y)\,dx + \int -\sec^2 y\,dy = C$$

$$x\tan y - x^2 y - xy - \tan y = C$$

Example 1.74 : *Solve* $\dfrac{dy}{dx} = \dfrac{x - 2y + 5}{2x + y - 1}$

Solution : $(x - 2y + 5)\,dx - (2x + y - 1)\,dy = 0$

Here $$M = x - 2y + 5; \quad N = -(2x + y - 1)$$

$$\therefore \qquad \frac{\partial M}{\partial y} = -2, \frac{\partial N}{\partial x} = -2$$

Thus the equation is exact. Hence

General solution is

$$\int\limits_{y\text{ const}} M\,dx + \int\limits_{\text{Free }x} N\,dy = C$$

$$\int (x - 2y + 5)\,dx + \int (-y + 1)\,dy = C$$

$$\Rightarrow \qquad \frac{x^2}{2} - 2xy + 5x - \frac{y^2}{2} + y = C$$

$$x^2 - 4xy + 10x - y^2 + 2y = C_1$$

Note : The above problem can be solved by Non-homogeneous type 2.

Exercise 1.7

1) $(1 + xy^2)dx + (1 + x^2y)dy = 0$ **[Ans. :** $x + \frac{x^2y^2}{2} + y = C$]

2) $(x\sqrt{x^2 + y^2} - y)\,dx + (y\sqrt{x^2 + y^2} - x)\,dy = 0$ **[Ans. :** $(x^2 + y^2)^{3/2} - 3xy = C$]

3) $(ye^{xy} - \tan x)\,dx + (xe^{xy} - \sec y)\,dy = 0$ **[Ans. :** $e^{xy} - \log \sec x - \log(\sec y + \tan y) = C$]

4) $[y\sin(xy) + xy^2\cos(xy)]dx + [x\sin(xy) + x^2y\cos(xy)]dy = 0$ **[Ans. :** $xy \sin xy = C$]

5) $[2x + \cosh(xy)]dx + \frac{[xy\cosh(xy) - \sinh(xy)]}{y^2}dy = 0$ **[Ans. :** $x^2 + \frac{\sinh(xy)}{y} = C$]

6) $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$ **[Ans. :** $y \sin x + x \sin y + xy = C$]

7) $(2xy + e^y)\,dx + (x^2 + xe^y)\,dy = 0$ **[Ans. :** $x^2y + xe^y = C$]

8) $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$ **[Ans. :** $(e^y + 1)\sin x = C$]

9) $\frac{dy}{dx} = \frac{1 + y^2 + 3x^2y}{1 - 2xy - x^3}$ **[Ans. :** $x^3y + xy^2 + x - y = C$]

10) $(x^2 + 2ye^{2x})\,dy + (2xy + 2y^2e^{2x})\,dx = 0$ **[Ans. :** $x^2y + xy^2e^{2x} = C$]

11) $\frac{dy}{dx} = \frac{x + y - 2}{y - x - 4}$ **[Ans. :** $x^2 + 2xy - y^2 - 4x + 8y = C$]

12) $\left(\frac{1}{x^2} + \frac{3y^2}{x^4}\right)dx = \frac{2y}{x^3}dy$ **[Ans. :** $\frac{1}{x} + \frac{y^2}{x^3} = C$]

13) $\frac{dy}{dx} = \frac{2x - y}{x - y}$ **[Ans. :** $xy - x^2 - \frac{y^2}{2} = C$]

14) $x(x^2 + y^2 + a^2)\,dx + y(x^2 + y^2 - b^2)\,dy = 0$ **[Ans. :** $x^4 + 2x^2y^2 + 2a^2x^2 + y^4 - 2b^2y^2 = C$]

15) $\dfrac{x^2dy - y^2dx}{(x-y)^2} = 0$	[Ans. : $y + \dfrac{y^2}{x-y} = C$]
16) $\dfrac{1}{2x}\dfrac{dy}{dx} + \dfrac{x+y}{x^2+y^2} = 0$	[Ans. : $2x^3 + 3x^2y + y^3 = C$]
17) $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$	[Ans. : $x^3 - 6x^2y + 6xy^2 + y^3 = C$]
18) $\dfrac{dy}{dx} = \dfrac{2x^2 + 3x - 2y + 3}{2y^2 - 3y + 2x - 3}$	[Ans. : $4x^3 + 9x^2 - 12xy + 18x - 4y^3 + 9y^2 + 18y = C$]
19) $\left(\dfrac{y^2}{1+x^2} - 2y\right)dx + (2y\tan^{-1}x - 2x + \sinh y)dy = 0$	[Ans. : $y^2\tan^{-1}x - 2xy + \cosh y = C$]
20) $(2xy^4 + \sin y)dx + (4x^2y^3 + x\cos y)dy = 0$	[Ans. : $x^2y^4 + x\sin y = C$]
21) $\left(\dfrac{1}{x^2} + \dfrac{3y^2}{x^4}\right)dx = \dfrac{2y}{x^3}dy$ given $y(1) = \sqrt{3}$	[Ans. : $\left(\dfrac{1}{x} + \dfrac{y^2}{x^3}\right) = 4$]
22) $y\,dx = (\sin y - x)dy$	[Ans. : $xy + \cos y = C$]
23) $\cos y - x\sin y\dfrac{dy}{dx} = \sec^2 x$	[Ans. : $\tan x - x\cos y = C$]
24) $\dfrac{dy}{dx} = \dfrac{4x - 2y + 1}{2x - 6y + 2}$	[Ans. : $2xy + 2y - 3y^2 - x - 2x^2 = C$]
25) $(\sin x\cos y + e^{2x})\,dx + (\cos x\sin y + \tan y)dy = 0$	[Ans. : $\cos x\cos y + \dfrac{e^{2x}}{2} + \log\sec y = C$]
26) $\left[\log(x^2+y^2) + \dfrac{2x^2}{x^2+y^2}\right]dx + \dfrac{2xy}{x^2+y^2}dy = 0$	[Ans. : $x\log(x^2+y^2) = C$]
27) Solve $(x + y - 2)\,dx + (x - y + 4)\,dy = 0$	[Ans. : $x^2 - y^2 + 2xy - 4x + 8y = 2C = C_1$]

Equations Reducible to Exact Form using Integrating Factor

Integrating factor :

A function f(x, y) is said to be integrating factor (I.F.) for the equation $M\,dx + N\,dy = 0$, if by multiplying this factor the equation becomes exact. i.e. An I.F. is a multiplying factor by which the equation can be made exact.

Rules for finding integrating factors of the equation $Mdx + Ndy = 0$ when it is not exact.

Rule 1 :

If the given D.E. is homogeneous and $xM + yN \neq 0$, then I.F. $= \dfrac{1}{x \cdot M + y \cdot N}$.

Example 1.75 : *Solve* $(3xy^2 - y^3)dx + (xy^2 - 2x^2y)dy = 0$

Solution : The given equation is homogeneous where

$$M = 3xy^2 - y^3;$$

$$N = xy^2 - 2x^2y$$

$$\therefore \qquad \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{3x^3y^2 - xy^3 + xy^3 - 2x^2y^2} = \frac{1}{x^2y^2}$$

Multiplying the equation by $\frac{1}{x^2y^2}$, we get,

$$\left(\frac{3}{x} - \frac{y}{x^2}\right)dx + \left(\frac{1}{x} - \frac{2}{y}\right)dy = 0 \text{ i.e. } M'\,dx + N'\,dy = 0$$

which is exact. General solution is,

$$\int_{y\ \text{const.}} M'\,dx + \int_{\text{free } x} N'\,dy = C$$

$$\int\left(\frac{3}{x} - yx^{-2}\right)dx + \int\left(-\frac{2}{y}\right)dy = C$$

$$3\log x + \frac{y}{x} - 2\log y = C$$

is the general solution.

Example 1.76 : *Solve* $x^2y\,dx - \left(x^3 + y^3\right)dy = 0$

Solution : The given equation is homogeneous.

Here $\qquad M = x^2y$;

$\qquad N = -(x^3 + y^3)$

$$\therefore \qquad \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{x^3y - x^3y - y^4} = -\frac{1}{y^4}$$

Multiplying the equation by $-\frac{1}{y^4}$, we get,

$$\left(-\frac{x^2}{y^3}\right)dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0 \text{ i.e. } M'\,dx + N'\,dy = 0$$

Which is exact. General solution is,

$$\int_{y\ \text{const.}} M'\,dx + \int_{\text{free } x} N'\,dy = C$$

$$\int\left(-\frac{x^2}{y^3}\right)dx + \int\frac{1}{y}\,dy = C$$

$$\Rightarrow \qquad -\frac{x^3}{3y^3} + \log y = C$$

$$\log y - \frac{x^3}{3y^3} = C$$

Example 1.77 : *Solve* $\left(x^2y - 2xy^2\right) dx - \left(x^3 - 3x^2y\right) dy = 0$

Solution : The given equation is homogeneous where

$$M = x^2y - 2xy^2;$$

$$N = -\left(x^3 - 3x^2y\right)$$

$$\therefore \quad \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2}$$

$$= \frac{1}{x^2y^2}$$

Multiplying the equation by $\frac{1}{x^2y^2}$, we get,

$$\left(\frac{1}{y} - \frac{2}{x}\right) dx - \left(\frac{x}{y^2} - \frac{3}{y}\right) dy = 0 \text{ i.e. } M'\,dx + N'\,dy = 0$$

which is exact. General solution is,

$$\int\limits_{y\ \text{const.}} M'\,dx + \int\limits_{\text{free } x} N'\,dy = C$$

$$\int\left(\frac{1}{y} - \frac{2}{x}\right) dx + 3\int \frac{1}{y}\, dy = C$$

i.e. $$\frac{x}{y} - 2\log x + 3\log y = C$$

$$\frac{x}{y} + \log\left(\frac{y^3}{x^2}\right) = \log C_1 \Rightarrow \log\frac{y^3}{C_1x^2} = -\frac{x}{y}$$

$$\Rightarrow \quad y^3 = C_1x^2e^{-x/y}$$

Example 1.78 : *Solve* $\left(x^2 - 3xy + 2y^2\right) dx + \left(3x^2 - 2xy\right) dy = 0$

Solution : The given equation is homogeneous where

$$M = x^2 - 3xy + 2y^2;$$

$$N = 3x^2 - 2xy$$

$$\text{I.F.} = \frac{1}{Mx + Ny}$$

$$= \frac{1}{x^3}$$

Multiplying the equation by $\frac{1}{x^3}$, we get,

$$\left(\frac{1}{x} - \frac{3y}{x^2} + \frac{2y^2}{x^3}\right) dx + \left(\frac{3}{x} - \frac{2y}{x^2}\right) dy = 0$$

i.e. $$M'\,dx + N'\,dy = 0$$

Which is exact. General solution is,

$$\int_{y\ \text{const.}} M' \, dx + \int_{\text{free } x} N' \, dy = C$$

$$\int \left(\frac{1}{x} - 3yx^{-2} + 2y^2x^{-3} \right) dx = C$$

$$\Rightarrow \qquad \log x + 3yx^{-1} + \frac{2y^2}{-2} x^{-2} = C$$

$$\log x + \frac{3y}{x} - \frac{y^2}{x^2} = C$$

Exercise 1.8

1) $(x^3 + y^3)dx - xy^2dy = 0$ **[Ans. : $\log \frac{x}{y} - \frac{1}{3} \frac{y^3}{x^3} = C$]**

2) $x(x - y)\frac{dy}{dx} = y(x + y)$ $\left(I.F. = \frac{1}{2xy^2} \right)$ **[Ans. : $xy^2 = C$]**

3) $(2x - y)^{e^{\frac{y}{x}}} dx + \left(y + xe^{\frac{y}{x}} \right) dy = 0$ **[Ans. : $2x^2e^{y/x} + y^2 = C$]**

4) $y - xp = x + yp,$ $\left(p = \frac{dy}{dx} \right)$ **[Ans. : $\frac{1}{2}\log(x^2 + y^2) = C + \tan^{-1}\left(\frac{x}{y}\right) - \frac{6}{\sqrt{7}} \tan^{-1}\left(\frac{x + 2y}{x\sqrt{7}}\right)$]**

5) $(3xy - 2ay^2)dx - (x^2 - 2axy)\, dy = 0$ **[Ans. : $\log \frac{x}{y} + \frac{2ay}{x} = C$]**

Rule 2 :

If the equation M dx + N dy = 0 can be written as

$$y \, f_1 (xy) \, dx + x \, f_2 (xy) \, dy = 0$$

i.e. $\quad M = y \, f_1 (xy),$

$\quad N = x \, f_2 (xy)$

Then $\frac{1}{Mx - Ny} = \frac{1}{xy \, (f_1 - f_2)}$ is an integrating factor.

Note :

$f_1 (xy)$, $f_2 (xy)$ are functions of (xy).

Example 1.79 : *Solve* $(x^2y^2 + 2) \, y \, dx + (2 - 2x^2y^2) \, xdy = 0$

Solution :

$$M = (x^2y^2 + 2) \, y$$

$$N = (2 - 2x^2y^2) \, x$$

$$\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{xy \, (x^2y^2 + 2 - 2 + 2x^2y^2)} = \frac{1}{3x^3y^3}$$

Multiplying the equation by I.F. we get,

$$\therefore \quad \frac{y(x^2y^2+2)}{3x^3y^3}dx + \frac{x(2-2x^2y^2)}{3x^3y^3}dy = 0 \text{ is exact.}$$

$$\text{i.e. } \left[\frac{1}{3x} + \frac{2}{3}\frac{1}{x^3y^2}\right]dx + \left[\frac{2}{3x^2y^3} - \frac{2}{3y}\right]dy = 0 \text{ is exact}$$

i.e. $$M'\,dx + N'\,dy = 0$$

G.S. is, $$\int\limits_{y\text{ const.}} M'\,dx + \int\limits_{\text{free } x} N'\,dy = C$$

$$\frac{1}{3}\int\left[\frac{1}{x} + \frac{2}{3y^2}\frac{1}{x^3}\right]dx - \frac{2}{3}\int\frac{dy}{y} = C$$

i.e. $$\frac{1}{3}\log x - \frac{2}{6x^2y^2} - \frac{2}{3}\log y = C$$

i.e. $$\log x - \frac{1}{x^2y^2} - 2\log y = C_1$$

Example 1.80 : *Solve* $y(1+xy)\,dx + x(1+xy+x^2y^2)\,dy = 0$

Solution : Given equation is not exact. To find, I.F.

We have $$\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{-x^3y^3}$$

Multiplying by $\frac{-1}{x^3y^3}$ to given equation, we get

$$\left(-\frac{1}{x^3y^2} - \frac{1}{x^2y}\right)dx + \left(\frac{-1}{x^2y^3} - \frac{1}{xy^2} - \frac{1}{y}\right)dy = 0 \qquad \dots (1)$$

i.e. $$M'\,dx + N'\,dy = 0$$

Equation (1) is exact.

Its G.S. is given by

$$\int\limits_{y\text{ const.}} M'\,dx + \int\limits_{\text{free } x} N'\,dy = C$$

$$-\frac{1}{y^2}\int\frac{1}{x^3}dx - \frac{1}{y}\int\frac{1}{x^2}dx - \int\frac{1}{y}dy = C$$

$$\frac{1}{2y^2x^2} + \frac{1}{xy} - \log y = C$$

Example 1.81 : *Solve* $(xy\sin xy + \cos xy)\,ydx + (xy\sin xy - \cos xy)\,xdy = 0$

Solution : Here $$\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{2xy\cos xy}$$

Multiplying the equation by $\frac{1}{2xy\cos xy}$, we get

$$\underbrace{\frac{1}{2}\left(y\tan xy+\frac{1}{x}\right)}_{M'}dx+\underbrace{\frac{1}{2}\left(x\tan xy-\frac{1}{y}\right)}_{N'}dy=0 \quad \dots (1)$$

Equation (1) is exact. ∴ General solution is

$$\int_{y\text{ const}} M'\,dx+\int_{\text{free }x} N'\,dy = C$$

$$\text{Or}\int\left(y\tan xy+\frac{1}{x}\right)dx+\int\left(x\tan xy-\frac{1}{y}\right)dy = C$$

$$y\frac{\log\sec xy}{y}+\log x+x\frac{\log\sec xy}{x}-\log y = \log C$$

$$\therefore \quad \log\sec^2 xy+\log\frac{x}{y} = \log C$$

$$\therefore \quad \frac{x}{y}\sec^2 xy = C$$

Exercise 1.9

1) $(x^2y^2+xy+1)y\,dx+(x^2y^2-xy+1)x\,dy$ [Ans. : $xy+\log x-\frac{1}{xy}-\log y=C$]

2) $(x^2y^2+5xy+2)y\,dx+(x^2y^2+4xy+2)x\,dy=0$ [Ans. : $x^5y^4 = Ce^{\frac{2}{xy}-xy}$]

3) $y(xy-3)dx+x(3xy+7)dy=0$ [Ans. : $y^7(xy+5)^8 = Cx^3$]

4) $y(xy+2x^2y^2)dx+x(xy-x^2y^2)dy=0$ [Ans. : $x^2 = Cy\,e^{\frac{1}{xy}}$]

5) $y(xy-3)dx+x(3xy-3)dy=0$ [Ans. : $\log(xy^3)+\frac{3}{xy} = C$]

6) $(1+xy)y\,dx+(1-xy)x\,dy=0$ [Ans. : $\log\frac{x}{y}-\frac{1}{xy} = C$]

7) $y(xy+2x^2y^2)dx+x(xy-x^2y^2)dy=0$ [Ans. : $x^2 = Cye^{1/xy}$]

8) $(x^3y^3+x^2y^2+xy+1)y\,dx+(x^3y^3-x^2y^2-xy+1)x\,dy=0$ [Ans. : $xy-\frac{1}{xy}-2\log y = C$]

Rule 3 :

Let M dx + N dy = 0 be the given differential equation.

If $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \quad \text{(function of x only)}$$

Then integrating factor = $e^{\int f(x)\,dx}$

Example 1.82 : *Solve* $\left(x^4 e^x - 2mxy^2\right) dx + 2mx^2 y\, dy = 0$

Solution :

Here, $$M = x^4 e^x - 2mxy^2 \; ;$$

$$N = 2mx^2 y$$

$$\therefore \quad \frac{\partial M}{\partial y} = -4mxy \; ;$$

$$\frac{\partial N}{\partial x} = 4mxy$$

Thus $$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Equation is not exact.

But, $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-8mxy}{2mx^2 y} = -\frac{4}{x} = \text{Function of x alone} = \phi(x)$$

$$\therefore \quad \text{I.F.} = e^{-4\int \frac{1}{x} dx} = e^{-4\log x} = x^{-4} = \frac{1}{x^4}.$$

Multiplying given equation by $\frac{1}{x^4}$ we get,

$$\therefore \quad \underbrace{\frac{x^4 e^x - 2mxy^2}{x^4}}_{M'} dx + \underbrace{\frac{2mx^2 y}{x^4}}_{N'} dy = 0 \quad \text{is exact.}$$

General solution is

$$\int_{y\text{ const}} M'\,dx + \int_{\text{free } x} N'\,dy = C$$

$$\int e^x dx - 2my^2 \int \frac{1}{x^3} dx = C$$

$$\therefore \quad e^x + \frac{my^2}{x^2} = C$$

Example 1.83 : *Solve* $\left(xy^2 - e^{1/x^3}\right) dx - x^2 y dy = 0$

Solution :

Here $$M = xy^2 - e^{1/x^3} ;$$

$$N = -x^2 y$$

$$\therefore \quad \frac{\partial M}{\partial y} = 2xy ;$$

$$\frac{\partial N}{\partial x} = -2xy$$

The equation is not exact.

But $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2xy + 2xy}{-x^2 y} = -\frac{4}{x} = f(x)$$

$$\therefore \quad \text{I.F.} = e^{\int\left(-\frac{4}{x}\right)dx} = e^{-4 \log x} = \frac{1}{x^4}$$

Multiplying the given differential equation by $\frac{1}{x^4}$

$$\left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{1/x^3}\right) dx - \frac{y}{x^2} dy = 0 \text{ which is exact.}$$

i.e. $$M' dx + N' dy = 0$$

General solution is

$$\int_{y \text{ const}} M' dx + \int_{\text{free } x} N' dy = C$$

$$\int\left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{1/x^3}\right) dx = C$$

i.e. $$\left(-\frac{y^2}{2x^2}\right) + \left(\frac{e^{1/x^3}}{3}\right) = C$$

$$\left[\because \frac{d}{dx} \frac{1}{x^3} = -\frac{3}{x^4} \text{ and } \int e^{f(x)} f'(x) = e^{f(x)}\right]$$

Example 1.84 : *Solve* $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right) dx + \left(\frac{x + xy^2}{4}\right) dy = 0$

Solution :

Here $$M = y + \frac{y^3}{3} + \frac{x^2}{2} ;$$

$$N = \frac{x + xy^2}{4}$$

$$\frac{\partial M}{\partial y} = 1 + y^2$$

$$\frac{\partial N}{\partial x} = \frac{1 + y^2}{4}$$

Given equation is not exact consider,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{(1 + y^2) - \left(\frac{1 + y^2}{4}\right)}{x\left(\frac{1 + y^2}{4}\right)} = \frac{3}{x} = f(x)$$

$$\therefore \qquad \text{I.F.} = e^{\int f(x)\,dx} = e^{\int \frac{3}{x}\,dx} = e^{3\log x} = x^3$$

Multiplying by x^3 to given equation,

$$\left(yx^3 + \frac{x^3y^3}{3} + \frac{x^5}{2}\right)dx + \left(\frac{x^4 + x^4y^2}{4}\right)dy = 0 \qquad \dots (1)$$

i.e. $\qquad M'\,dx + N'\,dy = 0$

(1) is exact and its G.S. is,

$$\int\limits_{y\text{ const.}} M'\,dx + \int\limits_{\text{free } x} N'\,dy = C$$

$$y\int x^3dx + \frac{y^3}{3}\int x^3dx + \frac{1}{2}\int x^5dx = C$$

$$3yx^4 + x^4y^3 + x^6 = C_1$$

Example 1.85 : *Solve* $(x^2 + y^2 + x)\,dx + xy\,dy = 0$

Solution :

$$M = x^2 + y^2 + x$$

$$N = xy$$

$$\therefore \qquad \frac{\partial M}{\partial y} = 2y, \qquad \frac{\partial N}{\partial x} = y$$

$$\therefore \qquad \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y - y}{xy} = \frac{1}{x} = f(x)$$

$$\therefore \qquad \text{I.F.} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Multiplying given equation by x,

$$\left(x^3 + xy^2 + x^2\right) dx + x^2 y \, dy = 0 \text{ which is exact.}$$

i.e. $$M' \, dx + N' \, dy = 0$$

General solution is

$$\int\limits_{y \text{ const}} M' \, dx + \int\limits_{\text{free } x} N' \, dy = C$$

$$\int \left(x^4 + xy^2 + x^2\right) dx = C$$

$$\Rightarrow \qquad \frac{x^4}{4} + \frac{x^2 y^2}{2} + \frac{x^3}{3} = C$$

Example 1.86 : *Solve* $\left(x \sec^2 y - x^2 \cos y\right) dy = \left(\tan y - 3x^4\right) dx$

Solution :

$$\left(\tan y - 3x^4\right) dx + \left(x^2 \cos y - x \sec^2 y\right) dy = 0 \qquad \ldots (1)$$

Equation (1) is not exact. To find I.F. consider

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{\sec^2 y - 2x \cos y + \sec^2 y}{x \left(x \cos y - \sec^2 y\right)} = \frac{-2 \left[x \cos y - \sec^2 y\right]}{x \left[x \cos y - \sec^2 y\right]}$$

$$= -\frac{2}{x} = f(x)$$

$$\therefore \qquad \text{I.F} = e^{\int \frac{2}{x} dx} = e^{-2 \log x}$$

$$\therefore \qquad \text{I.F.} = \frac{1}{x^2}$$

Multiplying to equation (1) by I.F.

$$\left(\frac{\tan y}{x^2} - 3x^2\right) dx + \left(\cos y - \frac{\sec^2 y}{x}\right) dy = 0$$

i.e. $$M' \, dx + N' \, dy = 0 \qquad \ldots (2)$$

Equation (2) is exact, G.S. is

$$\int\limits_{y \text{ const}} M' \, dx + \int\limits_{\text{free } x} N' \, dy = C$$

$$\tan y \int \frac{1}{x^2} dx - 3 \int x^2 dx + \int \cos y \, dy = C$$

$$-\frac{\tan y}{x} - x^3 + \sin y = C$$

Example 1.87 : *Solve* $\left(2x\sinh\frac{y}{x} + 3y\cosh\frac{y}{x}\right)dx - 3x\cosh\frac{y}{x}dy = 0$

Solution :

Here
$$M = 2x\sinh\frac{y}{x} + 3y\cosh\frac{y}{x};$$

$$N = -3x\cosh\frac{y}{x}$$

$$\therefore \quad \frac{\partial M}{\partial y} = 2x\cosh\frac{y}{x}\left(\frac{1}{x}\right) + 3\cosh\frac{y}{x} + 3y\sinh\frac{y}{x}\left(\frac{1}{x}\right)$$

$$= 5\cosh\frac{y}{x} + \frac{3y}{x}\sinh\frac{y}{x}$$

$$\frac{\partial N}{\partial x} = -3x\sinh\frac{y}{x}\left(-\frac{y}{x^2}\right) - 3\cosh\frac{y}{x}$$

$$= 3\frac{y}{x}\sinh\frac{y}{x} - 3\cosh\frac{y}{x}$$

$$\therefore \quad \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{8\cosh\frac{y}{x}}{-3x\cosh\frac{y}{x}} = -\frac{8}{3x} = f(x)$$

$$\therefore \quad \text{I.F.} = e^{\int -\frac{8}{3x}dx} = e^{-\frac{8}{3}\log x} = x^{-8/3} = f(x)$$

Multiplying the given equation by $x^{8/3}$ we have

$$\left[\left(2x^{-5/3}\sinh\frac{y}{x} + 3x^{-8/3}\,y\cosh\frac{y}{x}\right)dx - 3x^{-5/3}\cosh\frac{y}{x}dy = 0\right]$$

i.e. $\quad M'\,dx + N'\,dy = 0$

Which is exact.

Here it is difficult to integrate the term $\int M\,dx$

$\therefore$ We use formula $\displaystyle\int_{x\,=\,\text{const.}} N'\,dy + \int(\text{terms of } M' \text{ free from } y)\,dx = C$

General solution is

$$\int -3x^{-5/3}\cosh\frac{y}{x}\,dy = C$$

$$\Rightarrow \quad -3x^{-5/3}\left(\sinh\frac{y}{x}\right)(x) = C$$

$$x^{-2/3}\sinh\frac{y}{x} = C_1$$

Exercise 1.10

1) $(x-y)^2\,dx + 2xy\,dy = 0$ [Ans. : $\frac{y^2}{x} + \log x = C$]

2) $(y-2x^3)\,dx - x(1-xy)\,dy = 0$ [Ans. : $\frac{y}{x} + x^2 - \frac{y^2}{2} = C$]

3) $(x^2+y^2+1)\,dx - 2xy\,dy = 0$ [Ans. : $x^2 - y^2 - 1 = Cx$]

4) $(2x\log x - xy)\,dy + 2y\,dx = 0$ [Ans. : $2y\log x - \frac{y^2}{2} = C$]

5) $(20x^2 + 8xy + 4y^2 + 3y^3)\,y\,dx + 4(x^2 + xy + y^2 + y^3)\,x\,dy = 0$
[Ans. : $4x^5y + 2x^4y^2 + \frac{4}{3}x^3y^3 + y^4x^3 = C$]

6) *Show that the differential equation* $\psi(x)dy + \{1 + y\phi(x)\}dx = 0$ *has an integrating factor which is a function of x only. Find it and solve the equation.* [Ans. : I.F. $= \frac{1}{\Psi(x)} e^{\int \frac{\phi(x)}{\psi(x)}dx}$

G.S. is $\int \frac{1}{\psi(x)} e^{\int \frac{\phi(x)}{\psi(x)}dx}\,dx + y e^{\int \frac{\phi(x)}{\psi(x)}dx} = C$]

Rule 4 :

If the given differential equation is of the form M dx + N dy = 0 and

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y) \text{ [Function of y only]}$$

Then $$\text{I.F.} = e^{\int f(y)\,dy}$$

Example 1.88 : *Solve* $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

Solution :

Here
$$M = 3x^2y^4 + 2xy\,;$$
$$N = 2x^3y^3 - x^2$$
$$\therefore \quad \frac{\partial M}{\partial y} = 12x^2y^3 + 2x\,;$$
$$\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$
$$\therefore \quad \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-6x^2y^3 - 4x}{y(3x^2y^3 + 2x)} = \frac{-2(3x^2y^3 + 2x)}{y(3x^2y^3 + 2x)}$$
$$= -\frac{2}{y} = f(y)$$

$$\therefore \qquad \text{I.F.} = e^{\int -\frac{2}{y}dy} = e^{-2\log y} = \frac{1}{y^2} = f(y)$$

Multiplying the given equation by $\frac{1}{y^2}$ we have,

$$\underbrace{\left(3x^2y^2 + \frac{2x}{y}\right)}_{M'} dy + \underbrace{\left(2x^3y - \frac{x^2}{y^2}\right)}_{N'} dy = 0$$

Which is exact. General solution is,

$$\int\limits_{y\text{ const}} M'\,dx + \int\limits_{\text{free } x} N'\,dy = C$$

$$\int\left(3x^2y^2 + \frac{2x}{y}\right) dx = C$$

$$\frac{3x^3}{3}y^2 + \frac{2x^2}{2y} = C$$

$$x^3y^3 + x^2 = Cy$$

Example 1.89 : *Solve* $\left(\frac{y}{x}\sec y - \tan y\right)dx + (\sec y \log x - x)\,dy = 0$

Solution :

Here $\quad M = \frac{y}{x}\sec y - \tan y\,;$

$$N = \sec y \log x - x$$

$$\frac{\partial M}{\partial y} = \frac{1}{x}\sec y + \frac{y}{x}\sec y \tan y - \sec^2 y$$

$$\frac{\partial N}{\partial x} = \frac{\sec y}{x} - 1, \quad \left(\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right)$$

$$\therefore \qquad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -1 - \frac{y}{x}\sec y \tan y + \sec^2 y$$

$$= \tan y\left(\tan y - \frac{y}{x}\sec y\right)$$

$$\therefore \qquad \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{\tan y\left(\tan y - \frac{y}{x}\sec y\right)}{\frac{y}{x}\sec y - \tan y} = -\tan y = f(y)$$

$$\therefore \qquad \text{I.F.} = e^{-\int \tan y \cdot dy} = e^{-\log \sec y} = \cos y$$

Multiplying given equation by I.F. we get,

$$\therefore \cos y\left(\frac{y}{x}\sec y - \tan y\right)dx + \cos y\,(\sec y \log x - x)\,dx = 0$$

$$\underbrace{\left(\frac{y}{x} - \sin y\right)}_{M'}dx + \underbrace{(\log x - x\cos y)}_{N'}dy = 0$$

General solution is

$$\int_{y\text{ const.}} M'\,dx + \int_{\text{free } x} N'\,dy = C$$

$$y\int\frac{1}{x}dx - \sin y\int dx = C$$

i.e. $y\log x - x\sin y = C$

Example 1.90 : *Solve $y \log y\, dx + (x - \log y)\, dy = 0$*

Solution :

Here $\quad M = y\log y\,;$

$$N = x - \log y$$

$$\therefore \quad \frac{\partial M}{\partial y} = 1 + \log y$$

$$\therefore \quad \frac{\partial N}{\partial x} = 1$$

$$\therefore \quad \frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \frac{1 - 1 - \log y}{y\log y} = -\frac{1}{y} = f(y)$$

$$\therefore \quad \text{I.F.} = e^{-\int 1/y\,dy} = e^{-\log y} = e^{\log (y)^{-1}} = y^{-1} = \frac{1}{y}$$

Multiplying the given differential equation by $\frac{1}{y}$ we get,

$$\underbrace{\log y\,dx}_{M'} + \underbrace{\left(\frac{x}{y} - \frac{\log y}{y}\right)}_{N'}dy = 0$$

Which is exact. General solution is

$$\int_{y\text{ const.}} M'\,dx + \int_{\text{free } x} N'\,dy = C$$

$$\int \log y \; dx + \int \left(-\frac{\log y}{y}\right) dy = C$$

$$\left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$x \log y - \frac{(\log y)^2}{2} = C$$

Example 1.91 : *Solve* $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$

Solution :

Here, $\quad M = y^4 + 2y, \quad N = xy^3 + 2y^4 - 4x$

$$\frac{\partial M}{\partial y} = 4y^3 + 2;$$

$$\frac{\partial N}{\partial x} = y^3 - 4$$

$$\therefore \quad \frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \frac{-3(y^3+2)}{y(y^3+2)} = -\frac{3}{y} = f(y)$$

$$\therefore \quad \text{I.F.} = e^{-3\int \frac{1}{y} dy} = e^{-3\log y} = y^{-3} = \frac{1}{y^3}$$

Multiplying by I.F. to the equation we get,

$$\therefore \quad \underbrace{\frac{(y^4+2y)}{y^3}}_{M'} dx + \underbrace{\frac{(xy^3+2y^4-4x)}{y^3}}_{N'} dy = 0$$

It is exact differential equation.

General solution is

$$\int_{y\text{ const.}} M'\,dx + \int_{\text{free } x} N'\,dy = C$$

$$\left(y + \frac{2}{y^2}\right)\int dx + 2\int y\,dy = C$$

i.e.
$$\left(y + \frac{2}{y^2}\right)x + y^2 = C$$

Example 1.92 : *Solve* $y\left(2x^2y + e^x\right)\, dx = \left(e^x + y^3\right)\, dy$

Solution : Given

$$\underbrace{(2x^2y^2 + ye^x)}_{M}\, dx + \underbrace{(-e^x - y^3)}_{N}\, dy = 0 \qquad \dots (1)$$

$$\therefore \qquad \frac{\partial M}{\partial y} = 4x^2y + e^x;$$

$$\frac{\partial N}{\partial x} = -e^x$$

(By using rule 4) consider,

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \frac{-2(2x^2y + e^x)}{y(2x^2y + e^x)} = -\frac{2}{y} = f(y)$$

$$\therefore \qquad \text{I.F.} = e^{-2\int \frac{1}{y} dy} = e^{-2\log y} = \frac{1}{y^2}$$

Multiplying by I.F. to equation (1), equation (1) becomes

$$\underbrace{\left(2x^2 + \frac{e^x}{y}\right)}_{M'} dx + \underbrace{\left(-\frac{e^x}{y^2} - y\right)}_{N'} dy = 0 \qquad \dots (2)$$

Equation (2) is exact and G.S. is

$$\int_{y\ \text{const.}} M'\, dx + \int_{\text{free } x} N'\, dy = C$$

$$\int 2x^2 dx + \frac{1}{y}\int e^x dx - \int y\, dy = C$$

$$\frac{2x^3}{3} + \frac{e^x}{y} - \frac{y^2}{2} = C$$

Exercise 1.11

1) $xy \log\frac{x}{y} dx + \left(y^2 - x^2 \log\frac{x}{y}\right) dy = 0$ **[Ans. :** $\frac{x^2}{2y^2}\left(\log\frac{x}{y} - \frac{1}{2}\right) + \log y = C$**]**

2) $y(2xy + e^x)\, dx = e^x dy$ **[Ans. :** $x^2 + \frac{e^x}{y} = C$**]**

3) $(2x^4e^y + 2xy^3 + y)\, dx + (x^2y^4e^y - x^2y^2 - 3x)\, dy = 0$ **[Ans. :** $x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = C$**]**

4) $\frac{dy}{dx}(x+2y^3)=y+2x^3y^2$	[Ans. : $\frac{x}{y}+\frac{x^4}{2}-y^2=C$]
5) $x^2y^3\,dx+(x^3y-2)\,dy=0$	[Ans. : $3x^3y-2y-6=Cy^{3/y}$]
6) $y(x^2y+e^x)\,dx-e^x dy=0$	[Ans. : $\frac{x^3}{3}+\frac{e^x}{y}=C$]
7) $(2x+e^x\log y)\,y\,dx+e^x dy=0$	[Ans. : $x^2+e^x\log y=C$]

Rule 5 :

If the equation M dx + N dy = 0 can be written as

$x^a y^b(mydx+nxdy)+x^{a_1}y^{b_1}(m_1ydx+n_1xdy)=0$

Where a, b, m, n, a_1, b_1, m_1, n_1 are all constants. Then I.F. will be $x^h y^k$ where h and k are constants such that after multiplying the equation by integrating factor, the condition of exactness is satisfied.

Example 1.93 : *Solve* $\left(y^3-2x^2y\right)dx+\left(2xy^2-x^3\right)dy=0$

Solution : Here

$$M = y^3-2x^2y,\ N=2xy^2-x^3$$

$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ $\therefore$ Given equation is not exact.

The given equation can be written as,

$$(y^3dx+2xy^2dy)-(2x^2ydx+x^3dy) = 0$$

i.e. $y^2(ydx+2xdy)+x^2(-2y\,dx-xdy)=0$

which is of the form

$x^a y^b(mydx+nxdy)+x^{a_1}y^{b_1}(m_1ydx+n_1xdy)=0$

Where a = 0, b = 2, m = 1, n = 2, a_1 = 2, b_1 = 0, m_1 = –2, n_1 = – 1

$\therefore$ The I.F. must be $x^h y^k$

Multiplying given D.E. by the I.F. = $x^h y^k$, we get

$x^h y^k(y^3-2x^2y)dx+(2xy^2-x^3)x^h y^k\,dy=0$

$\Rightarrow (x^h y^{k+3}-2x^{h+2}y^{k+1})dx+(2x^{h+1}y^{k+2}-x^{h+3}y^k)dy=0$ ... (1)

i.e. $M_1dx+N_1dy = 0$

$$\frac{\partial M_1}{\partial y} = (k+3)\,x^h y^{k+2}-2(k+1)\,x^{h+2}y^k$$

$$\frac{\partial N_1}{\partial x} = 2(h+1)\,x^h y^{k+2} - (h+3)\,x^{h+2} y^k$$

Equation (1) is exact if $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ which requires,

$$k+3 = 2(h+1) \; ; -2(k+1) = -(h+3)$$

Solving we get h = 1, k = 1.

$\therefore$ I.F. $= x^h y^k = xy$

Multiplying the given equation by xy we get,

$$(xy^4 - 2x^3y^2)dx + (2x^2y^3 - x^4y)dy = 0 \text{ which is exact.}$$

$$\therefore \text{It's G.S. is} \int_{\text{y constant}} (xy^4 - 2x^3y^2)dx + \int_{\text{free x}} 0\, dy = C$$

$$y^4x^2 - y^2x^4 = 2C = C_1$$

Example 1.94 : *Solve* $\left(2x^2y^2 + y\right) dx - \left(x^3y - 3x\right) dy = 0$

Solution : Here

$$M = 2x^2y^2 + y, \; N = -x^3y + 3x$$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ Given equation is not exact.

It can be written as

$$x^2y\,(2y\,dx - x\,dy) + (y\,dx - 3x\,dy) = 0$$

which is of the form

$$x^a y^b (my\,dx + nx\,dy) + x^{a_1} y^{b_1} (m_1 y dx + n_1 x dy) = 0$$

$\therefore$ The I.F. $= x^h y^k$

Multiplying given equation by I.F. $= x^h y^k$, we get

$x^h y^k (2x^2y^2 + y)dx + (-x^3y + 3x)x^h y^k\, dy = 0$

$\Rightarrow (2x^{h+2}y^{k+2} + x^h y^{k+1})dx + (-x^{h+3}y^{k+1} + 3x^{h+1}y^k)dy = 0$

i.e. $\quad M_1 dx + N_1 dy = 0 \qquad \ldots (1)$

Equation (1) is exact if $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

i.e. $$2(k+2)\,x^{h+2}y^{k+1} + (k+1)\,x^h y^k = -(h+3)x^{h+2}y^{k+1} + 3(h+1)\,x^h y^k$$

$$\Rightarrow \quad 2(k+2) = -(h+3) \text{ and } k+1 = 3(h+1)$$

Solving we get $h = -\frac{11}{7}$ and $k = -\frac{19}{7}$

$$\therefore \quad \text{I.F.} = \frac{1}{x^{\frac{11}{7}}y^{\frac{19}{7}}}$$

Multiplying the given equation by I.F. $= \frac{1}{x^{\frac{11}{7}}y^{\frac{19}{7}}}$

We get $\left(2x^{\frac{3}{7}}y^{-\frac{5}{7}} + x^{-\frac{11}{7}}y^{-\frac{12}{7}}\right)dx - \left(x^{\frac{10}{7}}y^{-\frac{12}{7}} - 3x^{-\frac{4}{7}}y^{-\frac{19}{7}}\right)dy = 0$

which is exact D.E.

$\therefore$ It's G.S. is

$$\int\limits_{y\text{ constant}} \left(2x^{\frac{3}{7}}y^{-\frac{5}{7}} + x^{-\frac{11}{7}}y^{-\frac{12}{7}}\right)dx + \int\limits_{\text{free } x} 0\, dy = C$$

$$\Rightarrow \quad \frac{7}{5}x^{\frac{10}{7}}y^{-\frac{5}{7}} - \frac{7}{4}x^{-\frac{4}{7}}y^{-\frac{12}{7}} = C$$

Example 1.95 : *Solve* $(x^2y + y^4)\,dx + (2x^3 + 4xy^3)\,dy = 0$

Solution : Here

$$M = x^2y + y^4,\ N = 2x^3 + 4xy^3$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ Given equation is not exact.

The given equation can be written as

$$x^2(ydx + 2x\,dy) + y^3(ydx + 4x\,dy) = 0$$

Which is of the form

$x^a y^b(my\,dx + nx\,dy) + x^{a_1} y^{b_1}(m_1 y\,dx + n_1 x\,dy) = 0$

$\therefore$ The I.F $= x^h y^k$

Multiplying given equation by I.F. $= x^h y^k$

We get

$$(x^{h+2}y^{k+1} + x^h y^{k+4})dx + (2x^{h+3}y^k + 4x^{h+1}y^{k+3})dy = 0 \quad \dots (1)$$

i.e. $\quad M_1 dx + N_1 dy = 0$

Equation (1) is exact if

and $\quad \dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$

$$\Rightarrow \quad (k+1)\, x^{h+2}y^k + (k+4)\, x^h y^{k+3} = 2(h+3)\, x^{h+2}y^k + 4(h+1)\, x^h y^{k+3}$$

$$\Rightarrow \quad k+1 = 2(h+3) \text{ and } k+4 = 4(h+1)$$

Solving we get $h = \frac{5}{2}$, $k = 10$

$\therefore$ The $\quad$ I.F. $= x^{\frac{5}{2}}y^{10}$

Multiplying given D.E. by I.F. $= x^{\frac{5}{2}}y^{10}$

We get

$$\left(x^{\frac{9}{2}}y^{11} + x^{\frac{5}{2}}y^{14}\right)dx + \left(2x^{\frac{11}{2}}y^{10} + 4x^{\frac{7}{2}}y^{13}\right)dy = 0$$

which is exact.

$\therefore$ It is G.S. is

$$\int_{y\ \text{constant}} \left(x^{\frac{9}{2}}y^{11} + x^{\frac{5}{2}}y^{14}\right)dx + \int_{\text{free } x} 0\, dy = C$$

$$\Rightarrow \quad 7x^{\frac{11}{2}}y^{11} + 11x^{\frac{7}{2}}y^{14} = C_1$$

Exercise 1.12

1) $(xy^2 + 2x^2y^3)\,dx + (x^2y - x^3y^3)\,dy = 0$ **[Ans. : $5x^{-36/13}y^{24/13} - 12x^{-10/13}y^{-15/13} = C$]**

2) $(x^7y^2 + 3y)\,dx + (3x^8y - x)\,dy = 0$ **[Ans. : $2x^7y^3 - y^2 = Cx^6$]**

3) $(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$ **[Ans. : $x^2y^2\left(x^2 - y^2\right) = C$]**

4) $y(3y + 10x^2)\,dx - 2x(y + 3x^2)\,dy = 0$ **[Ans. : $x^3y^{-2} + 2x^5y^{-3} = C$]**

5) $(x^2y + y^4)\,dx + (2x^3 + 4y^3x)\,dy = 0$ **[Ans. : $\frac{1}{11}x^{11/2}y^{11} + \frac{1}{7}x^{7/2}y^{14} = C$]**

6) $(y^3 - 2x^2y)\,dx - (2xy^2 + x^3)\,dy = 0$ **[Ans. : $x^{-6/5}y^{12/5} + 3x^{4/5}y^{2/5} = C$]**

7) $(3x + 2y^2)\,y\,dx + 2x(2x + 3y^2)\,dy = 0$ **[Ans. : $x^3y^4 + x^2y^6 = C$]**

8) $(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0$	[Ans. : $5x^{-36/13} \cdot y^{24/13} - 12x^{10/13}y^{15/13} = C$]
9) $(y^2 + 2yx^2)dx + (2x^3 - xy)dy = 0$	[Ans. : $6\sqrt{xy} - x^{-3/2}y^{3/2} = C$]
10) $(2y + 6xy^2)dx + (3x + 8x^2y)dy = 0$	[Ans. : $x^2y^3 + 2x^2y^4 = C$]
11) $(3xy + 8y^5)dx + (2x^2 + 24xy^4)dy = 0$	[Ans. : $x^3y^2 + 4x^2y^6 = C$]

Linear Differential Equations of the first order :

Definition :

A differential equation is said to be linear if the degree of the differential equation is one.

Type 1 :

General form

A differential equation of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{... (1)}$$

where P, Q are functions of 'x' or constants, is called a linear differential equation of the first order in y.

Method of solution :

For Type 1 :

An I.F. of equation (1) is $e^{\int Pdx}$.

Therefore, the G.S. is given by

$$ye^{\int Pdx} = \int Q \cdot e^{\int Pdx}\, dx + C$$

Type 2 :

$\frac{dx}{dy} + P \cdot x = Q$ where P and Q are functions of 'y' or constants is also called a linear differential equation of the first order in x.

Method of solution for type 2

we write I.F. $= e^{\int P \cdot dy}$ and G.S. is given by

$$x \cdot e^{\int P \cdot dy} = \int Q\, e^{\int Pdy}\, dy + C$$

Example 1.96 : *Solve* $y^2 + \left(x - \frac{1}{y}\right)\frac{dy}{dx} = 0$

Solution : Note that the equation contains y^2 and so it cannot be linear in y. We try to see whether it is linear in x. We write the given equation as

$$y^2 \frac{dx}{dy} + x = \frac{1}{y}$$

$$\Rightarrow \qquad \frac{dx}{dy} + \frac{1}{y^2}\cdot x = \frac{1}{y^3} \qquad \text{(linear in x)}$$

$$P = \frac{1}{y^2},\ Q = \frac{1}{y^3}$$

$$\text{I.F.} = e^{\int \frac{1}{y^2}dy} = e^{-\frac{1}{y}}$$

Hence G.S. is,

$$x\cdot e^{-\frac{1}{y}} = \int \frac{1}{y^3} e^{-\frac{1}{y}} dy + C = \int \frac{1}{y}\cdot e^{-\frac{1}{y}}\left(\frac{1}{y^2}dy\right) + C$$

$$= \int(-t)e^t dt + C = -(te^t - e^t) + C \qquad \left(\because t = \frac{1}{y}\right)$$

$$= e^{-\frac{1}{y}}\left(1 - \frac{1}{y}\right) + C$$

$$\therefore \qquad x = 1 - \frac{1}{y} - Ce^{\frac{1}{y}} \quad \text{is required G.S.}$$

Example 1.97 : *Solve* $(1 + \sin y)\frac{dx}{dy} = 2y\cos y - x(\sec y + \tan y)$

Solution : Divide by 1 + sin y,

$$\frac{dx}{dy} + \left(\frac{\sec y + \tan y}{1 + \sin y}\right)x = \frac{2y\cos y}{1 + \sin y}$$

$$\frac{dx}{dy} + \left(\frac{\frac{1 + \sin y}{\cos y}}{1 + \sin y}\right)x = \frac{2y\cos y}{1 + \sin y}$$

$$\frac{dx}{dy} + (\sec y)\,x = \frac{2y\cos y}{1 + \sin y} \qquad \dots (1)$$

Equation (1) is linear in x with $P = \sec y$; $Q = \frac{2y\cos y}{1 + \sin y}$

$$\text{I.F.} = e^{\int \sec y\, dy} = e^{\log(\sec y + \tan y)} = \sec y + \tan y$$

G.S. of (1) is

$$x\,(\sec y + \tan y) = \int \frac{2y\cos y}{1+\sin y}\cdot(\sec y + \tan y)\,dy + C$$

$$= \int \left(\frac{2y\cos y}{1+\sin y}\right)\cdot\frac{(1+\sin y)}{\cos y}\,dy + C$$

$$= \int 2y\,dy + C = y^2 + C$$

$$\therefore \quad x\,(\sec y + \tan y) - y^2 = C$$

Example 1.98 : *Solve* $(1 + y^2) + \left(x - e^{-\tan^{-1} y}\right)\dfrac{dy}{dx} = 0$

Solution : The given equation can be written as

$$(1 + y^2)\frac{dx}{dy} + x = e^{-\tan^{-1} y}$$

$$\frac{dx}{dy} + \left(\frac{1}{1+y^2}\right)\cdot x = \frac{e^{-\tan^{-1} y}}{1+y^2} \text{ which is linear in x.}$$

$$\text{I.F.} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$$

$$\therefore \text{ G.S. is, } x\cdot e^{\tan^{-1} y} = \int \frac{e^{-\tan^{-1} y}}{1+y^2}\,e^{\tan y^{-1}}\,dy + C$$

$$= \int \frac{dy}{1+y^2} + C$$

$$\therefore \quad x\cdot e^{\tan^{-1} y} = \tan^{-1} y + C \text{ is the G.S.}$$

Example 1.99 : *Solve* $\cosh x\cdot\dfrac{dy}{dx} = 2\cosh^2 x\cdot\sinh x - y\sinh x$

Solution : The equation can be written as

$$\frac{dy}{dx} + \frac{\sinh x}{\cosh x}y = \frac{2\cosh^2 x\cdot\sinh x}{\cosh x}$$

$$\text{i.e.} \quad \frac{dy}{dx} + \tanh x\cdot y = 2\sinh x\cdot\cosh x \text{ is linear in y with}$$

$$P = \tanh x,\; Q = 2\sinh x\cosh x$$

$$\text{I.F.} = e^{\int P\cdot dx} = e^{\int \tanh x\cdot dx}$$

$$= e^{\log\cosh x} = \cosh x$$

$$y\,(\cosh x) = \int 2\sinh x\cdot\cosh^2 x\cdot dx + C \text{ is G.S.}$$

$$\left[\because \int [f(x)]^n\, f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$\therefore \frac{2}{3}\cosh^3 x + C \text{ is G.S.}$$

Example 1.100 : *Solve* $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$

Solution : The equation can be written as

$$\frac{dy}{dx} + \frac{1}{\sqrt{x}}y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

Here $P = \frac{1}{\sqrt{x}}, \quad Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int x^{-1/2}dx} = e^{2\sqrt{x}}$$

G.S. is $ye^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} e^{2\sqrt{x}}\,dx + C$

$$= \int x^{-1/2}dx + C$$

$\Rightarrow \quad ye^{2\sqrt{x}} = 2\sqrt{x} + C$

Example 1.101 : *Solve* $\frac{dy}{dx} + \left(\frac{x}{(1-x^2)^{3/2}}\right)\cdot y = \frac{x(1+\sqrt{1-x^2})}{(1-x^2)^2}$

Solution : Given differential equation is of the type

$$\frac{dy}{dx} + Py = Q,$$

Linear in y, $P = \frac{x}{(1-x^2)^{3/2}}$

$$Q = \frac{x(1+\sqrt{1-x^2})}{(1-x^2)^2}$$

$$\int P\,dx = \int \frac{x}{(1-x^2)^{3/2}}\,dx = -\frac{1}{2}\int (1-x^2)^{-3/2}(-2x\,dx)$$

$$= -\frac{1}{2}\left[\frac{(1-x^2)^{-1/2}}{-1/2}\right] = \frac{1}{\sqrt{1-x^2}}$$

$$\left[\text{Note } \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$\therefore \quad \text{I.F.} = e^{\frac{1}{\sqrt{1-x^2}}}$

G.S. $$ye^{\frac{1}{\sqrt{1-x^2}}} = \int \frac{x(1+\sqrt{1-x^2})}{(1-x^2)^2} e^{\frac{1}{\sqrt{1-x^2}}} dx + C$$

$$= \int \frac{(1+\sqrt{1-x^2})}{(1-x^2)^{1/2}} \cdot \frac{x}{(1-x^2)^{3/2}} \cdot e^{\frac{1}{\sqrt{1-x^2}}} dx$$

Put $$\frac{1}{\sqrt{1-x^2}} = u, \Rightarrow \frac{x\,dx}{(1-x^2)^{3/2}} = du$$

$$\int e^u(u+1)\,du = e^u u + C$$

Integrating by parts, we get,

$$y \cdot e^{\frac{1}{\sqrt{1-x^2}}} = e^{\frac{1}{\sqrt{1-x^2}}} \frac{1}{\sqrt{1-x^2}} + C$$

$$y = \frac{1}{\sqrt{1-x^2}} + Ce^{\frac{-1}{\sqrt{1-x^2}}}$$

Example 1.102 : *Solve* $\frac{dy}{dx} + y(1-x^2)^{-3/2} = \left(x+\sqrt{1-x^2}\right)(1-x^2)^{-2}$

Solution :

Here $$P = (1-x^2)^{-3/2}$$

$$Q = (x+\sqrt{1-x^2})(1-x^2)^{-2}$$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int (1-x^2)^{3/2} dx} = e^{\frac{x}{\sqrt{1-x^2}}}$$

G.S. is $$ye^{\frac{x}{\sqrt{1-x^2}}} = \int \frac{x+\sqrt{1-x^2}}{(1-x^2)^2} e^{\frac{x}{\sqrt{1-x^2}}} dx + C$$

$$= \int \left(1 + \frac{x}{\sqrt{1-x^2}}\right) \cdot e^{\frac{x}{\sqrt{1-x^2}}} \cdot \frac{1}{(1-x^2)^{3/2}} dx + C \quad \text{... (1)}$$

Put $$\frac{x}{\sqrt{1-x^2}} = u$$

$$\frac{1}{(1-x^2)^{3/2}} dx = du$$

Substituting in equation (1), we get

$$y\,e^{\frac{x}{\sqrt{1-x^2}}} = \int (1+u)e^u\,du + C$$

$$= u\,e^u + C$$

Substituting u we get

$$ye^{\frac{x}{\sqrt{1-x^2}}} = \frac{x}{\sqrt{1-x^2}} e^{\frac{x}{\sqrt{1-x^2}}} + C$$

i.e.
$$y = \frac{x}{\sqrt{1-x^2}} + Ce^{-\frac{x}{\sqrt{1-x^2}}}$$

Example 1.103 : *Solve* $x(x-1)\frac{dy}{dx} - (x-2)\, y = x^3(2x-1)$

Solution : The given equation can be written as

$$\frac{dy}{dx} - \frac{x-2}{x(x-1)}\, y = \frac{x^2(2x-1)}{x-1}$$

Here
$$P = -\frac{(x-2)}{x(x-1)}$$

$$Q = \frac{x^2(2x-1)}{x-1}$$

$\therefore$
$$\text{I.F.} = e^{\int P dx} = e^{-\int \frac{(x-2)}{x(x-1)} dx} = e^{\int \left(-\frac{2}{x} + \frac{1}{x-1}\right) dx}$$

$$= e^{-2\log x + \log(x-1)} = e^{\log\left(\frac{x-1}{x^2}\right)} = \frac{x-1}{x^2}$$

Hence, the general solution is,

$$y(\text{I.F.}) = \int Q(\text{I.F.})\, dx + C$$

$\Rightarrow$
$$y\left(\frac{x-1}{x^2}\right) = \int \frac{x^2\,(2x-1)}{x-1}\left(\frac{x-1}{x^2}\right) dx + C$$

$$\frac{y(x-1)}{x^2} = x^2 - x + C$$

Example 1.104 : *Solve* $x(1-x^2)\frac{dy}{dx} + (2x^2-1)\; y = x^3$

Solution : The equation can be written as

$$\frac{dy}{dx} + \frac{2x^2-1}{x(1-x^2)}\, y = \frac{x^3}{x(1-x^2)}$$

is linear independent variable y.

Here $\quad P = \dfrac{2x^2-1}{x(1-x^2)} = \dfrac{2x^2-1}{x(1-x)(1+x)}$

$$= -\frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)} \quad \text{(by partial fraction)}$$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int \left[-\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}\right] dx}$$

$$= e^{-\log x - \frac{1}{2}\log(1-x) - \frac{1}{2}\log(1+x)} = e^{-[\log x\sqrt{1-x^2}\,]}$$

$$= e^{\log [x\sqrt{1-x^2}\,]^{-1}} = \frac{1}{x\sqrt{1-x^2}}$$

G.S. is $\quad y\left(\dfrac{1}{x\sqrt{1-x^2}}\right) = \displaystyle\int \frac{1}{x\sqrt{1-x^2}}\,\frac{x^2}{1-x^2}\,dx + C$

$$\therefore \quad \frac{y}{x\sqrt{1-x^2}} = \int \frac{x}{(1-x^2)^{3/2}}\,dx + C$$

Use substitution $\quad u = \dfrac{1}{\sqrt{1-x^2}} \quad \therefore\ du = \dfrac{x\,dx}{(1-x^2)^{3/2}}$

$\therefore$ We get,

$$\frac{y}{x\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} + C$$

Example 1.105 : *Solve* $x^2(x^2-1)\dfrac{dy}{dx} + x(x^2+1)\,y = x^2-1$

Solution : $\dfrac{dy}{dx} + \left(\dfrac{x^2+1}{x(x^2-1)}\right) y = \dfrac{1}{x^2}$ which is linear in y.

$$P = \frac{x^2+1}{x(x^2-1)}, \quad Q = \frac{1}{x^2}$$

$$\int P\,dx = \int \frac{x^2+1}{x(x-1)(x+1)}\,dx = \int \left(\frac{1}{x-1}+\frac{1}{x+1}-\frac{1}{x}\right) dx \quad \text{(Partial fractions)}$$

$$= \log(x-1) + \log(x+1) - \log x = \log\left(\frac{x^2-1}{x}\right)$$

$$\therefore \quad \text{I.F.} = \int e^{P}\,dx = \frac{x^2-1}{x}$$

$$\therefore \quad \text{G.S. is } y\left(\frac{x^2-1}{x}\right) = \int \frac{1}{x^2}\left(\frac{x^2-1}{x}\right) dx + C = \int \left(\frac{1}{x}-\frac{1}{x^3}\right) dx + C$$

$$= \log x + \frac{1}{2x^2} + C$$

$\dfrac{y(x^2-1)}{x} - \log x - \dfrac{1}{2x^2} = C \quad$ is the required G.S.

Examples 1.106 : *Solve* $y\,e^{y} = \left(y^3 + 2\,x\,e^{y}\right)\dfrac{dy}{dx}$

Solution : The given equation can be written as

$$\frac{dx}{dy} = y^2\,e^{-y} + \frac{2x}{y}$$

$$\Rightarrow \quad \frac{dx}{dy} - \frac{2}{y}\,x = y^2\,e^{-y}$$ which is linear in x.

Here $\quad P = -\dfrac{2}{y}, \quad Q = y^2\,e^{-y}$

$$\therefore \quad \text{I.F.} = e^{\int P\,dy} = e^{-2\int \frac{1}{y}dy} = e^{-2\log y} = e^{\log(y)^{-2}} = \frac{1}{y^2}$$

G.S. is

$$x\left(\frac{1}{y^2}\right) = \int y^2\,e^{-y}\left(\frac{1}{y^2}\right)dy + C$$

$$\frac{x}{y^2} = \int e^{-y\,dy} + C$$

$$\Rightarrow \quad \frac{x}{y^2} = -e^{-y} + C$$

$$x + y^2\,e^{-y} = C\,y^2$$

Exercise 1.13

1) $\dfrac{dy}{dx} + y\cot x = \sin 2x$ **[Ans. : $y = \dfrac{2}{3}\sin^2 x + C\,\text{cosec}x$]**

2) $x\log x\dfrac{dy}{dx} + y = 2\log x.$ **[Ans. : $y\log x = (\log x)^2 + C$]**

3) $(x + 2y^3)\dfrac{dy}{dx} = y.$ **[Ans. : $x = y^3 + Cy$]**

4) $\dfrac{dy}{dx} = \dfrac{y}{2y\log y + y - x}.$ **[Ans. : $x = \dfrac{C}{y} + y\log y$]**

5) $(1 + x^2)\dfrac{dy}{dx} + y = e^{\tan^{-1}}x.$ **[Ans. : $2\,y\,e^{\tan^{-1}x} + C$]**

6) $(1 - x^2)\dfrac{dy}{dx} + 2xy = x\sqrt{1 - x^2}.$ **[Ans. : $y = \sqrt{1 - x^2} + C(1 - x^2)$]**

7) $\sin x\dfrac{dy}{dx} + 2y = \tan^3\dfrac{x}{2}.$ **[Ans. : $y\tan^{-1}\dfrac{x}{2} = \dfrac{\tan^5\frac{x}{2}}{5} + C$]**

8) $(1 + \sin y)\dfrac{dx}{dy} = [2y\cos y - x(\sec y + \tan y)].$ **[Ans. : $x(\sec y + \tan y) = y^2 + C$]**

9) $\sqrt{a^2 + x^2}\,\dfrac{dy}{dx} + y = [\sqrt{a^2 + x^2} - x].$ **[Ans. : $y\,(\sqrt{a^2 + x^2} + x) = a^2\log[\sqrt{a^2 + x^2} + x] + C$]**

10) $x^2\dfrac{dy}{dx} = 3x^2 - 2xy + 1.$ **[Ans. : $x^2\,y = x^3 + x + C$]**

11) $(x^2+1)^3 \dfrac{dy}{dx} + 4x(x^2+1)^2 y = 1.$	[Ans. : $(x^2+1)^2 y = \tan^{-1} x + C$]
12) $dx + x\,dy = e^{-y} \sec^2 y\,dy.$	[Ans. : $x e^y = x^3 + x + C$]
13) $(x+a) \dfrac{dy}{dx} - 3y = (x+a)^5.$	[Ans. : $2y = (x+a)^5 + 2C(x+a)^3$]
14) $\cosh x \dfrac{dy}{dx} + y \sinh x = 2\cosh^2 x \sinh x.$	[Ans. : $y \cos x = \dfrac{2}{3} \cosh^3 x + C$]
15) $(e^{-y} \sec^2 y - x)\,dy = dx.$	[Ans. : $x e^y = C + \tan y$]
16) $(1+x^2)\,dy = (\tan^{-1} x - y)\,dx$	[Ans. : $y = \tan^{-1} x - 1 + C e^{-\tan^{-1} x}$]
17) $y e^y = (y^3 + 2x e^y) \dfrac{dy}{dx}$	[Ans. : $\dfrac{x}{y^2} + e^{-y} = C$]
18) $\dfrac{dy}{dx} = \dfrac{e^x - 3xy}{x^2}$	[Ans. : $x^3 y = C + (x-1) e^x$]
19) $(2y + x^2)\,dx = x\,dy$	[Ans. : $y = x^2 \log(C x)$]
20) $\dfrac{dy}{dx} + \dfrac{y}{1-x} = x^2 - x$	[Ans. : $2y = (1-x)(C^2 - x^2)$]
21) $\sin x \dfrac{dy}{dx} + y \cos x = 2\sin^2 x \cdot \cos x$	[Ans. : $y \sin x = \dfrac{2}{3} \sin^3 x + C$]
22) $x \cos x \dfrac{dy}{dx} + (\cos x - x \sin x)\,y = 1$	[Ans. : $xy \cos x - x = C$]
23) $\cos x \dfrac{dy}{dx} + y = \sin x$	[Ans. : $y = 1 + (C - x)(\sec x - \tan x)$]
24) $(x^2+1)\dfrac{dy}{dx} = x^3 - 2xy + x$	[Ans. : $y(x^2+1) = \dfrac{x^4}{4} + \dfrac{x^2}{2} + C$]
25) $\dfrac{dy}{dx} + (1+2x)\,y = e^{-x^2}$	[Ans. : $y \cdot e^{x^2} + 2 = e^x + C$]
26) $(2y + x^2)\,dx = x\,dy$	[Ans. : $y = x^2 \log(C x)$]
27) $(1-x^2)\dfrac{dy}{dx} = 1 + xy$	[Ans. : $y\sqrt{1-x^2} = \sin^{-1} x + C$]
28) $(1+x^2)\dfrac{dy}{dx} + xy = 1$	[Ans. : $y\sqrt{1+x^2} = \log\left(x + \sqrt{1+x^2}\right) + C$]

Equation Reducible to Linear Form (Bernoulli's Equation)

The differential equation of the form

$$\frac{dy}{dx} + P \cdot y = Q \cdot y^n \qquad \dots (1)$$

where P and Q are functions of x alone (or may be constants) is called Bernoulli's differential equation.

To solve it, we divide (1) by y^n, then we have

$$y^{-n} \frac{dy}{dx} + P y^{1-n} = Q \qquad \dots (2)$$

Let $$y^{1-n} = u \Rightarrow (1-n)\, y^{-n} \frac{dy}{dx} = \frac{du}{dx}$$

$$y^{-n} \frac{dy}{dx} = \frac{1}{1-n} \frac{du}{dx}$$

Substituting these in (2).

$$\frac{1}{1-n} \frac{du}{dx} + Pu = Q$$

$$\frac{du}{dx} + P(1-n)\, u = Q(1-n)$$

i.e. $$\frac{du}{dx} + P_1 u = Q_1$$

where P_1 and Q_1 are functions of x.

which is linear differential equation and can be solved by usual method.

Example 1.107 : *Solve* $\dfrac{dy}{dx} - \dfrac{f(y)}{f'(y)} f'(x) = \dfrac{f(x)\, f'(x)}{f'(y)}$

Solution : The given equation can be written as,

$$f'(y) \frac{dy}{dx} - f(y)\, \phi'(x) = \phi(x)\, \phi'(x) \qquad \text{... (1)}$$

Let $$f(y) = u \Rightarrow f'(y) \frac{dy}{dx} = \frac{du}{dx}$$

Substituting in (1), we get,

$$\frac{du}{dx} - \phi'(x) \cdot u = \phi(x)\phi'(x) \qquad \text{(Linear equation)}$$

Here $P = -\phi'(x)$, $Q = \phi(x)\phi'(x)$

$$\text{I.F.} = e^{\int P\, dx} = e^{\int \phi'(x)\, dx} = e^{-\phi(x)}$$

G.S. is $$u\, e^{-\phi(x)} = \int \phi(x)\phi'(x)\, e^{-\phi(x)} dx + C$$

Let $$\phi(x) = t \Rightarrow \phi'(x)\, dx = dt$$

$$u\, e^{\phi(x)} = \int t\, e^{-t}\, dt + C$$

$$\Rightarrow \quad u\, e^{\phi(x)} = t\left(\frac{e^{-t}}{-1}\right) - (1)(e^{-t}) + C$$

$$f(y)\, e^{-\phi(x)} = -e^{-t}(t+1) + C$$

$$= -e^{-\phi(x)}[\phi(x) + 1] + C$$

Example 1.108 : *Solve* $2\frac{dy}{dx} - y \sec x = y^3 \tan x$

Solution : Divide by $-y^3$, we have,

$$-\frac{2}{y^3}\frac{dy}{dx} + \frac{1}{y^2}\sec x = -\tan x \qquad \dots (1)$$

Let $\frac{1}{y^2} = u \Rightarrow -\frac{2}{y^3}\frac{dy}{dx} = \frac{du}{dx}$

Substitute in (1),

$$\frac{du}{dx} + (\sec x)u = -\tan x \qquad \text{(Linear equation)}$$

Here $P = \sec x$; $Q = -\tan x$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int \sec x\,dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$$

G.S. is, $u(\sec x + \tan x) = -\int \tan x(\sec x + \tan x)\,dx + C$

$$\Rightarrow \quad \frac{1}{y^2}(\sec x + \tan x) = -\int \sec x \tan x\,dx + \int (1 - \sec^2 x)\,dx + C$$

$$= -\sec x - \tan x + x + C$$

Example 1.109 : *Solve* $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$

Solution : Divide by $y(\log y)^2$, then equation becomes

$$y^{-1}(\log y)^{-2}\frac{dy}{dx} + \frac{1}{x}(\log y)^{-1} = \frac{1}{x^2} \qquad \dots (1)$$

Let $(\log y)^{-1} = u$

$$\Rightarrow \quad -y^{-1}(\log y)^{-2}\frac{dy}{dx} = \frac{du}{dx}$$

Substitute in (1)

$$-\frac{du}{dx} + \frac{1}{x}u = \frac{1}{x^2}$$

$$\frac{du}{dx} - \frac{1}{x}u = -\frac{1}{x^2} \qquad \text{(Linear equation)}$$

Here $P = -\frac{1}{x}$; $Q = -\frac{1}{x^2}$

$$\text{I.F.} = e^{\int P\,dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

G.S. is, $u\left(\frac{1}{x}\right) = -\int \left(\frac{1}{x^2}\right)\frac{1}{x}\,dx + C$

$$(\log y)^{-1}\frac{1}{x} = \frac{1}{2x^2} + C$$

Example 1.110 : *Solve* $xy(1 + xy^2)\frac{dy}{dx} = 1$

Solution : The given equation can be written as,

$$\frac{dx}{dy} - yx = y^3x^2$$

Divide by x^2,

$$x^{-2}\frac{dx}{dy} - yx^{-1} = y^3 \qquad \ldots (1)$$

Let $u = x^{-1} \Rightarrow \frac{du}{dy} = -x^{-2}\frac{dx}{dy}$

Substituting in (1)

$$\frac{du}{dy} + yu = -y^3 \qquad \text{(Linear equation)}$$

Here $P = y$; $Q = -y^3$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int y\,dy} = e^{y^2/2}$$

G.S. is $ue^{y^2/2} = -\int y^3 e^{y^2/2}\,dy + C$

$$\Rightarrow \quad \frac{1}{x}e^{y^2/2} = -\int y^2 e^{y^2/2}\cdot y\,dy + C \qquad \ldots (2)$$

Put $\frac{y^2}{2} = t$, $\therefore$ $y\,dy = dt$, substituting in equation (2) we get,

$$\frac{1}{x}e^{y^2/2} = -\int 2te^t\,dt + C$$

$$= -2\left[te^t - (1)e^t\right] + C$$

$$= -2e^t(t-1) + C$$

$$\Rightarrow \quad \frac{1}{x}e^{y^2/2} = -2e^{y^2/2}\left[\frac{1}{2}y^2 - 1\right] + C$$

$$\frac{1}{x} = 2 - y^2 + Ce^{-y^2/2}$$

Example 1.111 : *Solve* $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)\, e^x \sec y$

Solution : Dividing by sec y, we have,

$$\cos y \frac{dy}{dx} - \frac{1}{(1+x)} \sin y = (1+x)\, e^x \qquad \dots (1)$$

Let $\sin y = u \Rightarrow \cos y \dfrac{dy}{dx} = \dfrac{du}{dx}$

Substitute in (1),

$$\frac{du}{dx} - \frac{1}{(1+x)} u = (1+x)\, e^x \qquad \text{(Linear equation)}$$

Here $P = -\dfrac{1}{1+x}$; $Q = (1+x)\, e^x$

$$\text{I.F.} = e^{-\int \frac{1}{1+x} dx} = e^{-\log(1+x)} = \frac{1}{1+x}$$

G.S. is, $u \dfrac{1}{1+x} = \displaystyle\int (1+x)\, e^x \frac{1}{(1+x)} dx + C$

$$\Rightarrow \quad \frac{\sin y}{1+x} = e^x + C$$

Example 1.112 : *Solve* $\dfrac{dy}{dx} - y \tan x = y^4 \sec x$

Solution : Dividing given equation by y^4, we get

$$y^{-4} \frac{dy}{dx} - y^{-3} \tan x = \sec x \qquad \dots (1)$$

Let $y^{-3} = u \Rightarrow -3y^{-4} \dfrac{dy}{dx} = \dfrac{du}{dx}$

Substitute in (1),

$$-\frac{1}{3}\frac{du}{dx} - (\tan x)\, u = \sec x$$

$$\frac{du}{dx} + (3 \tan x)\, u = -3 \sec x, \qquad \text{(Linear equation)}$$

$\therefore$ $P = 3 \tan x$; $Q = -3 \sec x$

$$\text{I.F.} = e^{3\int \tan x\, dx} = e^{3 \log \sec x} = \sec^3 x$$

G.S. is, $u \sec^3 x = \displaystyle\int (-3 \sec x) \sec^3 x\, dx + C$

$$\Rightarrow \qquad y^{-3}\sec^3 x = -3\int (1+\tan^2 x)\sec^2 x\, dx + C$$

Put $\tan x = t$, $\sec^2 x\, dx = dt$

$$= -3\int (1+t^2)\, dt + C$$

$$= -3\left(t + \frac{t^3}{3}\right) + C$$

$$= -3\tan x - \tan^3 x + C$$

$$\Rightarrow \qquad y^{-3}\sec^3 x + 3\tan x + \tan^3 x = C$$

Example 1.113 : *Solve* $\frac{dy}{dx} = -e^{x-y}\left(e^x + e^y\right)$

Solution : The given equation can be written as,

$$e^y \frac{dy}{dx} + e^x \cdot e^y = -e^{2x} \qquad \text{... (1)}$$

Let $\qquad e^y = u \Rightarrow e^y \frac{dy}{dx} = \frac{du}{dx}$

Substituting in (1),

$$\frac{du}{dx} + e^x \cdot u = -e^{2x} \qquad \text{(Linear differential equation)}$$

Here $P = e^x$, $Q = -e^{2x}$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int e^x dx} = e^{e^x}$$

G.S. is, $\qquad u e^{e^x} = -\int e^{2x} e^{e^x} dx + C$

$$\Rightarrow \qquad e^y e^{e^x} = -\int e^x e^{e^x} e^x dx + C$$

Put $\quad e^x = t$, $e^x dx = dt$

$$e^y e^{e^x} = -\int t e^t dt + C$$

$$\Rightarrow \qquad e^y e^{e^x} = -[t(e^t) - (1)e^t] + C$$

$$= -e^t(t-1) + C$$

$$= -e^{e^x}(e^x - 1) + C$$

$$\Rightarrow e^{e^x}(e^y + e^x - 1) = C$$

Example 1.114 : *Solve* $\left(1 + x^2\right)\frac{dy}{dx} = xy - y^2$

Solution : Dividing by $-(1+x^2)\,y^2$,

$$-y^{-2}\frac{dy}{dx}+\frac{x}{1+x^2}\cdot y^{-1} = \frac{1}{1+x^2} \quad \text{... (1)}$$

Let $y^{-1} = u \Rightarrow -y^{-2}\frac{dy}{dx} = \frac{du}{dx}$

Substituting in (1),

$$\frac{du}{dx}+\frac{x}{1+x^2}\cdot u = \frac{1}{1+x^2} \quad \text{(Linear equation)}$$

Here $P = \frac{x}{1+x^2}$, $Q = \frac{1}{1+x^2}$

$$\text{I.F.} = e^{\int \frac{x}{1+x^2}dx} = e^{\frac{1}{2}\log(1+x^2)}$$

$$\therefore \text{ G.S. is } u\sqrt{1+x^2} = \int \frac{1}{1+x^2}\sqrt{1+x^2}\,dx + C$$

$$= \int \frac{1}{\sqrt{1+x^2}}dx + C$$

$$\Rightarrow \quad y^{-1}\sqrt{1+x^2} = \sinh^{-1}x + C$$

Example 1.115 : *Solve* $\left(xy^2 - e^{1/x^3}\right)dx - x^2y\,dy = 0$

Solution : The equation can be written as,

$$x^2y\frac{dy}{dx} - xy^2 = -e^{1/x^3}$$

Divide by x^2

$$y\frac{dy}{dx} - \frac{1}{x}y^2 = -\frac{1}{x^2}e^{1/x^3} \quad \text{... (1)}$$

Let $y^2 = u \Rightarrow y\frac{dy}{dx} = \frac{1}{2}\frac{du}{dx}$

Substituting in (1)

$$\frac{1}{2}\frac{du}{dx} - \frac{1}{x}u = -\frac{1}{x^2}e^{1/x^3}$$

$$\frac{du}{dx} - \frac{2}{x}u = -\frac{2}{x^2}e^{1/x^3} \quad \text{(Linear equation)}$$

Here $P = -\frac{2}{x}$; $Q = -\frac{2}{x^2}e^{1/x^3}$

$$\text{I.F.} = e^{-\int 2/x\,dx} = e^{-2\log x} = e^{\log(x)^{-2}} = \frac{1}{x^2}$$

$\therefore$ G.S. is, $$u\left(\frac{1}{x^2}\right) = -2\int \frac{1}{x^2} e^{1/x^3} \frac{1}{x^2} dx + C$$

Put $\frac{1}{x^3} = t$

$$\therefore \quad \frac{-3}{x^4} dx = dt$$

$$\frac{dx}{x^4} = \frac{dt}{-3}$$

$$\therefore \quad u\left(\frac{1}{x}\right) = \frac{2}{3}\int e^t dt + C$$

$$\Rightarrow \quad \frac{y^2}{x^2} = \frac{2}{3} e^t + C$$

$$y^2 = \frac{2}{3} x^2 e^{1/x^3} + C$$

Example 1.116 : *Solve* $(x^3y^2 + xy)\,dx = dy$

Solution : The equation can be written as,

$$\frac{dy}{dx} = x^3y^2 + xy$$

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{x}{y} = x^3 \qquad \dots (1)$$

Let $$-\frac{1}{y} = u \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{du}{dx}$$

Substitute in (1)

$$\frac{du}{dx} + xu = x^3 \qquad \text{(Linear equation)}$$

Here $P = x$, $Q = x^3$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int x\,dx} = e^{x^2/2}$$

$\therefore$ G.S. is, $$ue^{x^2/2} = \int x^3 e^{x^2/2} dx + C$$

Let $$\frac{1}{2}x^2 = t \Rightarrow x\,dx = dt$$

$$ue^{x^2/2} = 2\int t\,e^t dt + C$$

$$-\frac{1}{y} e^{x^2/2} = 2\,[te^t - (1)\,e^t] + C$$

$$= 2e^t (t - 1) + C$$

$$-\frac{1}{y} = 2\left[\frac{1}{2}x^2 - 1\right] + Ce^{-x^2/2}$$

$$\frac{1}{y} = (2 - x^2) + Ce^{-x^2/2}$$

Example 1.117 : *Solve* $\sin y\frac{dy}{dx} = \cos x\left(2\cos y - \sin^2 x\right)$

Solution : Put $\cos y = u$

$$\sin y\frac{dy}{dx} = -\frac{du}{dx}$$

Equation becomes

$$-\frac{du}{dx} = 2u\cos x - \sin^2 x\cos x$$

$$\frac{du}{dx} + (2\cos x)u = \sin^2 x\cos x$$

which is linear in u.

$\therefore$ $P = 2\cos x$, $Q = \sin^2 x\cos x$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int 2\cos x\,dx} = e^{2\sin x}$$

G.S. is, $u\cdot e^{2\sin x} = \int \sin^2 x\cos x\cdot e^{2\sin x}dx + C$

Put $\sin x = t$ $\therefore$ $\cos x\,dx = dt$

$$ue^{2\sin x} = \int e^{2t}\cdot t^2\cdot dt + C$$

$$= t^2\left(\frac{e^{2t}}{2}\right) - (2t)\left(\frac{e^{2t}}{4}\right) + (2)\left(\frac{e^{2t}}{8}\right) + C$$

$$u\cdot e^{2\sin x} = \frac{e^{2t}}{4}(2t^2 - 2t + 1) + C$$

$$u = \frac{1}{4}(2\sin^2 x - 2\sin x + 1) + C\cdot e^{-2\sin x}$$

$$4\cos y = 2\sin^2 x - 2\sin x + 1 + C_1 e^{-2\sin x}$$

Example 1.118 : *Solve* $\frac{dy}{dx} = \frac{y^3}{e^{2x} + y^2}$

Solution :

$$\frac{e^{2x} + y^2}{y^3} = \frac{dx}{dy}$$

$$\frac{dx}{dy} - \frac{1}{y} = \frac{e^{2x}}{y^3}$$

$$e^{-2x}\frac{dx}{dy} - \frac{e^{-2x}}{y} = \frac{1}{y^3}$$

Put $e^{-2x} = u$

$$\therefore \qquad e^{-2x}\frac{dx}{dy} = \frac{-1}{2}\frac{du}{dy}$$

$$-\frac{1}{2}\frac{du}{dy} - \frac{u}{y} = \frac{1}{y^3}$$

$$\frac{du}{dy} + \left(\frac{2}{y}\right)u = -\frac{2}{y^3}$$

Which is linear in u, $P = \frac{2}{y}$, $Q = -\frac{2}{y^3}$

$$\text{I.F.} = e^{\int \frac{2}{y}dy} = e^{2\log y} = y^2$$

G.S. is,

$$u \cdot y^2 = \int -\frac{2}{y^3} \cdot y^2 dy + C$$

$$= -2\int \frac{1}{y}dy + C$$

$$u \cdot y^2 = -2\log y + C$$

$$e^{-2x}y^2 + \log y^2 = C$$

Example 1.119 : *Solve* $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

Solution : Given differential equation can be written as

$$\frac{dy}{dx} - xy = -y^3 e^{-x^2}$$

This is Bernoulli's equation. Divide by y^3

$$y^{-3}\frac{dy}{dx} - x \cdot y^{-2} = -e^{-x^2}$$

Put $\qquad y^{-2} = u \qquad \therefore \quad -2y^{-3}\frac{dy}{dx} = \frac{du}{dx}$

$$-\frac{1}{2}\frac{du}{dx} - x \cdot u = -e^{-x^2}$$

$$\therefore \qquad \frac{du}{dx} + (2x)u = 2e^{-x^2}$$

$P = 2x$, $Q = 2e^{-x^2}$

$$\therefore \quad \text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

$$\therefore \quad u \cdot e^{x^2} = \int 2e^{-x^2} \cdot e^{x^2} dx + C = \int 2\,dx + C$$

$$ue^{x^2} = 2x + C$$

$$\frac{e^{x^2}}{y^2} = 2x + C \quad \text{is the required G.S.}$$

Example 1.120 : *Solve* $\cos x \dfrac{dy}{dx} + y \sin x = \sqrt{y \sec x}$

Solution :

$$\frac{dy}{dx} + y \cdot \tan x = \sqrt{y} \sec^{3/2} x \qquad \text{(Bernoulli's differential equation)}$$

$$y^{-1/2} \frac{dy}{dx} + y^{1/2} \tan x = \sec^{3/2} x$$

Put $\quad y^{1/2} = u$

$$\therefore \quad y^{-1/2} = 2\frac{dy}{dx} = 2\frac{du}{dx}$$

$$2\frac{du}{dx} + u \cdot \tan x = \sec^{3/2} x$$

$$\frac{du}{dx} + \left(\frac{1}{2} \tan x\right) u = \frac{1}{2} \sec^{3/2} x$$

$$P = \frac{1}{2} \tan x, \; Q = \frac{1}{2} \sec^{3/2} x$$

Consider $\quad e^{\int P\,dx} = e^{\frac{1}{2}\int \tan x\,dx}$

$$e^{\int P\,dx} = e^{\frac{1}{2}\log \sec x} = e^{\log \sqrt{\sec x}}$$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\log \sqrt{\sec x}} = \sqrt{\sec x}$$

$$\therefore \text{G.S. is} \quad u\sqrt{\sec x} = \int \frac{1}{2} \sec^{3/2} x \sqrt{\sec x}\,dx + C$$

$$u\sqrt{\sec x} = \frac{1}{2}\int \sec^2 x\,dx + C = \frac{1}{2}\tan x + C$$

$$\sqrt{y \sec x} = \frac{1}{2} \tan x + C$$

Example 1.121 : *Solve* $\dfrac{dy}{dx} = 1 - x(y - x) - x^3 (y - x)^2$

Solution : Put $y - x = u$,

$$\therefore \qquad \frac{dy}{dx} - 1 = \frac{du}{dx}$$

$$\frac{du}{dx} = -xu - x^3u^2$$

$$\frac{du}{dx} + x \cdot u = -x^3u^2$$

This is Bernoulli's differential equation. Divide by u^2.

$$u^{-2}\frac{du}{dx} + xu^{-1} = -x^3$$

Put $u^{-1} = t$

$$\therefore \qquad u^{-2}\frac{du}{dx} = -\frac{dt}{dx}$$

$$-\frac{dt}{dx} + xt = -x^3$$

$$\frac{dt}{dx} - xt = x^3$$

$P = -x$, $Q = x^3$ and

$$\text{I.F.} = e^{\int -x\,dx} = e^{-\frac{x^2}{2}}$$

G.S. is

$$\therefore \qquad t \cdot e^{-\frac{x^2}{2}} = \int x^3 e^{-\frac{x^2}{2}} dx + C$$

Let $x^2 = 2V$, $xdx = dV$

$$\therefore \qquad t\,e^{-x^2/2} = \int 2\,V\,e^{-V}\,dV + C = 2\,[-V\,e^{-V} - e^{-V}] + C$$

$$te^{-x^2/2} = -2e^{-V}(V+1) + C = -e^{-x^2/2}(x^2+2) + C$$

$$\frac{e^{-x^2/2}}{u} = -e^{-x^2/2}(x^2+2) + C$$

$$\frac{1}{y-x} + x^2 + 2 = Ce^{x^2/2}$$

Example 1.122 : ***Solve*** $\frac{dy}{dx} - xy = y^2 e^{\frac{-x^2}{2}} \cdot \log x$

Solution : The equation is of Bernoulli's form. Dividing by y^2, we get

$$y^{-2}\frac{dy}{dx} - xy^{-1} = e^{\frac{-x^2}{2}} \cdot \log x$$

Let $y^{-1} = u$

$$\therefore \quad -y^{-2}\frac{dy}{dx} = \frac{du}{dx}$$

The equation becomes

$$-\frac{du}{dx} - xu = e^{\frac{-x^2}{2}} \cdot \log x$$

i.e.
$$\frac{du}{dx} + xu = -e^{\frac{-x^2}{2}} \cdot \log x \qquad \text{(Linear in u)}$$

$P = x$, $Q = -e^{\frac{-x^2}{2}} \log x$

$$\therefore \quad \text{I.F.} = e^{\int x\,dx} = e^{\frac{x^2}{2}}$$

G.S. is
$$u \cdot e^{\frac{x^2}{2}} = -\int e^{\frac{x^2}{2}} \cdot e^{\frac{-x^2}{2}} \cdot \log x \; dx + C$$

$$= -\int \log x \cdot dx + C \qquad \text{(Integrating by parts)}$$

$$= -(x \log x - x) + C \qquad \left(\because u = y^{-1} = \frac{1}{y}\right)$$

G.S. is
$$\frac{1}{y} e^{\frac{x^2}{2}} + x \log - x = C$$

Equation of the type

The equation $f'(y)\frac{dy}{dx} + P.f(y) = Q$; P, Q are functions of x can be reduced to linear differential equation by substituting f(y) = u and equation becomes $\frac{du}{dx} + Pu = Q$.

Similarly the equation, $f'(x)\frac{dx}{dy} + P\,f(x) = Q$ can be reduced to linear by substituting f(x) = u.

Example 1.123 : *Solve* $\sin y \frac{dy}{dx} = (1 - x \cos y) \cos y$

Solution : The equation can be written as

$$\sin y \frac{dy}{dx} = \cos y - x \cos^2 y$$

Dividing throughout by $\cos^2 y$, we get

$$\frac{\sin y}{\cos^2 y}\frac{dy}{dx} = \frac{\cos y}{\cos^2 y} - x$$

i.e. $$\sec y \cdot \tan y \frac{dy}{dx} - \sec y = -x \qquad \text{... (1)}$$

Let $\sec y = u$

$$\therefore \quad \sec y \cdot \tan y \frac{dy}{dx} = \frac{du}{dx}$$

(1) Becomes,

$$\frac{du}{dx} - u = -x \qquad \text{(Linear in u)}$$

$\therefore \quad P = -1, Q = -x$

$$\text{I.F.} = e^{-\int dx} = e^{-x}$$

G.S. is $$ue^{-x} = -\int xe^{-x} \cdot dx + C$$

i.e. $$ue^{-x} = -[x(-e^{-x}) - (1)(e^{-x})] + C \qquad (\because u = \sec y)$$

G.S. is $$\sec y = x + 1 + Ce^{x}$$

Example 1.124 : *Solve* $\frac{dy}{dx} + \frac{1}{x} \tan y = \frac{1}{x^2} \tan y \cdot \sin y$

Solution : Dividing the equation by $\tan y \cdot \sin y$, we get

$$\frac{1}{\tan y \cdot \sin y} \frac{dy}{dx} + \frac{1}{x} \frac{\tan y}{\tan y \cdot \sin y} = \frac{1}{x^2}$$

i.e. $$\cot y \cdot \text{cosec}\, y \frac{dy}{dx} + \frac{1}{x} \text{cosec}\, y = \frac{1}{x^2}$$

Let $\text{cosec}\, y = u$

$$-\text{cosec}\, y \cdot \cot y \frac{dy}{dx} = \frac{du}{dx}$$

Equation becomes

$$-\frac{du}{dx} + \frac{1}{x} u = \frac{1}{x^2}$$

i.e. $$\frac{du}{dx} - \frac{1}{x} u = -\frac{1}{x^2} \quad \text{is linear in u.}$$

$P = \frac{-1}{x}, Q = \frac{-1}{x^2}$

$$\text{I.F.} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$$

$$\text{G.S. is} \qquad u\frac{1}{x} = -\int \frac{1}{x^2}\frac{1}{x}\,dx + C = \frac{1}{2x^2} + C$$

$$\text{i.e.} \qquad \frac{1}{x \sin y} = \frac{1}{2x^2} + C \text{ is G.S.}$$

Example 1.125 : $\dfrac{dy}{dx} - x^3 \cos^2 y = -x \sin 2y$

Solution : The equation can be written as

$$\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$$

$$\therefore \qquad \frac{1}{\cos^2 y}\frac{dy}{dx} + x\left(\frac{2\sin y \cos y}{\cos^2 y}\right) = x^3$$

$$\text{i.e.} \qquad \sec^2 y \frac{dy}{dx} + 2 \tan y \cdot x = x^3$$

Put $\tan y = u$

$$\therefore \qquad \sec^2 y \cdot \frac{dy}{dx} = \frac{du}{dx}$$

Hence the equation becomes

$$\frac{du}{dx} + 2ux = x^3 \text{ is linear in u.}$$

$P = 2x,\ Q = x^3$

$$\therefore \qquad \text{I.F.} = e^{2\int x\,dx} = e^{x^2}$$

$$\therefore \text{ G.S. is} \qquad u \cdot e^{x^2} = \int x^3 e^{x^2}\,dx + C \qquad \dots (1)$$

$$\text{Put} \qquad x^2 = t$$

$$\therefore \qquad 2x\,dx = dt$$

$\therefore$ Equation (1) becomes

$$u\,e^{x^2} = \int t\,e^t \frac{dt}{2} + C = \frac{1}{2}[te^t - e^t] + C$$

$$\text{G.S. is} \qquad \tan y \cdot e^{x^2} = \frac{1}{2}\left(x^2 - 1\right) e^{x^2} + C$$

$$2 \tan y = x^2 - 1 + C_1 e^{-x^2} \qquad (2C = C_1)$$

Example 1.126 : $\left(y + e^y - e^{-x}\right) dx + x\left(1 + e^y\right) dy = 0$

Solution : Put $y + e^y = v$

$$\therefore \qquad (1 + e^y)\frac{dy}{dx} = \frac{dv}{dx}$$

The given differential equation is,

$$(y + e^{y} - e^{-x}) + x(1 + e^{y})\frac{dy}{dx} = 0$$

$$v - e^{-x} + x\frac{dv}{dx} = 0$$

$$\Rightarrow \qquad \frac{dv}{dx} + \frac{1}{x} \cdot v = \frac{1}{x}e^{-x} \quad \text{which is linear}$$

$P = \frac{1}{x}, Q = \frac{1}{x}e^{-x}$

$$\therefore \qquad \text{I.F.} = e^{\int \frac{1}{x}} = e^{\log x} = x$$

$$\therefore \text{ G.S. is} \qquad vx = \int \frac{1}{x}e^{-x}\, x dx + C$$

$$(y + e^{y})x = -e^{-x} + C$$

Note : The given differential equation is also exact.

Example 1.127 : $\frac{x^{n+1}}{y^{n+2}}dy - \frac{x^{n}}{y^{n+1}}dx + x^{m}dx = 0$

Solution : Write the differential equation as

$$y^{-n-2}\frac{dy}{dx} - \frac{1}{x}y^{-n-1} = -x^{m-n-1}$$

Put $y^{-n-1} = v$

Then
$$(-n-1)y^{-n-2} \cdot \frac{dy}{dx} = \frac{dv}{dx}$$

$$y^{-n-2}\frac{dy}{dx} = \left(\frac{-1}{n+1}\right)\frac{dv}{dx}$$

Then the equation becomes

$$-\left(\frac{1}{n+1}\right)\frac{dv}{dx} - \frac{1}{x} \cdot v = -x^{m-n-1}$$

$$\Rightarrow \qquad \frac{dv}{dx} + \frac{n+1}{x}v = (n+1)x^{m-n-1}$$

which is linear.

$$P = \frac{n+1}{x}$$

$$Q = (n+1)x^{m-n-1}$$

$$\text{I.F.} = e^{\int (n+1)/x dx} = e^{(n+1)\log x} = x^{n+1}$$

$$\therefore \text{G.S. is} \quad vx^{n+1} = C + \int (n+1)(x^{m-n-1}x^{n+1}\,dx = C + (n+1)\int x^m\,dx$$

$$\Rightarrow \quad y^{-n-1}x^{n+1} = C + (n+1)\frac{x^{m+1}}{m+1}$$

$$\Rightarrow \quad \frac{x^{n+1}}{y^{n+1}} = C + \frac{n+1}{m+1}x^{m+1}$$

Exercise 1.14

1) $x^2\dfrac{dy}{dx} = \sin^2 y - (\sin y \cos y)\, x$

2) $\dfrac{dy}{dx} + \dfrac{x}{1-x^2}\,y = x\sqrt{y}$ **[Ans. : $3\sqrt{y} + (1 - x^2) = C_1(1 - x^2)^{1/4}$]**

3) $\cos x \cdot \dfrac{dy}{dx} = y(\sin x - y)$ **[Ans. : $\dfrac{1}{y}\sec x = \tan x + C$]**

4) $x\dfrac{dy}{dx} + y = y^2 \log x$ **[Ans. : $\dfrac{1}{y} = Cx + \log(e^x)$]**

5) $x\dfrac{dy}{dx} + 3y = x^4 e^{x^{1/2}} \cdot y^3$ **[Ans. : $y^2x^6\left(e^{x^{1/2}} + C\right) = 1$]**

6) $(\sec x \tan x \tan y - e^x)\,dx + \sec^2 y \sec x\,dy = 0$ **[Ans. : $\sec x \tan y = e^x + C$]**

7) $\tan y\dfrac{dy}{dx} + \tan x = \cos y \cos^2 x$ **[Ans. : $\sec x \sec y = \sin x + C$]**

8) $2\dfrac{dy}{dx} + \dfrac{x}{\sqrt{1+y^2}} = \dfrac{(\sqrt{1-y^2} - y)}{x}$ **[Ans. : $x^2\left\{\sqrt{1 + y + y^2}\right\} = y + C$]**

9) $\dfrac{dy}{dx} = \dfrac{x^2 + y^2 + 1}{2xy}$ **[Ans. : $y^2 = x^2 + Cx - 1$]**

10) $xy\dfrac{dy}{dx} + y^2 = gx$ *if $y = 0$ when $x = 0$* **[Ans. : $y^2 = \dfrac{2}{3}gx$]**

11) $\dfrac{dy}{dx}\cos x + y \sin x = (y \sec x)^{1/2}$

12) $\dfrac{dr}{d\theta} = \dfrac{r\sin\theta - r^2}{\cos\theta}$ **[Ans. : $\dfrac{1}{r} = \sin\theta + C\cos\theta$]**

13) $e^y\left(1 + \dfrac{dy}{dx}\right) = e^x$ **[Ans. : $e^y e^x = \dfrac{e^{2x}}{2} + C$]**

14) $\dfrac{dz}{dx} + \dfrac{z}{x}\log z = \dfrac{z}{x^2}(\log x)^2$ **[Ans. : $x \log z = \dfrac{(\log x)^3 + C}{3}$]**

15) $3y^2\dfrac{dy}{dx} + 2xy^3 = 4xe^{-x^2}$ **[Ans. : $y^3 e^{x^2} = 2x^2 + C$]**

16) $\dfrac{dy}{dx} + xy = y^2 e^{\frac{x^2}{2}} \log x$ **[Ans. : $y^{-1}e^{-\frac{x^2}{2}} = -x\log x + x + C$]**

17) $2x\dfrac{dy}{dx} + y - 2x(x+1)y^3 = 0$ **[Ans. : $\dfrac{1}{xy^2} = -2(x \log x) + C$]**

18) $\frac{dx}{dy} - x\tan y = x^4 \sec y$ **[Ans. : $x^{-3}\sec^3 y = -3\tan y + \tan^3 y + C$]**

19) $x\frac{dy}{dx} + 3y = x^4 e^{1/x^2} y^3$ **[Ans. : $x^6 y^2 \left(C + e^{1/x^2}\right) = 1$]**

20) $\frac{dy}{dx} = \frac{\sin^2 y}{x^2} - \frac{\sin y \cos y}{x}$ **[Ans. : $\cot y = \frac{1}{2x} + Cx$]**

21) $x^2 \frac{dy}{dx} = e^y - x$ **[Ans. : $Cx^2 + 2x e^{-y} = 1$]**

22) $3\frac{dx}{dy} + \frac{x}{y+1} = \frac{3(y+1)}{x^2}$ **[Ans. : $x^3(y+1) = C + (y+1)^3$]**

23) $\tan x \cos y \, dy + \sin y \, dx + e^{\sin x} dx = 0$ **[Ans. : $\sin x \sin y = C - e^{\sin x}$]**

24) $3x(1-x^2)y^2 \frac{dy}{dx} + (2x^2 - 1)y^3 = ax^3$ **[Ans. : $y^3 = ax + Cx\sqrt{1-x^2}$]**

25) $y + 2\frac{dy}{dx} = y^3(x-1)$ **[Ans. : $y^2\left(x + Ce^x\right) = 1$]**

26) $y\,dy = (x - y^2)\,dx$ **[Ans. : $y^2 = x - \frac{1}{2} + Ce^{-2x}$]**

27) $xy^2 \frac{dy}{dx} - y^3 = x^2$ **[Ans. : $y^3 = x^2(Cx - x)$]**

University Questions

May - 2003

1. *Form the differential equation for which general solution is* $y = C_1 e^{3x} + C_2 e^{4x}$ **[4 Marks]**
2. *Solve any four of the following* **[16 Marks]**

   i) $\frac{dy}{dx} = 1 - x(y-x) - x^3(y-x)^2$

   [Ans. : $\frac{1}{(y-x)^2} = -(x^2+1) + Ce^{x^2}$]

   ii) $\frac{dy}{dx} = \frac{\tan y - 2xy - y}{x^2 - x\tan^2 y + \sec^2 y}$

   [Ans. : $(\tan y - y)x - x^2 y + \tan y = C$]

   iii) $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

   [Ans. : $x^3y^2 + \frac{x^2}{y} = C$]

   iv) $x^3 \frac{dy}{dx} = y^3 + y^2\sqrt{y^2 - x^2}$

   v) $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

Dec. - 2003

1. *Find the differential equation of all circles touching Y-axis at the origin and centres on X-axis.* **[3 Marks]**

2. *Solve any four* **[16 Marks]**

i) $\sin y \dfrac{dy}{dx} = (1 - x\cos y)\cos y$

[Ans. : $\dfrac{e^x}{\cos y} = e^x(x-1) + C$]

ii) $(xy^2 + 2x^2y^3)\,dx + (x^2y - x^3y^2)\,dy = 0$ **[Ans. : $-\dfrac{2}{xy} + \log\dfrac{x}{y^2} = C$]**

iii) $(6x - 4y + 1)\dfrac{dy}{dx} = 3x - 2y + 1$

[Ans. : $(3x - 2y) + \dfrac{1}{4}\log[4(3x-2y)+1] = -2x + C$]

iv) $(2x^2y^2 + y)\,dx - (x^3y - 3x)\,dy = 0$

v) $y\,dx = (2y\log y + y - x)\,dy$ **[Ans. : $xy = 2\left[\dfrac{y^3\log y}{3} - \dfrac{y^3}{9}\right] + C$]**

May - 2004

1. *Solve (any three) :* **[12 Marks]**

i) $\dfrac{y}{x}\dfrac{dy}{dx} = \sqrt{1 + x^2 + y^2 + x^2y^2}$

[Ans. : $2(1 + y^2)^{1/2} = x\sqrt{1+x^2} + \log(x\sqrt{1+x^2}) + C$]

ii) $(3x + 6y - 1)\,dy = (x + 2y - 3)\,dx$ **[Ans. : $\dfrac{3}{5}\left[(x+2y) + \dfrac{16}{15}\log(5(x+2y)-7)\right] = x + C$]**

iii) $\dfrac{dy}{dx} + \dfrac{y}{(1-x)\sqrt{x}} = 1 - \sqrt{x}$

[Ans. : $2\log\sqrt{1+x^2} + 2\sqrt{x} - 2\tan^{-1}\sqrt{x} + C$]

iv) $\dfrac{dy}{dx} = \dfrac{\tan y - 2xy - y}{x^2 - x\tan^2 y + \sec^2 y}$

[Ans. : $\log(\sec y) - x^2y - xy - \tan y = C$]

2. *Form the differential equation whose general solution is* $y = A\,e^{-9t}\cos(3t + B)$ **[5 Marks]**

[Ans. : $\dfrac{d^2y}{dt^2} + 18\dfrac{dy}{dt} + 90\,y = 0$]

3. *Obtain the differential equations whose solution is* $Ax^2 + By^2 = 1$. **[5 Marks]**

[Ans. : $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx} - \dfrac{y}{x}\right)\dfrac{dy}{dx} = 0$]

4. *Solve any three of the following :*

i) $\left(\dfrac{x+y-a}{x+y+b}\right)\dfrac{dy}{dx} = \dfrac{x+y+a}{x+y-b}$

[12 Marks]

[Ans. : $(x+y) - \dfrac{(a+b)}{2}\log[(x+y)^2 + ab] = 2x + C$]

ii) $(y^3 - 2x^2y)dx + (2xy^2 - x^3)dy = 0$ **[Ans. : $xy = \sin x + C\cos x$]**

iii) $\dfrac{dy}{dx} = e^{x-y}(e^x - e^y)$

[Ans. : $V = (e^x - 1) + C\,e^{-e^x}$ *where* $V = e^y$]

iv) $x\cos x\dfrac{dy}{dx} + (x\sin x + \cos x)y = 1$

Dec. - 2004

1. *Solve any three* **[12 Marks]**

i) $x^4\dfrac{dy}{dx} + x^3y - \sec(xy) = 0$

[Ans. : $\sin(xy) + \dfrac{1}{2x^2} = C$]

ii) $(y^4 - 2x^3 y)\,dx + (x^4 - 2xy^3)\,dy = 0$ **[Ans. : $2y^2x^{-1} + 2y^{-1}x^2 = C_1$]**

iii) $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$ **[Ans. : $y\sin x + x(\sin y + y) = C$]**

iv) $(y - 2x^3)\,dx - x(1 - xy)\,dy = 0$ **[Ans. : $y\log x - \dfrac{2x^3}{3} - y = C$]**

2. *Form the differential equation whose general solution is* $y = ae^{-2x} + be^{-3x}$ **[5 Marks]**
[Ans. : $y_2 + 5y_1 + 6y = 0$]

3. *Obtain the differential equation whose general solution is* $y = A\cos(\log x) + B\sin(\log x)$ **[5 Marks]**
[Ans. : $x^2y_2 + xy_1 + y = 0$]

4. *Solve any three* **[12 Marks]**

i) $y(2x^2y + e^x)\,dx = (e^x + y^3)\,dy$ **[Ans. : $2y\left[-x^2e^{-x} - 2x\,e^{-x} - 2e^{-x} + y\,x\right] - y = C$]**

ii) $\dfrac{dy}{dx} = \dfrac{x + 2y - 3}{3x + 6y - 1}$ **[Ans. : $\dfrac{3}{5}\left[(x + 2y) + \dfrac{2}{5}\log\left[5(x + 2y) - 7\right]\right] = x + C$]**

iii) $\cos y - x\sin y\dfrac{dy}{dx} = \sec^2 x$ **[Ans. : $\cos y\; x = \tan x + C$]**

iv) $\dfrac{dy}{dx} = (x - y + 1)^2 + (x - y)$ **[Ans. : $\log\left(\dfrac{x - y}{1 + x - y}\right) + x = C$]**

May - 2005

1. *Solve any three* **[12 Marks]**

i) $\dfrac{dy}{dx} = \sin(x + y) + \cos(x + y)$

ii) $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

iii) $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$

iv) $(2y + 6xy^2)\,dx + (3x + 8x^2y)\,dy = 0$

2. *Form the differential equation whose general solution is y = A cos (log x) + B sin (log x) where A and B are arbitrary constants.* **[5 Marks]**

3. *Solve any three* **[12 Marks]**

i) $\dfrac{dy}{dx} = \dfrac{(3x - 4y - 2)}{(6x - 8y - 5)}$

ii) $\dfrac{dy}{dx} = \dfrac{(4x^3y^2 + y\cos xy)}{(2x^4y + x\cos xy)}$

iii) $\left(\dfrac{y}{x}\sec y - \tan y\right)dx = (x - \sec y\log x)\,dy$

iv) $(x^3y^3 + xy)\dfrac{dy}{dx} = 1$

4. *Obtain the differential equation whose general solution is* $(x-A)^2+(y-B)^2=r^2$ *where A and B are arbitrary constants.* **[5 Marks]**

Dec. - 2005

1. *Solve :*

   i) $\dfrac{dy}{dx}=\dfrac{x^2-3xy+2y^2}{2xy-x^2}$ **[4 Marks]**

   ii) $\dfrac{dy}{dx}=\tan^2(x+y)$ **[4 Marks]**

   iii) $(x^2y+y^4)\,dx+(2x^3+4xy^3)\,dy=0$ **[4 Marks]**

   iv) $(1+\sin y)\dfrac{dx}{dy}=2y\cos y-x(\sec y+\tan y)$ **[4 Marks]**

2. *Form the differential equation of which general solution is* $y=\log\cos(x-a)+b$ **[5 Marks]**

3. *Obtain the differential equation whose general solution is* $x=(c_1+c_2t)e^t$. **[5 Marks]**

4. *Solve the following :*

   i) $\dfrac{dy}{dx}=\dfrac{x-y+3}{2x-2y+5}$ **[4 Marks]**

   ii) $xy-\dfrac{dy}{dx}=y^3\cdot e^{-x^2}$ **[4 Marks]**

   iii) $(2x+e^x\log y)\,y\,dx+e^x\,dy=0$ **[4 Marks]**

   iv) $(y^4-2x^3y)\,dx+(x^4-2xy^3)\,dy=0$ **[4 Marks]**

May - 2006

1. *Form the differential equation by eliminating arbitrary constants A and B from the following equation* $y=A\cos(\log x)+B\sin(\log x)$. **[5 Marks]**

2. *Solve any three of the following* **[12 Marks]**

   i) $\dfrac{dy}{dx}=\dfrac{6x-4y+3}{3x-2y+1}$

   ii) $x\,dy-y\,dx=(x^2+y^2)(x\,dx+y\,dy)$

   iii) $3y^2\dfrac{dy}{dx}+2xy^3=4xe^{-x^2}$

   iv) $(x^2y^2+xy+1)\,y\,dx+(x^2y^2-xy+1)\,x\,dy=0$

3. *Obtain the differential equation whose general solution is* $y=Ae^{2x}+Be^{3x}$ *where A and B are arbitrary constant.* **[5 Marks]**

4. *Solve any three of the following* **[12 Marks]**

   i) $\dfrac{dy}{dx}=\sin(x+y)+\cos(x+y)$

   ii) $\dfrac{dy}{dx}+\dfrac{4x^3y^2+y\cos xy}{2x^4y+x\cos xy}=0$

iii) $(2y+6xy^2)\,dx+(3x+8x^2y)\,dy=0$

iv) $(x^2+1)\dfrac{dy}{dx}+4xy=\dfrac{1}{(x^2+1)^2}$

Dec. - 2006

1. *Solve any three of the following :* **[12 Marks]**

i) $\left(x\tan\dfrac{y}{x}-y\sec^2\dfrac{y}{x}\right)dx+x\sec^2\dfrac{y}{x}\,dy=0$

ii) $\dfrac{dy}{dx}+\dfrac{y\cos x+\sin y+y}{\sin x+x\cos y+x}=0$

iii) $\dfrac{dy}{dx}-y\tan x=y^4\sec x$

iv) $(20x^2+8xy+4y^2+3y^3)\,y\,dx+4(x^2+xy+y^2+y^3)\,x\,dy=0$

2. *Form the differential equation whose general solution is* $y=e^x[A\cos x+B\sin x]$ *, where A and B are arbitrary constants.* **[5 Marks]**

3. *Solve any three of the following :* **[12 Marks]**

i) $(x^2y^2+5xy+2)\,y\,dx+(x^2y^2+4xy+2)\,x\,dy=0$

ii) $(3xy+8y^5)\,dx+(2x^2+24xy^4)\,dy=0$

iii) $x^2(x^2-1)\dfrac{dy}{dx}+x(x^2+1)\,y=x^2-1$

iv) $\dfrac{dy}{dx}=\dfrac{x+y+1}{2x+2y+1}$

4. *Form the differential equation whose general solution is* $xy=ae^x+be^{-x}+x^2$ *where a and b are arbitrary constants.* **[5 Marks]**

May - 2007

1. *Solve any three :* **[12 Marks]**

i) $ye^{x/y}dx=(xe^{x/y}+y^2)\,dy$

ii) $(x^4e^x-2mxy^2)\,dx+2mx^2y\,dy=0$

iii) $(1+x^2)\dfrac{dy}{dx}+xy=1$

iv) $\dfrac{dy}{dx}=\dfrac{x+2y-3}{2x+y-3}$

v) $\cos y-x\sin y\dfrac{dy}{dx}=\sec x$

2. *Form a differential equation of which general solution is,* $y=\log\cos(x-a)+b$*, where a and b are arbitrary constants.* **[5 Marks]**

3. *Solve any three :* **[12 Marks]**

i) $(y^4-2x^3y)\,dx+(x^4-2xy^3)\,dy=0$

ii) $\tan y\dfrac{dy}{dx}+\tan x=\cos y\cos^2x$

iii) $x \cos x \cos y + \sin y \frac{dy}{dx} = 0$

iv) $(2y + 6xy^2)\,dx + (3x + 8x^2y)\,dy = 0$

v) $e^{x+y}\left(x\frac{dy}{dx} + y\right) = e^{xy}\left(1 + \frac{dy}{dx}\right)$

4. *Form a differential equation of which general solution is,* $y = 4(x - A)^2$, *where A is arbitrary constant.* **[5 Marks]**

Dec. - 2007

1. *Solve any three :* **[12 Marks]**

i) $\frac{dy}{dx} + x\tan(y - x) = 1$

ii) $\left[\log(x^2 + y^2) + \frac{2x^2}{x^2 + y^2}\right]dx + \frac{2xy}{x^2 + y^2}dy = 0$

iii) $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)\,dy = 0$

iv) $(1 + \sin y)\frac{dx}{dy} = 2y\cos y - x(\sec y + \tan y)$

v) $\frac{dy}{dx} = \frac{3x - 4y - 2}{6x - 8y - 5}$

2. *Form the differential equation whose general solution is* $y = e^x(A\cos x + B\sin x)$, *where A and B are arbitrary constants.* **[5 Marks]**

3. *Solve any three :* **[12 Marks]**

i) $(x^2y - 2xy^2)dx = (x^3 - 3x^2y)dy$

ii) $(y^2 + 2xy)dx + (2x^2 + 3xy)dy = 0$

iii) $x\frac{dy}{dx} + 3y = x^4e^{1/x^2}y^3$

iv) $(x\sec^2 y - x^2\cos y)dy = (\tan y - 3x^4)dx$

v) $\left(x\tan\left(\frac{y}{x}\right) - y\sec^2\left(\frac{y}{x}\right)\right)dx + x\sec^2\left(\frac{y}{x}\right)dy = 0$

4. *Form the differential equation whose general solution is* $y = e^x(ax^2 + bx + c)$, *where a, b, c are arbitrary constants.* **[5 Marks]**

May - 2008

1. *Form the differential equation for which* $xy = ae^x + be^{-x} + x^3$ *is a solution.* **[5 Marks]**

2. *Solve any three :* **[12 Marks]**

i) $\frac{dy}{dx} = \frac{x\sin x}{2e^y\sinh y}$

ii) $(1 + y^2)\,dx + (x - \tan^{-1} y)\,dy = 0$

iii) $(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$

iv) $\frac{dy}{dx} = \frac{x - y + 1}{x + y - 3}$

3. *Solve any three* **[12 Marks]**

   i) $\dfrac{dy}{dx} = 1 + 3x^2 e^{-y}$

   ii) $(2x + e^x \log y)\, y\, dx + e^x dy = 0$

   iii) $\left[\log(x^2 + y^2) + \dfrac{2x^2}{x^2 + y^2}\right] dx + \dfrac{2xy}{x^2 + y^2} dy = 0$

   iv) $(xdy - ydx) + \dfrac{1}{x^2 + y^2}(xdy + ydy) = 0$

4. *Form the differential equation of family of circles of fixed radius a with centres on x-axis.* **[5 Marks]**

Dec. - 2008

1. *Form the differential equation for which* $y = Ax + \dfrac{B}{x}$ *is the solution.* **[4 Marks]**

2. *Solve any three :* **[12 Marks]**

   i) $(xy^3 + y)\, dx + 2(x^2y^2 + x + y^4)\, dy = 0$

   ii) $x \cos x \dfrac{dy}{dx} + y(x \sin x + \cos x) = 1$

   iii) $(2x + y - 3)\, dy - (x + 2y - 3)\, dx = 0$

   iv) $(1 + xy)y\, dx + (1 - xy)x\, dy = 0$

3. *Form the differential equation from* $x = (A + Bt)\, e^t$. **[4 Marks]**

4. *Solve any three* **[12 Marks]**

   i) $x\dfrac{dy}{dx} + 3y = x^4 e^{1/x^2} y^3$

   ii) $x(x - y)\, dy + y^2 dx = 0$

   iii) $y^2(x^2 + 2)\, dx + (x^3 + y^3)(ydx - xdy) = 0$

   iv) $(y^2 + 2yx^2)\, dx + (2x^3 - xy)\, dy = 0$

2 Applications of Differential Equations

2.1 Introduction

The differential equation treated as powerful tool for the engineers. It plays an important role in the study of Engineers problems viz. Electrical circuits, rate measure, structural mechanics, Heat and mass transfer, Mass spring system etc.

The approach of engineering student to the differential equation solely differs from that of a student of Mathematics by dividing the problem in three stages.

a) **Modeling** : The formation of Mathematical Model from the given physical, Engineering or read world problem.

b) **Solving** : Obtaining solution of Mathematical Model by applying appropriate Mathematical or numerical Methods.

c) **Interpreting** : Understanding and application (Physical interpretation) of the solution in the real world situation.

In this chapter, we shall discuss some applications of first order, first degree differential equations.

2.2 Orthogonal Trajectories

Orthogonal trajectory : A curve which cuts every member of a given family of curves at right angles is called **orthogonal trajectory** of the family.

- Two families of the curves are said to be orthogonal if every member of either family cuts each member of other family at right angles.
- The orthogonal trajectories occur very frequently in electrostatics, equipotential lines, lines of electric force etc. Also, in two dimensional problems in the flow of heat, the lines of heat flow in a body usually perpendicular to the isothermal curves at the same temperature.

2.3 Working Rule for Finding Orthogonal Trajectories : of a Given Family of Curves

A) For cartesian curves $f(x,y,c) = 0$

Step I : Differentiating equation $F(x, y, c) = 0$

w.r.t. 'x' to eliminate c and forming differential equation

$$\phi\left(x, y, \frac{dy}{dx}\right) = 0 \quad ...(1)$$

Step II : Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ and forming differential equations for family of orthogonal trajectories as

$$\phi\left(x, y, -\frac{dx}{dy}\right) = 0 \quad ...(2)$$

Step III : Solving differential equation (2), we get the family of orthogonal trajectories.

Illustrated Examples

Example 2.1 : *Find orthogonal trajectories of the family of parabolas $y = ax^2$*

Solution : We have $y = ax^2$ ...(1)

Differentiating w.r.t x, we get,

$$\frac{dy}{dx} = 2ax \quad ...(2)$$

Eliminating 'a' from equations (1) and (2) we get,

$$\frac{dy}{dx} = \frac{2y}{x} \quad ...(3)$$

Which is the differential equation for the family (1)

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in equation (3), we get

$$-\frac{dx}{dy} = \frac{2y}{x}$$

$$\Rightarrow \quad x\,dx + 2y\,dy = 0 \quad ...(4)$$

Which is differential equation for the family of orthogonal trajectories integrating equation (4), we get,

$$\frac{x^2}{2} + y^2 = c^2$$

or $$\frac{x^2}{\left(\sqrt{2}\,c\right)^2} + \frac{y^2}{c^2} = 1$$

Which is family of ellipses as shown in the figure 2.1.

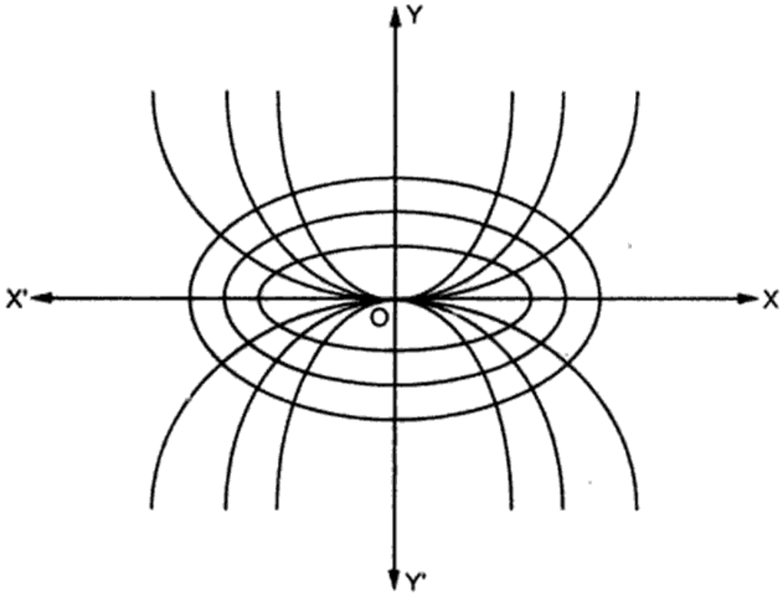


Fig. 2.1

Example 2.2 : *Find the orthogonal trajectories of the family of*

i) $xy = c^2$ *(Hyperbolas)*

ii) $y = mx$ *(Straight lines)*

iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2 + \lambda} = 1$ *(Ellipses, λ is a parameter)*

iv) $y^2 = 4ax$ *(Parabolas)*

Solution : i) Given $x\,y = c^2$ ...(1)

Differentiating w.r.t x, we get

$$x\frac{dy}{dx} + y(1) = 0$$

or $\frac{dy}{dx} = -\frac{y}{x}$ which is differential equation for the family (1). ... (2)

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (2), we get,

$$-\frac{dx}{dy} = -\frac{y}{x}$$

$$\Rightarrow \qquad xdx - ydy = 0 \qquad ...(3)$$

Which is differential equation for the family of orthogonal trajectories.

On integrating (3), we get,

$$\frac{x^2}{2} - \frac{y^2}{2} = k^1$$

$\Rightarrow \qquad x^2 - y^2 = k^2$, which is the required family of orthogonal trajectories.

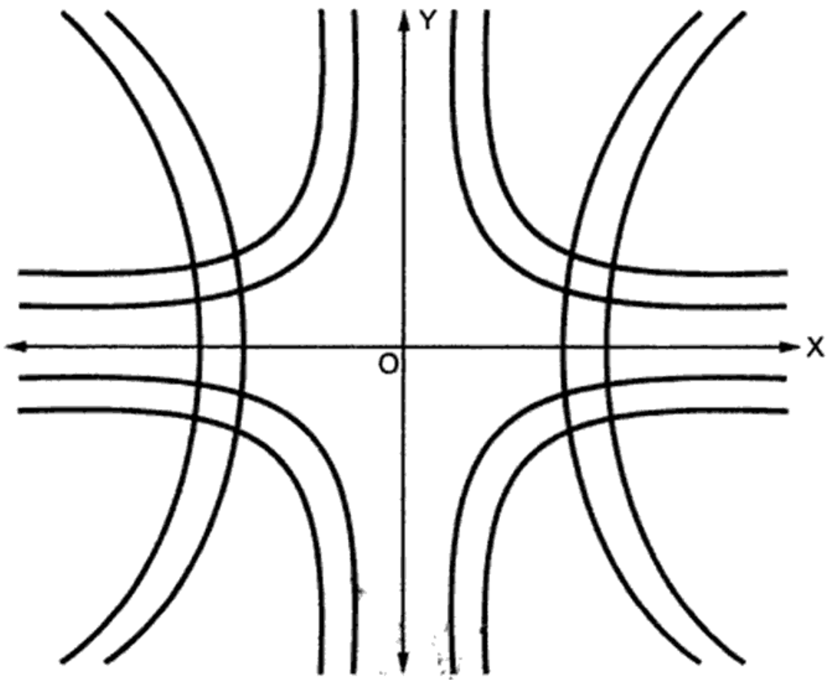


Fig. 2.2

ii) Given $\qquad y = mx \qquad ...(1)$

Differentiating (1) w.r.t. x,

$$\frac{dy}{dx} = m \qquad ...(2)$$

From (1) and (2), we get $\frac{dy}{dx} = \frac{y}{x}$

Which is differential equation for family (1),

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get,

$$-\frac{dx}{dy} = \frac{y}{x}$$

or $\qquad xdx + ydy = 0$

On integrating, we get,

$x^2 + y^2 = k^2$, which is the required equation orthogonal trajectories.

iii) Given $$\frac{x^2}{a^2} + \frac{y^2}{b^2 + \lambda} = 1 \qquad \text{...(1)}$$

Differentiating (1) w.r.t. x, we get,

$$\frac{2x}{a^2} + \frac{2y}{b^2 + \lambda}\frac{dy}{dx} = 0$$

or $$\frac{x}{a^2} + \frac{y}{b^2 + \lambda}\frac{dy}{dx} = 0 \qquad \text{... (2)}$$

To eliminate λ from (1) and (2), we equate the values of $b^2 + \lambda$,

From equation (1),

$$\frac{y^2}{b^2 + \lambda} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$$

or $$b^2 + \lambda = \frac{a^2 y^2}{a^2 - x^2} \qquad \text{...(A)}$$

From equation (2), $$b^2 + \lambda = -\frac{a^2 y}{x}\frac{dy}{dx} \qquad \text{...(B)}$$

From equation (A) and (B)

$$\frac{a^2 y^2}{a^2 - x^2} = -\frac{a^2 y}{x}\frac{dy}{dx}$$

or $$\frac{xy}{a^2 - x^2} + \frac{dy}{dx} = 0 \qquad \text{...(3)}$$

Which is the differential equation for the family (1)

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (3)

$$\therefore \quad \frac{xy}{a^2 - x^2} - \frac{dx}{dy} = 0$$

Or $y\,dy - \left(\frac{a^2 - x^2}{x}\right) dx = 0$ (V.S. form)

On integrating, we get,

$$\frac{y^2}{2} - a^2 \log x + \frac{x^2}{2} = c$$

or $x^2 + y^2 = 2a^2 \log x + c_1$ which is the required equation of orthogonal trajectories.

iv) Given $y^2 = 4ax$ ...(1)

Differentiating equation (1) w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a(1)$$

or $$\frac{dy}{dx} = \frac{2a}{y} \qquad ...(2)$$

From equation (1) and (2),

$$\frac{dy}{dx} = \frac{2}{y} \times \frac{y^2}{4x} = \frac{y}{2x}$$

$$\frac{dy}{dx} = \frac{y}{2x} \qquad ..(3)$$

Which is differential equation for the family (1)

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get,

$$-\frac{dx}{dy} = \frac{y}{2x}$$

$$\Rightarrow \quad 2x\,dx + y\,dy = 0$$

On integrating, we get

$$x^2 + \frac{y^2}{2} = k$$

Or $$2x^2 + y^2 = c^2$$

Which is the required equation of orthogonal trajectories of (1).

B) For polar curves : F(r,θ,c) = 0

Step I : Differentiating $F(r,\theta, c) = 0$ w.r.t θ and forming differential equation for the given family as,

$$\phi\left(r,\theta,\frac{dr}{d\theta}\right) = 0 \qquad ...(1)$$

Step II : Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ we get differential equation for the family of orthogonal trajectories as,

$$\phi\left(r,\theta,-r^2\frac{d\theta}{dr}\right) = 0 \qquad ...(2)$$

Step III : Solving differential equation (2) we get the equations for the family of orthogonal trajectories.

Example 2.3 : *Find the orthogonal trajectories of*

(i) $r = a(1 - \cos\theta)$

(ii) $r^2 = a^2 \cos 2\theta$

Solution : (i) Given $r = a(1 - \cos\theta)$ ...(1)

Differentiating (1) w.r.t θ, we get,

$$\frac{dr}{d\theta} = a\sin\theta \quad \text{...(2)}$$

From (1) and (2),

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$$

Or $\frac{1}{r}\frac{dr}{d\theta} = \cot\frac{\theta}{2}$ which is differential equation for the family (1) ... (3)

Replace $\frac{dr}{d\theta}$ by $\left(-r^2\frac{d\theta}{dr}\right)$ in (3), we get,

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \cot\frac{\theta}{2}$$

Or $$r\frac{d\theta}{dr} + \cot\frac{\theta}{2} = 0$$

Or $$\frac{dr}{r} + \tan\frac{\theta}{2}\,d\theta = 0 \text{ (V.S. form)} \quad \text{...(4)}$$

Which is the differential equation for the family of orthogonal trajectories.

Integrating (4), we get,

$$\int\frac{dr}{r} + \int\tan\frac{\theta}{2}\,d\theta = \log c$$

$$\log r - 2\log\cos\frac{\theta}{2} = \log c$$

$$\log r = \log\left(c\cos^2\frac{\theta}{2}\right)$$

Or $$r = c(1+\cos\theta)$$

Which is the orthogonal trajectories of the family (1)

(ii) Given $$r^2 = a^2 \cos 2\theta$$

$$\therefore \quad r = a\sqrt{\cos 2\theta} \qquad ...(1)$$

Differentiating equation (1) w.r.t. θ, we get

$$\frac{dr}{d\theta} = a\frac{1}{2\sqrt{\cos 2\theta}}(-2\sin 2\theta)$$

Or $$\frac{dr}{d\theta} = -a\frac{\sin 2\theta}{\sqrt{\cos 2\theta}} \qquad ...(2)$$

Eliminating (a) from equations (1) and (2)

$$\frac{dr}{d\theta} = \frac{-\sin 2\theta}{\sqrt{\cos 2\theta}} \times \frac{r}{\sqrt{\cos 2\theta}}$$

$$\frac{dr}{d\theta} = -r\tan 2\theta \qquad ...(3)$$

Which is differential equation for the family (1).

Replacing $\frac{dr}{d\theta}$ by " $-r^2\frac{d\theta}{dr}$" we get

$$-r^2\frac{d\theta}{dr} = -r\tan 2\theta$$

Or $$r\frac{d\theta}{dr} = \tan 2\theta$$

Or $$\int\frac{d\theta}{\tan 2\theta} = \int\frac{dr}{r} + k$$

$$\frac{1}{2}\log\sin 2\theta = \log r + k$$

Or $$\log\sin 2\theta = \log r^2 + \log c$$

Or $$\log\sin 2\theta = \log cr^2$$

Or $$r^2 = \frac{1}{c}\sin 2\theta$$

$$r^2 = c_1\sin 2\theta \qquad ...(4)$$

Which is the equation of family of orthogonal trajectories for (1).

Exercise 2.1

Find the orthogonal trajectories of the family of **[Ans. :**

i) $2x^2 + y^2 = cx$ $[x^2 = -y^2 \log (cy)]$

ii) $2x^2 - y^2 = kx^4$ $\left[y = c - \frac{1}{4} \log |x|\right]$

iii) $ay^2 = x^3$ $\left[x^2 + \frac{3}{2} y^2 = c\right]$

iv) $x^2 + (x-a)^2 = a^2$ $[(x-k)^2 + y^2 = k^2]$

v) $r = a(1 + \cos\theta)$ $[r = c(1 - \cos\theta)]$

vi) $r = a\theta$ $[r = c e^{-\theta^2/2}]$

vii) $r = \frac{2a}{1 + \cos\theta}$ $\left[r = \frac{2c}{1 - \cos\theta}\right]$

viii) $x^2 = 4a(y + a)$ $[x^2 = 4c(y + c)]$

2.4 Newton's Law of Cooling

Newton's law of cooling states that the rate of change of temperature of a body is proportional to the difference between the temperature of the body and the surrounding medium.

Let 'θ' be the temperature of a body at any time t and 'θ_0' be the temperature of the surrounding medium then according to Newton's law of cooling

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\therefore \frac{d\theta}{dt} = -k(\theta - \theta_0)$$ where k is a proportionality constant.

Negative sign is attached because as t increases θ decreases.

Note :

1) If it is a process of heating then $\frac{d\theta}{dt} = k(\theta - \theta_0)$

2) If the body is cooling at the rate $k\theta$ and heated at the rate αt then net rate of change is $\frac{d\theta}{dt} = -k\theta + \alpha t$

Example 2.4 : *A body of temperature 100°C is placed in a room whose temperature is 25°C and cool to 50°C in 5 minutes. What will be its temperature after a further interval of time 5 minutes.*

Solution : Let θ be the temperature of body of any time 't,' and θ_0 be the temperature of surrounding medium,

$$\theta_0 = 25°C$$

By Newton's law of cooling,

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$= -k(\theta - 25)$$

Separating variables, we get,

$$\frac{d\theta}{\theta - 25} = -\,kdt \qquad \text{... (1)}$$

We have, in 5 minutes body cools down from 100°C to 50°C.

i.e. at $t = 0, \quad \theta = 100\ °C$

at $t = 5, \quad \theta = 50\ °C$

$$\therefore \quad \int_{100}^{50} \frac{d\theta}{\theta - 25} = -\,k\int_0^5 dt$$

i.e. $\log\left[(\theta - 25)\right]_{100}^{50} = -\,k[t]_0^5$

i.e. $\log\,[25] - \log\,[75] = -\,5k$

i.e. $$k = \frac{1}{5}\log\frac{75}{25} = \frac{1}{5}\log 3 \qquad \text{... (2)}$$

Next, let in 10 minutes body cools from 100 °C to θ °C (say)

Equation (1) becomes, $$\int_{100}^{\theta} \frac{d\theta}{\theta - 25} = -\,k\int_0^{10} dt$$

$$\log(\theta - 25) - \log 75 = -\,10\,k = -\,2\log 3 = \log\frac{1}{9}$$

$$\frac{\theta - 25}{75} = \frac{1}{9}$$

$$\theta - 25 = \frac{75}{9}$$

$$\theta = 25 + 8.3333$$

$$= 33.3333°C$$

Example 2.5 : *A body originally at 80 °C cools to 60 °C in 20 minutes the temperature of air being 40 °C, what will be the temperature of the body after 40 minutes.* **[May-2006]**

Solution : Let 'θ' be the temperature of a body at any time 't'.

By Newton's law of cooling,

$$\frac{d\theta}{dt} = -k(\theta - 40) \qquad (\because \theta_0 = 40)$$

$$\therefore \quad \frac{d\theta}{\theta - 40} = -k\,dt \qquad \text{... (1)}$$

At t = 0, θ = 80°C and when t = 20, 0 = 60°C integrating (1) with these limits, we get,

$$\int_{80}^{60} \frac{d\theta}{\theta - 40} = -k\int_{0}^{20} dt$$

$$\log\left[(\theta - 40)\right]_{80}^{60} = -k[t]_0^{20}$$

$$\log[20] - \log[40] = -20k$$

$$\log\frac{1}{2} = -20k$$

$$\Rightarrow \quad k = \frac{1}{20}\log 2 \qquad \text{... (2)}$$

Let θ = T when t = 40. But θ = 80 when t = 0.

∴ Equation (1) becomes

$$\int_{80}^{T} \frac{d\theta}{\theta - 40} = -k\int_{0}^{40} dt$$

$$\therefore \quad \log\left(\frac{T-40}{40}\right) = -40k = -2\log 2 = \log\frac{1}{4} \qquad (\because \text{from (2)})$$

$$\text{Or} \quad \frac{T-40}{40} = \frac{1}{4}$$

$$\Rightarrow \quad T - 40 = 10$$

$$\Rightarrow \quad T = 50\,°C$$

Example 2.6 : *Temperature of water initially is 100 °C and that of surrounding is 20 °C if water cools down to 60 °C in first 20 minutes. During what time will it cool to 30 °C ?*

Solution : Let 'θ' be the temperature of a body at any time 't'.

By Newton's law of cooling,

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$= -k(\theta - 20) \quad (\because \theta_0 = 20\ ^\circ C)$$

$$\therefore \quad \frac{d\theta}{\theta - 20} = -kdt \quad \text{(V.S. form)} \qquad \dots (1)$$

At $t = 0, \ \theta = 100\ ^\circ C$ and

and $t = 20, \theta = 60\ ^\circ C$, $\therefore$ from (1),

$$\therefore \quad \int_{100}^{60} \frac{d\theta}{\theta - 20} = -k\int_{0}^{20} dt$$

$$\log\left[(\theta - 20)\right]_{100}^{60} = -20k$$

$$\text{i.e.} \quad \log\frac{40}{80} = -20k$$

$$\therefore \quad k = \frac{1}{20}\log 2 = 0.03466 \qquad \dots (2)$$

Next let at $t = 0$, $\theta = 100$ and at $t = T, \theta = 30$

$$\therefore \quad \int_{100}^{30} \frac{d\theta}{\theta - 20} = -k\int_{0}^{T} dt$$

$$\log\left[(\theta - 20)\right]_{100}^{30} = -kT$$

$$\text{i.e.} \quad \log 10 - \log 80 = -kT$$

$$\text{i.e.} \quad \log 8 = +kT$$

$$T = \frac{1}{0.03466}\log 8$$

$$= 59.99 \text{ minutes}$$

$$\approx 1 \text{ hours}$$

Example 2.7 : *The temperature of body decreases at a rate* $k\theta$, *where* θ *is the amount of the body hotter than the surrounding air. The body is heated by a source which makes the body's temperature increase at a rate 'at'. Where 't' is time and 'a' is constant. If this source is applied at t = 0, and the body is kept at the temperature of the surrounding air, show that*

$$\theta = \frac{a}{k}\left(t - \frac{1}{k} + \frac{1}{k}e^{-kt}\right)$$

(May-2003, Dec.-2001)

Solution : Let T = temperature of a body

T_A = temperature of a surrounding air

$$\therefore \quad \theta = T - T_A$$

and
$$\frac{d\theta}{dt} = \frac{dT}{dt} \qquad \text{...(1)}$$

Now, the temperature of body decreases at the rate of $k\theta$.

$$\therefore \quad \frac{dT}{dt} = -k\theta \qquad \text{...(2)}$$

The rate of increase of temperature of body heating by source is,

$$\frac{dT}{dt} = at \qquad \text{...(3)}$$

From (2) and (3) The net rate of change temperature (due to both effects) is,

$$\frac{dT}{dt} = -k\theta + at$$

Or
$$\frac{d\theta}{dt} = -k\theta + at \qquad \therefore \text{ from (1)}$$

Or
$$\frac{d\theta}{dt} + k\theta = at$$

Which is linear differential equation.

$$\text{I.F.} = e^{\int k\,dt} = e^{kt}$$

and it's general solution is

$$\theta\left(e^{kt}\right) = \int (at)\, e^{kt}\, dt + c$$

$$= at\,\frac{e^{kt}}{k} - (a)\,\frac{e^{kt}}{k^2} + c$$

$$\theta\, e^{kt} = \frac{a}{k^2}(kt - 1)\, e^{kt} + c \qquad \text{...(4)}$$

Given At $t = 0$, $T = T_A$ or $\theta = 0$

$\therefore$ From equation (4), $0 = \frac{-a}{k^2} + c \Rightarrow c = \frac{a}{k^2}$

$$\therefore \quad \theta = \frac{a}{k^2}\left(kt - 1 + e^{-kt}\right)$$

$$\therefore \quad \theta = \frac{a}{k}\left(t - \frac{1}{k} + \frac{1}{k}\, e^{-kt}\right)$$

Example 2.8 : *A metal ball is heated to a temperature of 100 °C and at time t = 0 it is placed in water which is maintained at 40 °C. If the temperature of the ball reduces to 60°C in 4 minutes, find the time at which the temperature of the ball is 50°C.* **(May-2006)**

Solution : Step 1 : Let the temperature of the ball be T °C at time t minutes then By Newton's Law of cooling

$$\frac{dT}{dt} = -k(T - T_0) \quad \ldots (1)$$

Use same notations θ and θ_0

Step 2 : Given $T_0 = 40$ °C is the temperature of water

$$\therefore \quad \frac{dT}{(T-40)} = -k\, dt$$

Integrating, $$\int \frac{dT}{(T-40)} = -k\int dt + C$$

$$\log(T-40) = -kt + \log C \quad \ldots (2)$$

$$-kt = \log\frac{T-40}{C} \quad \ldots (3)$$

Here we have two unknowns k and C. To find these two unknowns two conditions are given,

At $t = 0, \; T = 100,$

Substituting in (2) we get $\log C = \log 60$

$$\boxed{C = 60}$$

Thus substituting value of C in (3)

$$-kt = \log\frac{T-40}{60} \quad \ldots (4)$$

Also T = 60 at t = 4 substituting in equation (4)

$$\boxed{k = \frac{1}{4}\log 3}$$

Step 4 : Substituting value of k in (4)

$$-\frac{t}{4}\log 3 = \log\frac{T-40}{60} \quad \ldots (5)$$

is the relation in t and T.

Step 5 : Now to find the time at T = 50.

Substituting T = 50 in equation (5),

$$t = \frac{4\log 6}{\log 3} = 6.5 \text{ minutes.} \quad \ldots \text{Ans}$$

Example 2.9 : *According to Newton's law of cooling the rate at which substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is 30 °K and the substance cools from 37 °K to 34 °K in 15 minutes, find when the temperature will be 31 °K.*

Solution : Step 1 : Let T be the temperature of the substance at the t minutes.

Then from Newton's law we have,

$$\frac{dT}{dt} = -k(T - T_0)$$

where $T_0 = 30$ °C is the temperature of air.

$$\therefore \qquad \frac{dT}{T - 30^\circ} = -k\,dt$$

Step 2 : Integrating

As substance cools from 37 °K to 34 °K in 15 minutes integrating between the limits 0 to 15 we have,

$$\int_{37^\circ}^{34^\circ} \frac{dT}{T - 30^\circ} = -k \int_0^{15} dt$$

$$\log\left[(T - 30)\right]_{37}^{34} = -k\,[t]_0^{15}$$

$$\log 4 - \log 7 = -15\,k$$

$$k = \frac{1}{15}\log\left(\frac{7}{4}\right) \qquad \text{... (1)}$$

Step 3 : Now when temperature is 31 °K

$$\int_{370}^{310} \frac{dT}{T - 300} = -k\int_0^t dt$$

$$\log 10 - \log 70 = -kt \qquad \text{... (2)}$$

Substituting value of k in above equation (2) we get,

$$\log 7 = \frac{1}{15}\left(\log\frac{7}{4}\right)t$$

$$\therefore \qquad t = \frac{15\log 7}{\log 7/4} = 52 \text{ minutes.}$$

Example 2.10 : *The temperature θ of a body put in a medium at temperature $\theta_0\,(\theta_0 < \theta)$ satisfies the equation $\frac{d\theta}{dt} = k(\theta_0 - \theta)$, where 'k' is a positive constant. A thermometer which reads 75 °F indoors is taken outdoors. After 5 minutes it reads 65 °F. After another 5 minutes it reads 60°F. What is the outside temperature?*

Solution : Step 1 :

Given that, $$\frac{d\theta}{dt} = k(\theta_0 - \theta)$$

Here θ_0 is the temperature of air.

Step 2 : $$\frac{d\theta}{(\theta_0 - \theta)} = k\,dt$$

Integrating, we get,

$$\int \frac{d\theta}{\theta_0 - \theta} = \int k\,dt + c$$

$$-\log(\theta_0 - \theta) = kt + c \quad \text{... (1)}$$

Step 3 : Initially at $t = 0, \theta = 75$

$$\therefore \quad c = -\log(\theta_0 - 75)$$

Substituting in equation (1)

$$\therefore \quad kt = \log\left(\frac{\theta_0 - 75}{\theta_0 - \theta}\right) \quad \text{... (2)}$$

Again at $t = 5, \; \theta = 65$

and $t = 10, \theta = 60$

We have, from (2)

$$5k = \log\left(\frac{\theta_0 - 75}{\theta_0 - 65}\right) \quad \text{... (3)}$$

and $$10\,k = \log\left(\frac{\theta_0 - 75}{\theta_0 - 60}\right) \quad \text{... (4)}$$

Eliminating k from (3) and (4), we have

$$\frac{10}{5}\log\left(\frac{\theta_0 - 75}{\theta_0 - 65}\right) = \log\left(\frac{\theta_0 - 75}{\theta_0 - 60}\right)$$

$$\log\left(\frac{\theta_0 - 75}{\theta_0 - 65}\right)^2 = \log\left(\frac{\theta_0 - 75}{\theta_0 - 60}\right)$$

$$\left(\frac{\theta_0 - 75}{\theta_0 - 65}\right)^2 = \left(\frac{\theta_0 - 75}{\theta_0 - 60}\right)$$

$$(\theta_0 - 75)(\theta_0 - 60) = (\theta_0 - 65)^2$$

$$(\theta_0)^2 - 135\theta_0 + 4500 = (\theta_0)^2 - 130\,\theta_0 + 4225$$

$$\theta_0 = 55\,^{\circ}F$$

Example 2.11 : *If the dead body is located in a room that is kept at a constant temperature 68°F. For some time after death, the body will radiate heat into the cooler room causing the body temperature decrease. Assuming that the victim's temperature was a normal 98.6°F at the time of death. Find the time of the murder, if the doctor arrived at 9.40 P.M. and immediately measured the dead body temperature as 94.4°F and made another measurement at 11.00 the body temperature as 89.2°F.*

Solution : Let T be temperature of body.

T_0 be room temperature = 68 °F.

According to Newton's law of cooling;

$$\frac{dT}{dt} = -k(T - T_0)$$

$$= -k(T - 68)$$

Or $$\frac{dT}{T - 68} = -kdt \qquad ...(1)$$

From given conditions,

$$t = 0, \quad T = 94.4 \text{ °F}$$

$$t = \mathbf{80}, \quad T = 89.2 \text{ °F}$$

From equation (1)

$$\int_{94.4}^{89.2} \frac{dT}{T-68} = -k\int_0^{80} dt = -k(t)_0^{80}$$

$$[\log(T - 68)]_{94.4}^{89.2} = -80\ k$$

$$\mathbf{\log 21.2 - \log 26.4 = -80\ k}$$

$$\Rightarrow \quad 80\ k = \log\left(\frac{26.4}{21.2}\right)$$

$$k = \frac{\log(1.242)}{80}$$

$$= \mathbf{0.0027}$$

Again from (1), at the time of death T = 98.6 °F so that,

$$\int_{94.4}^{98.6} \frac{dT}{T-68} = -k\int_0^t dt$$

$$(T-68)_{94.4}^{98.6} = -kt$$

$$\Rightarrow \quad t = \frac{1}{k}\log(1.1590)$$

$$= \frac{1}{0.0027}\log(1.1590)$$

$$t = 54.6799 \text{ minutes} \approx 54$$

$\therefore$ The death occurred 54 minutes before first measurement 9.40 P.M. Which is chosen as time zero. This puts the murder at about 8.46 P.M.

Exercise 2.2 :

1) *If the temperature of the body drops from 100 °C to 80 °C in one minute when the temperature of the surrounding is 20 °C, what will be the temperature of the body at the end of second minute (θ = 40 °C).*

2) *Water at temperature 100 °C cools in first 10 minutes to 88 °C in a room at temperatures 25 °C. Find the temperature of water 20 minutes.* **[Ans. : 77.92 °C]**

3) *When a thermometer is placed in a hot liquid bath at temperature T the temperature θ indicated by the thermometer rises at the rate $T - \theta$ for a both at 95 °C the thermometer reads 15 °C at a certain instant ($t = 0$) and 35 °C at t = 10 sec. What will be temperature at t = 20 sec?* **[Ans. : 50 °C]**

4) *If the temperature of the air is 300 k and the substance cools from 370 k to 340 k in 15 minutes, Find when the temperature will be 310 k.* **[Ans. : 52 minutes]**

5) *If a thermometer is taken outdoors where the temperature is 0 °C, from a room in which the temperature is 21 °C and reading drops to 10 °C in 1 minute. How long after its removal will the reading be 5 °C.* **[May 2000] [Ans. : 1 minutes. 56 sec.]**

2.5 Simple Electrical Circuits

A differential equation for R-L circuit or R-C circuit can be formed by using Kirchoff's law's.

a) Kirchoff's voltage law : In a closed loop the algebraic sum of the potential drops or voltage drops is equal to the total electromotive force applied.

b) Current law : At any junction or node of circuit, the current incomes is always equal to outgoing current.

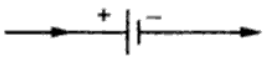
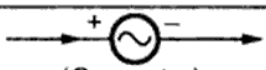
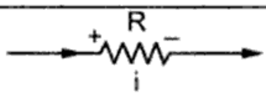
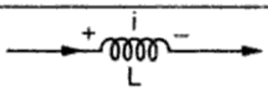
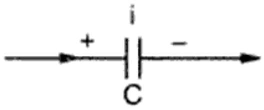
c) Active elements : An energy source is voltage source or a current is said to be active element.

Example : Battery or generator.

d) Passive element : A passive elements transform or store energy but not an energy source.

Example : circuit constants resistance, Inductance and capacitance.

e) Basic relations and table :

• Element	Symbol	Unit
• Quantity of electricity	$q = \int i dt$	Coulomb
• Current	$i = \frac{dq}{dt}$	Ampere
• Electromotive force (EMF) or voltage (E)		Volts
• Variable voltage	 (Generator)	Volts
• Resistance (R)		Ohms or Ω
• Inductance (L)		Henry
• Capacitance (C)		Farad and μF

f) If i is current flowing through a loop, then

i) Voltage drop across R is Ri

ii) Voltage drop across C is q/C or $\frac{1}{C} \int i dt$

iii) Voltage drop across L is $L \frac{di}{dt}$.

A) R-L Series Circuit : If R, L and voltage source E are connected is series and switch is closed, i is current flowing through a circuit, then

According to Kirchoff's voltage law :

Sum of the potential drops = total e.m.f.across R and L

$$L \frac{di}{dt} + Ri = E$$

Or
$$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$$

Which is linear differential equation.

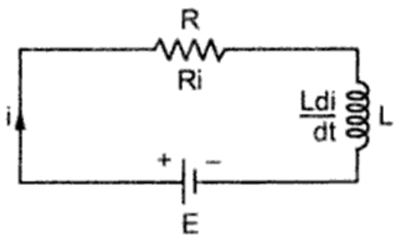


Fig. 2.3

$$\therefore \quad \text{I.F.} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L} t} \text{ and G.S. is}$$

$$i \text{ (I.F)} = \int \frac{E}{L} \text{(I.F.)} \, dt + k$$

$$i \, e^{\frac{R}{L} t} = \int \frac{E}{L} e^{\frac{R}{L} t} \, dt + k$$

$$= \frac{E}{L} \frac{e^{\frac{R}{L} t}}{R/L} + k$$

Or $$i = \frac{E}{R} + k\, e^{-\frac{R}{L}t}$$ Which gives current at any time 't'.

Observation : As $t \to \infty$, $e^{-\frac{R}{L}t} \to 0$ and $i = \frac{E}{R}$ which is maximum value of i.

B) R-C Series Circuit :

Let 'i' be the current flowing in the circuit containing resistance 'R' and capacitor 'C' in series with voltage source E(t), at any time 't' then by Kirchoff's voltage law :

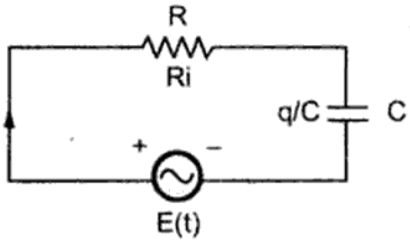


Fig. 2.4

$$Ri + \frac{q}{C} = E(t)$$

Or $$Ri + \frac{1}{C}\int i\,dt = E(t) \quad ...(1)$$

Differentiating (1) w.r.t. 't', we get,

$$R\frac{di}{dt} + \frac{i}{C} = E'(t)$$

Or $$\frac{di}{dt} + \frac{i}{RC} = \frac{E'(t)}{R}$$

Which is linear differential equation and can be solved exactly in a similar manner as discussed in (A) above.

C) L-C Series Circuits :

Let i be the current flowing in the circuit containing inductance 'L' and capacitor C in series without applied e.m.f.

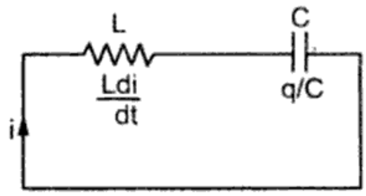


Fig. 2.5

Then by Kirchoff's voltage law :

$$L\frac{di}{dt} + \frac{q}{C} = 0$$

Or $$\frac{di}{dt} = -\frac{q}{LC}$$

Or $$i\frac{di}{dq} = -\frac{q}{LC} \qquad \left(\because \frac{di}{dt} = \frac{di}{dq}\frac{dq}{dt} = i\frac{di}{dq}\right)$$

Or $$i\,di = -\frac{q\,dq}{LC}$$, on integrating

$$i^2 = -\frac{q^2}{LC} + k \quad ...(1)$$

From the given initial conditions q or i can be obtained from equation (1).

Note :

1) $\int e^{at} \sin bt\, dt = \dfrac{e^{at}}{a^2 + b^2}(a \sin bt - b \cos bt) + c$

Or $\int e^{at} \sin bt\, dt = \dfrac{e^{at}}{\sqrt{a^2 + b^2}} \sin(bt - \phi) + c$

Where $\phi = \tan^{-1}\left(\dfrac{b}{a}\right)$

2) $\int e^{at} \cos bt\, dt = \dfrac{e^{at}}{a^2 + b^2}(a \cos bt + b \sin bt) + c$

Or $\int e^{at} \cos bt\, dt = \dfrac{e^{at}}{\sqrt{a^2 + b^2}} \cos(bt - \phi) + c$

Where $\phi = \tan^{-1}\left(\dfrac{b}{a}\right)$

Example 2.12 : *Find current 'i' in the circuit having resistance 'R' and condenser of capacity 'C' in series with e.m.f. E sin ωt.* **(Dec.- 2003, May-2006, Dec.-2007)**

Solution : By Kirchoff's law :

Sum of P.D'S = Total e.m.f. applied.

i.e. $Ri + \dfrac{q}{C} = E \sin \omega t$

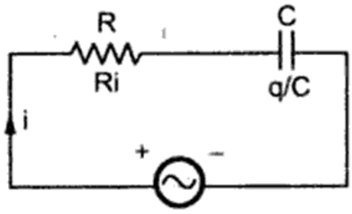


Fig. 2.6

$\Rightarrow \dfrac{dq}{dt} + \dfrac{q}{RC} = \dfrac{E}{R} \sin \omega t$ which is linear differential equation in q

$\therefore$ I.F. $= e^{\int \frac{1}{RC} dt} = e^{t/RC}$

$\therefore$ G.S. is

$$q\left(e^{t/RC}\right) = \int e^{t/RC} \frac{E}{R} \sin \omega t\, dt + k$$

$$= \frac{E}{R} \frac{e^{t/RC}}{\sqrt{\frac{1}{R^2 C^2} + \omega^2}} \sin(\omega t - \phi) + k \qquad \text{Where } \tan \phi = RC\omega$$

$$= EC \frac{e^{t/RC}}{\sqrt{1 + R^2 C^2 \omega^2}} \sin(\omega t - \phi) + k$$

Or $$q = \frac{EC}{\sqrt{1 + R^2 C^2 \omega^2}} \sin(\omega t - \phi) + k e^{-t/RC}$$

Now $$i = \frac{dq}{dt} = \frac{EC\omega}{\sqrt{1 + R^2 C^2 \omega^2}} \cos(\omega t - \phi) - \frac{k}{RC} e^{-t/RC} \quad \text{...Ans}$$

Example 2.13 : *A circuit consists of resistance 'R' ohms and a condenser of C farads connected to a constant e.m.f. E volts. If q/C is the voltage of the condenser at time 't' after closing the circuit. Show that the voltage at time 't' is* $E\left(1 - e^{-t/RC}\right)$. **(Dec.-2005, May-2005, May-2007)**

Solution : The differential equation for R-C circuit is

$$Ri + \frac{q}{C} = E$$

Or $$i + \frac{q}{RC} = \frac{E}{R}$$

Or $$\frac{dq}{dt} + \frac{q}{RC} = \frac{E}{R}$$ which is linear differential equation in q

$\therefore$ $$\text{I.F.} = e^{\int \frac{1}{RC} dt} = e^{t/RC}$$

and general solution is

$$q\, e^{t/RC} = \int \frac{E}{R} e^{t/RC}\, dt + k$$

$$= \frac{E}{R} \frac{e^{t/RC}}{1/RC} + k$$

Or $$\frac{q}{C} e^{t/RC} = E\, e^{t/RC} + k \qquad ...(1)$$

Given at $t = 0, \quad q = 0$

$\therefore$ $0 = E + k \Rightarrow k = -E$

From (1)

$$\frac{q}{C} = E\left(1 - e^{-t/RC}\right)$$

Example 2.14 : *The charge 'Q' on the plate of a condenser of capacity 'C' charged through a resistance 'R' by a steady voltage 'V' satisfies the differential equation.*

$R\frac{dQ}{dt} + \frac{Q}{C} = V$, *If* $Q = 0$ *at* $t = 0$, *show that* $Q = CV\left[1 - e^{-t/RC}\right]$

Find the current flowing into the plate. **(May-2004, Dec.-2005, Dec.-2001)**

Solution : Given $R\frac{dQ}{dt} + \frac{Q}{C} = V$

Or $$\frac{dQ}{dt} + \frac{Q}{RC} = \frac{V}{R}$$ Which is linear D.E. in Q

$\therefore$ $$\text{I.F.} = e^{\int \frac{1}{RC} dt} = e^{t/RC}$$

General solution is,

$$Q\, e^{t/RC} = \int \frac{V}{R} e^{t/RC}\, dt + k$$

$$= \frac{V}{R} \frac{e^{t/RC}}{1/RC} + k$$

Or $$Q = CV + k\, e^{-t/RC} \qquad \ldots(1)$$

Given for t = 0, Q = 0

From equation (1)

$$0 = CV + k \Rightarrow k = -CV$$

$$Q = CV - CV\, e^{-t/RC} = CV\left(1 - e^{-t/RC}\right)$$

$$\text{Current} = i = \frac{dQ}{dt} = CV\left[\frac{1}{RC} e^{-t/RC}\right]$$

$$= \frac{V}{R} e^{-t/RC}$$

Example 2.15 : ***When switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current i builds up at rate given by*** $L\frac{di}{dt} + Ri = E$. ***Find i as a function of 't'. How long will it be, before the current has reached one half it's maximum value. If E = 6 volts, R = 100 ohms and L = 0.1 henry.*** **(May-2003)**

Solution : Given

$$L\frac{di}{dt} + Ri = E$$

or $$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L} \quad \text{which is linear.}$$

$\therefore$ $$\text{I.F.} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L} t}$$

G.S. is

$$i\, e^{\frac{R}{L} t} = \int e^{\frac{R}{L} t} \frac{E}{L}\, dt + k$$

$$= \frac{E}{L} \frac{e^{\frac{R}{L} t}}{\frac{R}{L}} + k$$

Or $$i = \frac{E}{R} + k\, e^{-\frac{R}{L} t} \qquad \ldots(1)$$

At $t = 0, i = 0$

$\therefore$ From equation (1)

$$0 = \frac{E}{R} + k(1) \Rightarrow k = -\frac{E}{R}$$

$$\therefore \quad i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right) \quad \text{...(2)}$$

i_{max} as $t \to \infty$

$$\therefore \quad i_{max} = \frac{E}{R} \text{ from (1)}$$

One half of the maximum value of i is

$$\frac{1}{2} i_{max} = \frac{E}{2R}$$

$\therefore$ From equation (2)

$$\frac{E}{2R} = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$$

Or
$$\frac{1}{2} - 1 = -e^{-\frac{R}{L}t}$$

$$\frac{1}{2} = e^{-\frac{R}{L}t}$$

$$t = \frac{L}{R} \log 2 = \frac{0.1}{100} \log 2 = 0.0006931 \text{ sec.}$$

Example 2.16 : *A constant e.m.f. E volts is applied to a circuit containing a constant resistance R ohms in series and a constant inductance L henries. The current i at any time t is given by* $L\frac{di}{dt} + Ri = E$. *If the initial current is zero. Show that the current builds up to half its theoretical maximum value in* $\frac{L}{R}$ *log 2 seconds.*

(Dec.-2004, Dec.-2006)

Solution :

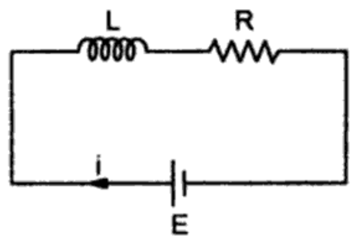


Fig. 2.7

Let i be the current at any time 't' then by Kirchoff's voltage law.

$$L\frac{di}{dt} + Ri = E$$

i.e. $$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \qquad \text{... (1)}$$

This is a linear differential equation.

$\therefore$ $$\text{I.F.} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$$

Solution of equation (1)

i.e. $$i\, e^{\frac{R}{L}t} = \int \frac{E}{L} e^{\frac{R}{L}t} dt + k$$

$$= \frac{E}{L}\frac{e^{\frac{R}{L}t}}{\frac{R}{L}} + k$$

$$i\, e^{\frac{R}{L}t} = \frac{E}{R} e^{\frac{R}{L}t} + k$$

Given at t = 0, i = 0

$\therefore$ From equation (1), $0 = \frac{E}{R} + k \Rightarrow k = -\frac{E}{R}$

$\therefore$ $$i\, e^{\frac{R}{L}t} = \frac{E}{R} e^{\frac{R}{L}t} - \frac{E}{R}$$

$\therefore$ $$i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right) \qquad \text{... (2)}$$

i - reaches theoretical maximum if $e^{-\frac{R}{L}t} \to 0$

i.e. $$t \to \infty$$

$\therefore$ $$i_{max} = \frac{E}{R}$$

Let at $$t = T,$$

$$i = \frac{i_{max}}{2} = \frac{E}{2R}$$

Substituting in equation (2), we get

$$\frac{E}{2R} = \frac{E}{R}\left(1 - e^{-\frac{R}{L}T}\right)$$

i.e. $$e^{-\frac{R}{L}T} = \frac{1}{2} \text{ i.e. } e^{\frac{R}{L}T} = 2$$

$$\frac{R}{L}T = \log 2$$

i.e. $$\boxed{T = \frac{L}{R}\log 2}$$

Example 2.17 : *In a circuit containing inductance L, resistance R and voltage E the current i is given by $E = Ri + L\frac{di}{dt}$, if L = 640 H, R = 250 Ω and E = 500 volts and i = 0 when t = 0, find the time that elapses before the current reaches 90 % of its maximum value.* **(May - 2004, May - 2006)**

Solution : Use the similar procedure as used in example 2.16 to prove

$$i = \frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right)$$

$$i_{max} = \frac{E}{R} = \frac{500}{250} = 2$$

$$90\ \%\ \text{of } i_{max} = \frac{9}{10}\times 2 = 1.8$$

Let at t = T, i = 90% of i_{max} = 1.8

$$\therefore \quad 1.8 = 2\left(1 - e^{-\frac{R}{L}T}\right)$$

$$e^{-\frac{R}{L}T} = 1 - \frac{9}{10} = \frac{1}{10}$$

$$\frac{RT}{L} = \log 10$$

$$T = \frac{L}{R}\log 10 = \frac{640}{250}\log 10 = 5.89 \text{ sec.}$$

Example 2.18 : *A voltage Ee^{-at} is applied at t = 0 to a circuit containing inductance L and resistance R. Show that current at any time 't' is given by $i = \frac{E}{R-aL}\left(e^{-at} - e^{-\frac{Rt}{L}}\right)$* **(1993, 1998, May - 2007)**

Solution : Let i be the current at any time 't'

∴ By Kirchoff's law,

$$L\frac{di}{dt} + Ri = Ee^{-at}$$

i.e. $$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L} e^{-at}$$

This is Linear differential equation in i

$$\therefore \quad \text{I.F.} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L} t}$$

$\therefore$ G.S. is
$$i e^{\frac{R}{L} t} = \int \frac{E}{L} e^{-at} e^{\frac{Rt}{L}} dt + k$$

$$= \frac{E}{L} \frac{e^{\left(\frac{R}{L} - a\right) t}}{\frac{R}{L} - a} + k$$

$$i e^{\frac{R}{L} t} = \frac{E}{R - La} e^{\left(\frac{R}{L} - a\right) t} + k \qquad \dots (1)$$

Now at t = 0, i = 0, from equation (1)

$$\therefore \quad 0 = \frac{E}{R - La} + k \Rightarrow k = -\frac{E}{R - La}$$

From equation(1), $i e^{\frac{R}{L} t} = \frac{E}{R - La} \left[e^{\left(\frac{R}{L} - a\right) t} - 1 \right]$

Dividing by $e^{Rt/L}$ we get

Or $$i = \frac{E}{R - La} \left[e^{-at} - e^{-\frac{R}{L} t} \right]$$

Example 2.19 : *The differential equation of a circuit containing inductance L, resistance R and voltage E sin pt is given by* $L\frac{di}{dt} + Ri = E \sin pt$. *If i = 0 at t = 0, show that* $i = \frac{E}{\sqrt{R^2 + L^2 p^2}} \left[\sin(pt - \phi) + e^{-\frac{Rt}{L}} \sin \phi \right]$, *where* $\phi = \tan^{-1}\left(\frac{LP}{R}\right)$

Solution : We have,

$$L \frac{di}{dt} + Ri = E \sin pt$$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L} \sin pt$$

This is linear differential equation.

$$\therefore \quad \text{I.F.} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L} t}$$

It's solution is

$$i\, e^{\frac{R}{L}t} = \int \frac{E}{L} \sin pt\, e^{\frac{R}{L}t}\, dt + k$$

$$= \frac{E}{L} \frac{e^{\frac{R}{L}t}}{\sqrt{p^2 + \frac{R^2}{L^2}}} \sin(pt - \phi) + k \qquad \text{... (1)}$$

Where $\phi = \tan^{-1}\left(\frac{LP}{R}\right)$

At $t = 0,\ i = 0$

$$0 = \frac{E}{\sqrt{R^2 + L^2P^2}} \sin(-\phi) + k$$

$\therefore$ $$k = \frac{E \sin\phi}{\sqrt{R^2 + L^2P^2}}$$

i.e. $$i = \frac{E}{\sqrt{R^2 + L^2P^2}} \left[\sin(pt - \phi) + e^{-\frac{Rt}{L}} \sin\phi\right]$$

Where $\phi = \tan^{-1}\left(\frac{LP}{R}\right)$

Example 2.20 : *Prove that the differential equation of a circuit containing a resistence R and a condenser of capacity C in series with e.m.f. E is* $E = Ri + \int \frac{i}{C}\, dt$. *Find the current i at any time t, when* $E = E_0 \sin \omega t$.

Solution : Let i be the current at any time t

$\therefore$ By Kirchoff's law,

$$Ri + \frac{1}{C}\int i\,dt = E_0 \sin \omega t$$

Differentiate w.r.t. t

$$\frac{di}{dt} + \frac{1}{RC} i = \frac{E_0\, \omega}{R} \cos \omega t$$

This is linear differential equation

$\therefore$ $$\text{I.F.} = e^{\int \frac{1}{RC} dt} = e^{\frac{1}{RC}t}$$

$\therefore$ And it's G.S. is

$$\left.\begin{aligned} i\, e^{\frac{1}{RC}t} &= \int \frac{E_0\omega}{R} e^{\frac{1}{RC}t} \cos \omega t\, dt + k \\ i\, e^{t/RC} &= \frac{E_0\, \omega}{R} \frac{e^{\frac{1}{RC}t}}{\sqrt{\frac{1}{R^2C^2} + \omega^2}} \cos(\omega t - \phi) + k \end{aligned}\right\} \qquad \text{... (1)}$$

Where $\phi = \tan^{-1}(RC\omega)$

We have at $t = 0,\ i = 0$

$\therefore$ From (1) $$0 = \frac{E_0 C\omega}{\sqrt{1+\omega^2R^2C^2}}\cos\phi + k$$

i.e. $$k = \frac{-E_0 C\omega}{\sqrt{1+\omega^2R^2C^2}}\cos\phi$$

Substituting in equation (1) we get,

$$i\,e^{\frac{1}{RC}t} = \frac{E_0 C\omega}{\sqrt{1+\omega^2R^2C^2}}\, e^{+\frac{1}{RC}t}\left[\cos(\omega t-\phi) - \cos\phi\, e^{-\frac{1}{RC}t}\right]$$

i.e. $$i = \frac{E_0 C\omega}{\sqrt{1+\omega^2R^2C^2}}\left[\cos(\omega t-\phi) - \cos\phi\, e^{-\frac{1}{RC}t}\right]$$

Example 2.21 : *If Q_0 be the initial charge of a condenser of capacity C having inductance L in series find the charge q and current i at any time 't'.*

Solution : Let i be the current at any time 't'

By Kirchoff's law,

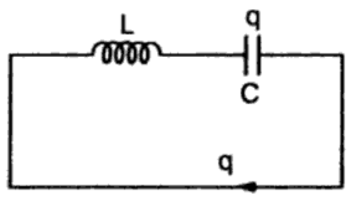


Fig. 2.8

$$L\frac{di}{dt} + \frac{1}{C}\int i\,dt = 0$$

But $$i = \frac{dq}{dt}$$

$\therefore$ $$\frac{di}{dt} = \frac{di}{dq}\frac{dq}{dt} = i\frac{di}{dq}$$

$$Li\frac{di}{dq} + \frac{1}{C}q = 0$$

Integrating,

$$\frac{i^2}{2} + \frac{1}{LC}\frac{q^2}{2} = \frac{k}{2}$$

$$\therefore \qquad i^2 + \frac{q^2}{LC} = k \qquad \text{... (1)}$$

We have at $\quad t = 0, \; q = Q_0$ and $i = 0$

Substituting in equation (1) we get,

$$\therefore \qquad k = \frac{Q_0^2}{LC}$$

Substituting in equation (1), we get,

$$i^2 = \frac{1}{LC}\left(Q_0^2 - q^2\right)$$

$$\frac{dq}{dt} = i = \pm \frac{1}{\sqrt{LC}} \sqrt{Q_0^2 - q^2}$$

Considering negative sign, because as t - increases, q - decreases.

$$\frac{-dq}{\sqrt{Q_0^2 - q^2}} = \frac{+1}{\sqrt{LC}} dt$$

Integrating, we get,

$$\cos^{-1}\left(\frac{q}{Q_0}\right) = \frac{+1}{\sqrt{LC}} t + k_1$$

Again at $\quad t = 0, \; q = Q_0$

$$\cos^{-1}(1) = 0 = k_1 \Rightarrow \boxed{k_1 = 0}$$

$$\therefore \qquad q = Q_0 \cos\left(\frac{1}{\sqrt{LC}} t\right)$$

And $$i = -\frac{Q_0}{\sqrt{LC}} \sin\left(\frac{1}{\sqrt{LC}} t\right)$$

Exercise 2.3 :

1) *A capacitor of capacitance C is charged through a resistance R by a battery which supplies a constant voltage E, the instantaneous charge Q on capacitor satisfies the differential equation* $R\frac{dQ}{dt} + \frac{Q}{C} = E$. *Find Q as a function of time 't' if the capacitor is initially uncharged, i.e, if* $Q_0 = 0$. *How long will it take before the charge on the capacitor is one half its final value ?*

[Ans. : $Q = EC\left(1 - e^{\frac{-t}{RC}}\right)$

$Q_{max} = EC$

Let at $t = T\left(Q \text{ becomes } \frac{Q_{max}}{2}\right) = 0.693RC$**]**

2) *When a switch is closed in a circuit containing a resistance R, an inductance L and a battery which supplies a constant voltage E, the current i builds up at the rate defined by the equation* $L\frac{di}{dt} + Ri = E$. *Find the current as a function of time ? How long will it take i to reach one - half its final value ?*

i) Evaluate $\lim_{t \to \infty} i(t)$.

ii) Find i if $i_0 = \frac{E}{R}$

$$\left[\text{Ans.}: i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)\right.$$

$$\lim_{t \to \infty} i(t) = i_{max} = \frac{E}{R}$$

$$\left.t\left(\frac{i_{max}}{2}\right) = \frac{L}{R}\log 2 = 0.693\frac{L}{R}\right]$$

3) *An inductance of 2 henries and resistance of 20 ohms are connected in series with an e.m.f. E volts if the current is zero when t = 0, find the current at the end of 0.01 sec if a) E = 100 volts b) E = 100 sin 150 t volts.* **[Ans. : a) 0.475 amp b) 0.299 amp]**

4) *When a resistance R ohms and a capacitance C farads are connected in series with an e.m.f. E volts, the current i amperes is given by* $R\frac{di}{dt} + \frac{i}{C} = \frac{dE}{dt}$. *If R = 1000 ohms.* $L = 50 \times 10^{-1}$ *farads,* $i = 10$ *amperes and t = 0. Find the current for t = 1 sec.* **[Ans. : 8.187 amperes]**

5) *A circuit consists of a resistance R ohms and an inductance L henry connected to a generator of* $E\cos(\omega t + \alpha)$.

Find the current in the circuit.

$$\left[\text{Ans.}: i = \frac{E}{\sqrt{R^2 + L^2\omega^2}}\left[\cos\left(\omega t + a - \tan^{-1}\frac{L\omega}{R}\right) - e^{\frac{R}{L}t}\left(\cos a - \tan^{-1}\frac{L\omega}{R}\right)\right]\right]$$

6) *A 20 ohms resistor is connected in series with a capacitor of 0.01 farad and e.m.f. E volts given by* $40e^{-3t} + 20e^{-6t}$. *If q = 0 at t = 0, Show that maximum change on the capacitor is 0.25 Coulomb.* **(Dec.-2007)**

7) *A resistance of 100 ohms, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in a circuit as a function of time.* **(May-2005)**

$$\left[\text{Ans.}: i = \frac{1}{5}\left(1 - e^{-200t}\right)\right]$$

8) ***The equation of L - R circuit is given by*** $L\frac{di}{dt} + Ri = 10\sin t$, ***if i = 0 at t = 0, express i as a function of t.*** **(Dec.-2004, Dec.-2005)**

$$\left[\text{Ans.}: i = \frac{10}{\sqrt{R^2 + L^2}}\left[\sin(t - \phi) + \sin\phi \cdot e^{-\frac{Rt}{L}}\right]\right.$$

$$\left.\text{where } \phi = \tan^{-1}\left(\frac{L}{R}\right)\right]$$

2.6 Applications in Mechanics

A) Rectilinear Motion :

A motion of a body or particle along straight line is called rectilinear motion.

Consider a particle of mass m moving along X-axis starting from 'O'. Let P be the position of a particle at a time 't' and OP = x, then.

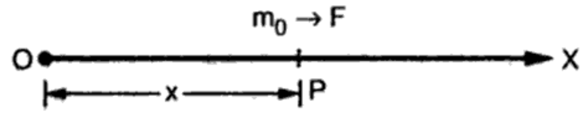


Fig. 2.9

i) **Velocity** = V = $\frac{dx}{dt}$ = Rate of change of displacement w.r.t. to time.

ii) **Acceleration** = a = $\frac{dV}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dx^2}{dt^2}$

Also $a = \frac{dV}{dt} = \frac{dV}{dx}\,\frac{dx}{dt} = V\frac{dV}{dx}$

iii) Newton's second law of motion :

It states that the rate of change of momentum (mass × velocity) at a body is proportional to the resultant force acting on a body.

If 'm' is constant then Newton's second law is

$$\boxed{m\,\frac{dV}{dt} = \text{Net force}}$$

iv) Newton's law of gravitation :

It states that the force of attraction between two particles is inversely proportional to the square of the distance between them.

i.e. $$\boxed{F \propto \frac{1}{r^2}}$$

Where 'r' is a distance between two particles.

v) Terminal velocity : If the particle is falling under gravity where retarding force is proportional to square of the velocity of a body does not increases indefinitely but it takes the limiting value i.e. $\lim\limits_{t \to \infty} V(t)$ is called terminal velocity.

vi) D-Alembert's principle :

The algebraic sum of forces acting on a body along a given direction is the product of mass and acceleration.

i.e. $$\boxed{m\,\frac{d^2x}{dt^2} = \sum F}$$

vii) Forces acting on a body are usually vertically downward, tension is elastic string or spring, forces of attraction, forces of resistance due to air, wind and friction etc.

Example 2.22 : *A body of mass 'm' falling from rest is subjected to the force of gravity and an air resistance proportional to the square of velocity kv^2. If it falls through a distance x and possesses a velocity v at that instant. Prove that* $\frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right)$, *where* $mg = ka^2$.

(Dec.-2000, May-1995, 1998, 2004, 2005, 2006)

Solution : The forces acting on body are

i) The resistance due to air i.e. kv^2 (vertically upward).

ii) The gravitational acceleration i.e. mg (vertically downward).

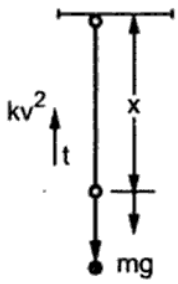


Fig. 2.10

By Newton's second law, the equation of motion is

$$mv\frac{dv}{dx} = mg - kv^2$$

$$= ka^2 - kv^2 \qquad (\because mg = ka^2)$$

Or
$$mv\frac{dv}{dx} = k(a^2 - v^2)$$

$$\Rightarrow \int \frac{v\,dv}{a^2 - v^2} = \frac{k}{m}\int dx + c$$

$$\text{Or } -\frac{1}{2}\log(a^2 - v^2) = \frac{kx}{m} + c \qquad ...(1)$$

Initially x = 0, when v = 0

$\therefore$ From equation (1), $\boxed{c = -\frac{1}{2}\log a^2}$

$$\therefore \frac{1}{2}\left[\log a^2 - \log(a^2 - v^2)\right] = \frac{kx}{m}$$

$$\Rightarrow \frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right)$$

Example 2.23 : *A body of mass m falls under gravity and retarding force due to air resistance is proportional to square of velocity. Find velocity and limiting speed and distance travelled as a function of time.*

Solution : Let x be the position of the particle at any time 't'

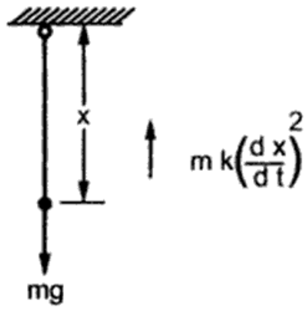


Fig. 2.11

Then forces acting on the body are

i) Force due to air resistance = $mk\left(\frac{dx}{dt}\right)^2$

ii) Force due to mass m = mg

∴ By D-Alembert's principle

$$m\frac{d^2x}{dt^2} = mg - km\left(\frac{dx}{dt}\right)^2$$

Let $$v = \frac{dx}{dt}$$

$$\frac{dv}{dt} = \frac{d^2x}{dt^2}$$

∴ $$\frac{dv}{dt} = g - kv^2$$

Separating variables $$\frac{dv}{g - kv^2} = dt$$

Integrating $$\int \frac{dv}{\frac{g}{k} - v^2} = \int k dt + c_1$$

Let $$\frac{g}{k} = \lambda^2$$

$$\int \frac{dv}{\lambda^2 - v^2} = kt + c_1$$

$$\frac{1}{2\lambda}\log\left(\frac{\lambda + v}{\lambda - v}\right) = kt + c_1 \qquad \because \tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

i.e. $$\frac{1}{\lambda}\tan^{-1}h\left(\frac{v}{\lambda}\right) = kt + c_1$$

We have at t = 0, v = 0

$$\therefore \quad 0 = c_1$$

$$\therefore \quad \frac{1}{\lambda}\tanh^{-1}\left(\frac{v}{\lambda}\right) = kt$$

$$\therefore \quad \frac{v}{\lambda} = \tanh(\lambda kt)$$

$$= \frac{e^{\lambda kt} - e^{-\lambda kt}}{e^{\lambda kt} + e^{-\lambda kt}}$$

$$\text{Limiting speed} = \lim_{t \to \infty} v(t) = \lambda = \sqrt{\frac{g}{k}}$$

Next $$\frac{1}{\lambda}\frac{dx}{dt} = \tanh(\lambda kt)$$

i.e. $$dx = \lambda \tanh(\lambda kt)\, dt$$

Integrating, $$x = \frac{\lambda}{k\lambda}\log\cosh(\lambda kt) + c_2$$

$$= \frac{1}{k}\log\cosh(\lambda kt) + c_2$$

We have at t = 0, x = 0

$$0 = 0 + c_2 \Rightarrow c_2 = 0$$

$$\therefore \quad x = \frac{1}{k}\log\cosh(\lambda kt)$$

Example 2.24 : *A particle of mass m is projected vertically upward with velocity V, assuming that the air resistance is k times the velocity,*

a) Show that particle will reach maximum height in time $\frac{m}{k}\log\left(1 + \frac{kV}{mg}\right)$

b) Also prove that distance travelled at any time 't' is $\left(\frac{mv}{k} + \frac{m^2 g}{k^2}\right)\left(1 - e^{-\frac{kt}{m}}\right) - \frac{gmt}{k}$

Solution : (a) Forces acting are

I) Air resistance = $k \dfrac{dx}{dt}$

II) Due to mass m = mg

Let x be the position of a particle at any time 't' then by Newton's second law

$$m \frac{d^2x}{dt^2} = -mg - k\frac{dx}{dt}$$

let $$\frac{dx}{dt} = v$$

$$\therefore \quad \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

i.e. $$m\frac{dv}{dt} = -mg - kv$$

$$m\frac{dv}{mg+kv} = -dt$$

$$m\frac{kdv}{mg+kv} = -kdt$$

$$\text{i.e.} \quad m\log(mg+kv) = -kt + c_1 \qquad \dots (1)$$

We have at $x = 0, \; v = V$

$$m\log(mg+kV) = c_1$$

Substituting in (1), we get

$$m\log(mg+kv) = -kt + m\log(mg+kV)$$

i.e $$t = \frac{-m}{k}\log\left(\frac{mg+kv}{mg+kV}\right) \qquad \dots (2)$$

When the particle reaches the maximum height v = 0

i.e. $$t = -\frac{m}{k}\log\left(\frac{mg}{mg+kV}\right)$$

$$= \frac{m}{k}\log\left(1+\frac{kV}{mg}\right)$$

(b) from equation (2)

$$\frac{mg+kv}{mg+kV} = e^{-\frac{kt}{m}}$$

$$\therefore \qquad \frac{k}{m} v = -g + \left(g + \frac{k}{m} V\right) e^{-\frac{kt}{m}}$$

$$\text{i.e.} \qquad \frac{dx}{dt} = \frac{-mg}{k} + \left(\frac{mg}{k} + V\right) \cdot e^{-\frac{kt}{m}}$$

Integrating,

$$x = \frac{-mg}{k} t + \left(\frac{mg}{k} + V\right) \frac{e^{-\frac{kt}{m}}}{-\frac{k}{m}} + c_2$$

At $t = 0, \; x = 0$

$$\therefore \qquad c_2 = \left(\frac{mg}{k} + V\right) \frac{m}{k}$$

$$\therefore \qquad x = -mgt + \left(\frac{mg}{k} + V\right) \frac{m}{k} (1 - e^{-kt})$$

$$= \left(\frac{m^2 g}{k^2} + \frac{mv}{k}\right) (1 - e^{-kt}) - \frac{mgt}{k}$$

Example 2.25 : *A particle of mass m falls from rest under gravity in a fluid whose resistance to motion at any instant is mk times the velocity where k is constant. Find the terminal velocity of the body and also time taken to acquire one half of its limiting speed.*

Solution : Let x be the position of a particle at any time t.

Forces acting on are,

I) Air resistance $= mk \frac{dx}{dt}$

II) Weight = mg

By Newton's 2nd Law.

$$m \frac{d^2x}{dt^2} = mg - mk \frac{dx}{dt}$$

$$\text{Let} \qquad \frac{dx}{dt} = V$$

$$\frac{dv}{dt} = g - kv$$

Separating variables

$$\frac{dv}{g - kv} = dt$$

$$-\frac{1}{k} \log (g - kv) = t + c \qquad \dots (1)$$

we have at t = 0, v = 0

$$\therefore \qquad c = -\frac{1}{k}\log(g)$$

substituting in equation (1)

$$\therefore \qquad t = \frac{1}{k}\log\left(\frac{g}{g-kv}\right) \qquad \text{... (2)}$$

$$\therefore \qquad -kt = \log\left(\frac{g-kv}{g}\right)$$

$$\text{Or} \qquad e^{-kt} = \frac{g-kv}{g} = 1-\frac{k}{g}v$$

$$\text{i.e.} \qquad v = \frac{g}{k}(1-e^{-kt})$$

$$\therefore \text{ Terminal velocity} = \underset{t\to\infty}{\text{Limit}}\ v(t) = \frac{g}{k} = V \text{ (say)}$$

$$\text{i.e.} \qquad k = \frac{g}{V}$$

Substituting in (2), we get a time required to attain a terminal velocity.

$$t \text{ (terminal velocity)} = \frac{V}{g}\log\left[\frac{g}{g-\frac{g}{V}v}\right]$$

$$= \frac{V}{g}\log\left[\frac{V}{V-v}\right]$$

$$= -\frac{V}{g}\log\left[1-\frac{v}{V}\right]$$

$$\text{Let at} \qquad t = T,$$

$$\text{When} \qquad v = \frac{1}{2}V$$

$$\therefore \qquad T = -\frac{V}{g}\log\left[1-\frac{1}{2}\right]$$

$$= -\frac{V}{g}\log\frac{1}{2} = \frac{V}{g}\log 2$$

Example 2.26 : *A moving body opposed by a force per unit mass of value cx and resistance per unit of mass of value* bv^2 *where x and v are displacement and velocity of the particle of that instant. Find the velocity of a particle in terms of x, if it starts from rest.* **[Dec.-2005]**

Solution : By Newton's 2nd law of motion, the equation of motion

is $$v \frac{dv}{dx} = -cx - bv^2$$

i.e. $$v\frac{dv}{dx} + bv^2 = -cx$$

Let $$v^2 = u$$

$$2v \frac{dv}{dx} = \frac{du}{dx}$$

$$\frac{1}{2}\frac{du}{dx} + bu = -cx$$

i.e. $$\frac{du}{dx} + 2bu = -2cx$$

This is L.D.E.

$$\text{I.F.} = e^{\int 2bdx} = e^{2bx}$$

And its solution is

$$u.e.^{2bx} = -\int 2cx \cdot e^{2bx}\,dx + A$$

$$= -2c\left[x\,\frac{e^{2bx}}{2b} - (1)\,\frac{e^{2bx}}{4b^2}\right] + A$$

At $$x = 0,\ u = v^2 = 0$$

$\therefore$ $$0 = \frac{c}{2b^2} + A \Rightarrow A = -\frac{c}{2b^2}$$

$\therefore$ $$v^2 = -\frac{c}{b}x + \frac{c}{2b^2} - \frac{c}{2b^2}\,e^{-2bx}$$

Example 2.27 : *A particle of mass m is projected vertically upward under gravity the resistance due to air being mk time the velocity. Show that the greatest height attained by the particle is* $\frac{v^2}{g}[\lambda - \log(1+\lambda)]$ *where V is the greatest velocity, which the above mass will attain when it falls freely and* λV *is the initial velocity.*

Solution : Let x be the position of a particle at any time 't'.

Forces acting on particle are

I) Weight mg

II) Air resistance mk $\frac{dx}{dt}$

Equation of motion is given by Newton's 2nd law of motion.

$$m\frac{d^2x}{dt^2} = -mg - mk\frac{dx}{dt}$$

Let $\qquad \dfrac{dx}{dt} = v$

$$\frac{dv}{dt} = v\frac{dv}{dx}$$

$$v\frac{dv}{dx} = -g - kv \qquad \text{... (1)}$$

When a particle falls freely then equation of motion becomes

$$m\frac{d^2x}{dt^2} = mg - mkv$$

i.e. $\qquad \dfrac{dv}{dt} = g - kv \qquad$ ... (2)

But when v becomes maximum $\dfrac{dv}{dt} = 0$ (acceleration = 0)

Let $\qquad v_{max} = V$ (given)

Substituting in (2)

$$0 = g - kV$$

i.e. $\qquad k = \dfrac{g}{V}$

Substituting in equation (1)

$$v\frac{dv}{dx} = -g - \frac{g}{V}v = \frac{-g}{V}(V+v)$$

$$\frac{vdv}{V+v} = -\frac{g}{V}dx$$

$$\frac{V+v-V}{V+v}dv = -\frac{g}{V}dx$$

$$\left(1 - \frac{V}{V+v}\right)dv = -\frac{g}{V}dx$$

Integrating

$$v - V\log(V+v) = -\frac{g}{V}x + c \qquad \text{... (3)}$$

We have at $\qquad x = 0,\ v = \lambda V \qquad$ (given)

$$\lambda V - V\log V(1+\lambda) = c$$

Substituting in equation (3), we get

$$v - V \log (V + v) = \frac{-g}{V} x + \lambda V - V \log V(1 + \lambda)$$

Let x = h = greatest height attained by a particle.

But at the greatest height, velocity v = 0

$$- V \log V = \frac{-g}{V} h + \lambda V - V \log V(1 + \lambda)$$

$$\frac{g}{V} h = \lambda V - V \log V(1 + \lambda) + V \log V$$

$$= \lambda V - V\left[\log(1 + \lambda)\right]$$

$$h = \frac{V^2}{g}\left[\lambda - \log(1 + \lambda)\right]$$

Example 2.28 : *A particle is projected vertically upward with velocity u and the air resistance produced retardation kv^2, where v is velocity at any instant. Show that the velocity V with which the particle will return to the point of projection is given by* $\frac{1}{V^2} = \frac{1}{u^2} + \frac{k}{g}$

Solution : Equation of motion is,

$$mv \frac{dv}{dx} = - mg - mkv^2$$

i.e.

$$v \frac{dv}{dx} = -(g + kv^2)$$

$$\frac{vdv}{g + kv^2} = - dx$$

Integrating,

$$\frac{1}{2k} \log(g + kv^2) = - x + c_1 \qquad \dots (1)$$

at

$$x = 0, \; v = u$$

$$\therefore \quad c_1 = \frac{1}{2k} \log (g + ku^2)$$

Substituting in equation (1)

$$\frac{1}{2k} \log (g + kv^2) = - x + \frac{1}{2k} \log (g + ku^2)$$

i.e. $$x = \frac{1}{2k} \log\left(\frac{g + ku^2}{g + kv^2}\right)$$

Let at x = h = maximum height attained by particle

$\therefore$ at $x = h, \ v = 0$

$$\therefore \quad h = \frac{1}{2k} \log\left(1 + \frac{k}{g} u^2\right) \quad \text{... (2)}$$

Once the particle reaches maximum height it starts falling under gravity

Equation of downward motion is,

$$mv \frac{dv}{dx} = mg - mk\,v^2$$

i.e. $$\frac{vdv}{g - kv^2} = dx$$

Integrating,

$$-\frac{1}{2k} \log\left(g - kv^2\right) = x + c_2$$

Again at x = 0, v = 0

$$\therefore \quad -\frac{1}{2k} \log(g) = c_2$$

$$\therefore \quad x = \frac{1}{2k} \log\left(\frac{g}{g - kv^2}\right)$$

And at x = h = point of projection

$$v = V \quad \text{(given)}$$

$$h = \frac{1}{2k} \log\left(\frac{g}{g - kV^2}\right) \quad \text{... (3)}$$

$\therefore$ From equation (2) and (3) we get,

$$h = \frac{1}{2k} \log\left(1 + \frac{k}{g} u^2\right) = \frac{1}{2k} \log\left(\frac{g}{g - k\,v^2}\right)$$

$$\therefore \quad 1 + \frac{ku^2}{g} = \frac{g}{g - kv^2}$$

$$\therefore \quad \frac{ku^2}{g} = \frac{g}{g - v^2k} - 1 = \frac{v^2k}{g - v^2k}$$

$$\therefore \quad \frac{g}{ku^2} = \frac{g}{kv^2} - 1$$

$$\therefore \quad \frac{g}{kv^2} = 1 + \frac{g}{ku^2}$$

Multiplying by k/g we get,

$$\frac{1}{v^2} = \frac{k}{g} + \frac{1}{u^2}$$

Example 2.29 : *The distance x descended by a parachuter satisfy the differential equation* $\left(\frac{dx}{dt}\right)^2 = k^2\left(1 - e^{-2g\frac{x}{k^2}}\right)$ *where k and g are constant. If x = 0 when t = 0. Show that* $x = \frac{k^2}{g}\log\cosh\left(\frac{gt}{k}\right)$.

(May-2005)

Solution : We have,

$$\frac{dx}{dt} = k\sqrt{1 - e^{\frac{-2gx}{k^2}}}$$

i.e.
$$\frac{dx}{\sqrt{1 - e^{\frac{-2gx}{k^2}}}} = kdt$$

Integrating

$$\int \frac{dx}{\sqrt{1 - e^{\frac{-2gx}{k^2}}}} = \int kdt + c$$

Put
$$1 - e^{\frac{-2gx}{k^2}} = u^2$$

$$e^{\frac{-2gx}{k^2}} \cdot \frac{2g}{k^2}\, dx = 2u\, du$$

i.e. $$(1 - u^2)\frac{g}{k^2}\, dx = u\, du$$

i.e.
$$dx = \frac{k^2}{g}\frac{u}{(1 - u^2)}\, du$$

∴
$$\int \frac{1}{u} \cdot \frac{k^2}{g}\frac{u}{1 - u^2}\, du = kt + c$$

i.e.
$$\frac{k^2}{g}\int \frac{1}{1 - u^2}\, du = kt + c$$

i.e.
$$\frac{k^2}{g}\frac{1}{2}\log\left(\frac{1 + u}{1 - u}\right) = kt + c$$

i.e.
$$\frac{k^2}{g}\tanh^{-1} u = kt + c$$

But when t = 0, x = 0 and hence u = 0

∴
$$c = 0$$

$$\frac{k}{g}\tanh^{-1} u = t$$

i.e. $$u = \tanh\left(\frac{gt}{k}\right)$$

$$1 - e^{\frac{-2gx}{k^2}} = \tanh^2\left(\frac{gt}{k}\right)$$

$$e^{\frac{-2gx}{k^2}} = 1 - \tanh^2\left(\frac{gt}{k}\right) = \text{sech}^2\left(\frac{gt}{k}\right)$$

$\therefore$ $$e^{\frac{2gx}{k^2}} = \cosh^2\left(\frac{gt}{k}\right)$$

$\therefore$ $$\frac{2gx}{k^2} = \log\cosh^2\left(\frac{gt}{k}\right) = 2\log\cos h\left(\frac{gt}{k}\right)$$

$\therefore$ $$x = \frac{k^2}{g}\log\cosh\left(\frac{gt}{k}\right)$$

Example 2.30 : *A particle is moving in a straight line with acceleration $k\left(x + \frac{a^4}{x^3}\right)$ directed towords origin. If it starts from rest at a distance 'a' from the origin, prove that it will arrive at the origin at the end of time $\frac{\pi}{4\sqrt{k}}$.* **(May-2004, May-2001)**

Solution : We have,

$$v\frac{dv}{dx} = -k\left(x + \frac{a^4}{x^3}\right)$$

i.e. $$vdv = -k\left(x + \frac{a^4}{x^3}\right)dx$$

Integrating, we get,

$$\frac{v^2}{2} = -k\left(\frac{x^2}{2} - \frac{a^4}{2x^2}\right) + c$$

we have at x = a, v = 0, $\therefore$ c = 0

$\therefore$ $$v^2 = \frac{k}{x^2}(a^4 - x^4)$$

$\therefore$ $$v = -\frac{\sqrt{k}}{x}\sqrt{a^4 - x^4}$$

Considering negative sign, because particle is directed towards origin

$\therefore$ $$v = \frac{dx}{dt} = -\sqrt{k}\,\frac{\sqrt{a^4 - x^4}}{x}$$

$$\int \frac{x}{\sqrt{a^4 - x^4}}\, dx = -\int \sqrt{k}\, dt + c_1$$

Put $\quad x^2 = a^2 u$

$$2x\, dx = a^2 du$$

$$x dx = \frac{a^2}{2} du$$

i.e. $\quad \displaystyle\int \frac{1}{a^2\sqrt{1 - u^2}} \cdot \frac{a^2}{2} du = -\sqrt{k}\ t + c_1$

i.e. $\quad \displaystyle\frac{1}{2} \sin^{-1} u = -\sqrt{k}\ t + c_1$

i.e. $\quad \displaystyle\frac{1}{2} \sin^{-1} \frac{x^2}{a^2} = -\sqrt{k}\ t + c_1 \qquad \ldots (1)$

We have at t = 0, x = a

$\therefore \quad \displaystyle\frac{1}{2} \sin^{-1}(1) = c_1 \Rightarrow \sin^{-1} = 2C_1$

$\therefore \quad \displaystyle 2C_1 = \frac{\pi}{2} \quad \therefore\ C_1 = \frac{\pi}{4}$

Substituting in equation (1)

$\therefore \quad \displaystyle\frac{1}{2} \sin^{-1} \frac{x^2}{a^2} = -\sqrt{k}\, t + \frac{\pi}{4}$

Let at t = T, particle reaches at x = 0

$$0 = -\sqrt{k}\, T + \frac{\pi}{4}$$

$\therefore \quad \displaystyle T = \frac{\pi}{4\sqrt{k}}$

Example 2.31 : *Assuming the resistance to the motion of a ship through water is $a^2 + b^2v^2$ where v is the velocity of the ship and a, b are constants, write down the differential equation for the retardation of ship moving with engine stopped. Prove further that the time in which the speed falls to one half of its original value is given by $\frac{W}{abg} \tan^{-1}\left[\frac{abu}{2a^2 + b^2u^2}\right]$, where u is the initial velocity.*

(Dec.-2001)

Solution : We have $\quad t = 0 \qquad$ (when engine is stopped)

Initial velocity = u

Let x be the distance travelled by a ship at any time 't' (after engine is stopped)

$\therefore$ By Newtons 2nd law of motion.

$$m\frac{d^2x}{dt^2} = -\left(a^2 + b^2v^2\right)$$

Let $$\frac{dx}{dt} = v, \quad m = \frac{W}{g}$$

where W = Weight of a ship

$$\frac{W}{g} \cdot \frac{dv}{dt} = -\left(a^2 + b^2v^2\right)$$

i.e. $$\frac{W}{g} \cdot \frac{dv}{a^2 + b^2v^2} = -dt$$

On integrating, we get,

$$\frac{W}{gab} \tan^{-1}\left(\frac{bv}{a}\right) = -t + c$$

We have at t = 0, v = u

$$\therefore \qquad c = \frac{W}{gab} \tan^{-1}\left(\frac{bu}{a}\right)$$

Substituting in equation (1), we get

$$t = \frac{W}{gab}\left[\tan^{-1}\left(\frac{bu}{a}\right) - \tan^{-1}\left(\frac{bv}{a}\right)\right]$$

$$= \frac{W}{gba} \tan^{-1} \frac{b}{a}\left[\frac{u - v}{1 + uv\,\dfrac{b^2}{a^2}}\right]$$

Let at t = T, $v = \frac{u}{2}$ = half of initial velocity

$$\therefore \qquad T = \frac{W}{gba} \tan^{-1}\left[\frac{b}{a} \cdot \frac{u - \dfrac{u}{2}}{1 + u \cdot \dfrac{u}{2} \cdot \dfrac{b^2}{a^2}}\right]$$

$$= \frac{W}{gba} \tan^{-1}\left[\frac{b}{a} \cdot \frac{u}{2 + u^2\,\dfrac{b^2}{a^2}}\right]$$

$$= \frac{W}{gba} \tan^{-1}\left[\frac{abu}{2a^2 + b^2u^2}\right]$$

Example 2.32 : *A bullet is fired into sand tank, its retardation is proportional to square root of its velocity. Show that in time $\frac{2\sqrt{V}}{k}$ the bullet will come to rest where V is the initial velocity.*

Solution : We have,

$$m\frac{dv}{dt} = -mk\sqrt{v}$$

i.e. $$\frac{dv}{\sqrt{v}} = -k\,dt$$

Integrating, we get,

$$2\sqrt{v} = -kt + c_1 \quad \text{... (1)}$$

at $t = 0,\ v = V$

$$2\sqrt{V} = c_1$$

Substituting in equation (1), we get,

$$2\sqrt{v} = -kt + 2\sqrt{V}$$

i.e. $$t = \frac{2}{k}\left(\sqrt{V} - \sqrt{v}\right)$$

We have at $t = T,\ v = 0$

$\therefore$ $$T = \frac{2\sqrt{V}}{k}$$

Example 2.33 : *Find the least velocity required to project a particle in vertically upward direction assuming that it is acted upon by gravitational attraction of the earth only.*

Solution : Let 'r' be the position of a particle from centre of earth at any time 't' when projected upward with initial velocity say V.

By Newtons law of gravitation $\frac{d^2r}{dt^2} \propto \frac{1}{r^2}$

i.e. $$\frac{dv}{dt} = -\frac{k}{r^2} \quad \text{(where k is constant)}$$

i.e. $$v\frac{dv}{dr} = -\frac{k}{r^2} \quad \text{... (1)}$$

Let R be the radius of the earth.

On the surface of earth.

i.e. at $r = R$, acceleration is due to the gravity only.

$$\therefore \quad v\frac{dv}{dr} = -g \qquad \text{... (2)}$$

From equation (1) equation and (2) $k = gR^2$

Substituting in equation (1) we get

$$v\frac{dv}{dr} = -\frac{gR^2}{r^2}$$

i.e. $$vdv = -\frac{gR^2}{r^2}\,dr$$

Integrating, $$\frac{v^2}{2} = \frac{gR^2}{r} + c \qquad \text{... (3)}$$

But for $$r = R,\ v = V$$

$$\therefore \quad \frac{V^2}{2} = gR + c$$

$$\Rightarrow \quad c = \frac{V^2}{2} - gR$$

Substituting in equation (3) we get

$$\frac{v^2}{2} = -\frac{gR^2}{r} + \frac{V^2}{2} - gR$$

i.e. $$v^2 = -\frac{2gR^2}{r} + V^2 - 2gR$$

The particle will not return back if v^2 remains positive.

As particle goes up $\frac{2gR^2}{r^2} \to 0$ as $r \to \infty$

$\therefore v^2$ remains positive if $V^2 - 2gR \geq 0$

i.e. $$V^2 \geq 2gr$$

$$V^2 \geq \sqrt{2gr}$$

least velocity of projection $= \sqrt{2gr}$

Exercise 2.4 :

1) *An equation in the theory of stability of an aeroplane is* $\frac{dv}{dt} = g\cos\theta - kv$, *where* v *is instantaneous velocity,* g, θ, k *are constants. When* $t = 0$, $v = 0$. *Show that* $v = \frac{g\cos\theta}{k}\left(1 - e^{-kt}\right)$

2) *A body of mass* m *is subjected to retardation proportion to square of velocity prove that* $x = \frac{m}{2k}\log\left(\frac{a^2}{a^2 - v^2}\right)$ *where* $mg = ka^2$.

(May-2006)

3) *A body starts from a rest is opposed by a force per unit mass of value cx and resistance per unit mass of value bv^2, where x and v are instantaneous displacement and velocity of a body. Show that* $v = \frac{c}{2b^2}\left(1 - e^{-2bx}\right) - \frac{cx}{b}$

4) *The distance x descended by a parachuter satisfies the differential equation* $v\frac{dv}{dx} = g\left(1 - \frac{v^2}{k^2}\right)$ *where v is instantaneous velocity, k, g are constants. Show that* $x = \frac{k^2}{g}\log\cosh\left(\frac{gt}{k}\right)$ **(Dec.-1993, 1998)**

5) *A particle of mass m is projected vertically upwards with initial velocity u, the air resistance being mk times the velocity. Show that the distance travelled by the particle at any time t is given by* $x = \left(\frac{g}{k^2} + \frac{u}{k}\right)\left(1 - e^{kt}\right) - \frac{g}{k}t$

6) *A vehicle starts from rest and its acceleration is given by* $k\left(1 - \frac{t}{T}\right)$ *where k and T are constants. Find the maximum speed and the distance travelled when maximum speed is attained.*

[Ans. : Maximum speed = $\mathbf{\frac{kT}{2}}$**, s =** $\mathbf{\frac{kT^2}{3}}$**]**

7) *A particle of mass m moves in a horizontal straight line with retardation* $\frac{k}{x^3}$ *directed towards origin at a distance x from origin. If initially the particle is at a distance 'a' from origin. Show that it will be at a distance* $\frac{a}{2}$ *from 'O' in time* $\frac{a^2\sqrt{3}}{2\sqrt{k}}$.

8) *A particle moves in a straight line in a resisting medium where retardation is proportional to cube of velocity. If u is initial velocity prove that* $v = \frac{u}{1 + kxu}$ *and* $t = \frac{x}{u} + \frac{kx^2}{2}$.

9) *The resistance to the motion of a car of mass m varies as the square of its speed of its speed and the effective horse power exerted at the road wheels is constant equals to p show that the distance in which the car can accelerate from* v_0 *to* v_1 *is given by* $\frac{m}{3k}\log\left(\frac{p - kv_0^3}{p - kv_1^3}\right)$

Hint : $mv\frac{dv}{dx} = F - kv^2$, *when* $F = \frac{p}{v}$

Let velocity changes from v_0 *to* v_1 *from the point* x_0 *to* x_1 . *To get required solution integrate equation as* $\int_{v_0}^{v_1} \frac{mv\,dv}{F - kv^2} = \int_{x_0}^{x_1} dx.$

10) *A particle moves in a straight line under the action of force of attraction varying inversely as the* $\frac{3}{2}$ *th power of the distance. Show that the velocity acquired by falling from an infinite distance to a distance 'a' from the centre is equal to the velocity which would be acquired in moving from distance 'd' to a distance* $\frac{a}{4}$

Hint : $v\frac{dv}{dx} = -kx^{-\frac{3}{2}}$

i.e. $vdv = -kx^{-\frac{3}{2}}dx$

$$\int_0^{v_1} vdv = -\int_{\infty}^{a} x^{-\frac{3}{2}}dx$$

$$\therefore v_1 = \frac{2\sqrt{\pi}}{a^{\frac{1}{4}}} \quad \text{... (i)}$$

Next $\int_0^{v_2} vdv = -\int_a^{\frac{0}{4}} x^{-\frac{3}{2}}dx$

$$\therefore \quad v_2 = \frac{2\sqrt{\pi}}{a^{\frac{1}{4}}} \quad \text{... (ii)}$$

from equation (i) and equation (ii) $v_1 = v_2$

11) *A particle of unit mass is projected vertically upward with velocity u. Assuming that the air resistance is k times the instantaneous velocity of a particle show that the particle will return to the point of projection with velocity V given by* $V + u = \frac{g}{k}\log\left(\frac{g + gu}{g - kV}\right)$ **(Dec.-2005)**

Hint : *Refer example 2.28.*

12) *The acceleration of a moving body is proportional to the cube of its velocity and is negative. Show that in time t body travels a distance* $x = \frac{\left(\sqrt{2kv_0^2 t + 1} - 1\right)}{kv_0}$ *where* v_0 *is initial velocity.*

13) *A paratrooper and his parachute weight 50 kg. At the instant parachute opens, he is travelling vertically downward at the speed 20 m/s. If the air resistance varies directly the instantaneous velocity and if it is 20 Newtons when the velocity is 10 m/s, find the limiting velocity, the position and the velocity of a paratrooper at any time 't'.*

Hint : $\frac{50}{g}\frac{dv}{dt} = 50 - kv$

$v = 5\left(5 - e^{-\frac{gt}{25}}\right)$, *Limiting velocity = 25 m/s*

$$x = 25t - \frac{125}{g}\left(1 - e^{-\frac{gt}{25}}\right)$$

14) *A body falling from rest is subjected to the force of gravity and air resistence of* $\frac{n^2}{g}$ *times the square of the velocity, prove that* $v^2 = \frac{g^2}{n^2}\left(1 - e^{-\frac{2n^2x}{g}}\right)$

15) *A particle of unit mass moves in a straight line under retardation which is k times its velocity. initially the particle is at a distance 'a' from a given point 'd' in the line and is moving towards 0 with velocity u. Prove that it will reach 0 in time* $\frac{1}{k}\log\left(\frac{u}{u - ak}\right)$

16) *A particle falls in a vertical line under gravity and the force of air resistance to its motion is proportional to velocity show that its velocity can not exceed a particular limit.*

Hint : $\frac{dv}{dt} = g - kv,\ v_{\max} = \frac{g}{k}$

17) *A moving body is opposed by a force proportional to the displacement and by a resistance proportional to the square of velocity. Find the velocity in form of x*

Hint : $v\frac{dv}{dx} = -k_1 x - k_2 v^2$

$$v^2 = \frac{k_1}{k_2}x + \frac{2mk_1}{k_2^2} + Ce^{-\frac{k_2}{2m}x}$$

18) *A body weighing w be falled from rest under the influence of gravity and retarding force due to air resistance, assumed to be proportional to the velocity of the body. Find the equations expressing the velocity of fall and distance fallen as function of time and verify that these reduce to ideal laws.* $v = gt$ *and* $s = \frac{1}{2}gt^2$

19) *A body of weight W falls from rest under the influence of gravity and a retarding force proportional to nth power of the velocity. Show that the velocity of the body approaches the limiting value* $v_\infty = n\sqrt{\frac{w}{k}}$ *where k is proportionality constant. Show also that the time T that it takes the body to reach one - half its limiting value is given by the equation.* $T = \alpha\frac{V_\infty}{g}$ *. And compute the value of* α *for*

a) $n = \frac{1}{4}$ *b)* $n = \frac{1}{3}$ *c)* $n = \frac{1}{2}$

d) $n = 2$ *e)* $n = 3$ *f)* $n = 4$

20) *A unit mass particle moves along the x-axis in a resisting medium and is proportional to the velocity. If the particle starts from the position* $x = x_0$ *with velocity* v_0 *find the limiting position approched by the particle take* $-kv$ *with* $k > 0$*, as the force of resistance.* **[Ans. :** $x_0 + \frac{v_0}{k}$**]**

2.7 Simple Harmonic Motion (SHM)

Definition

A particle is said to be in SHM if its acceleration is always directed towards a fixed point on the line and varies as the distance from the fixed point.

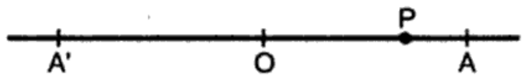


Fig. 2.12

Let O be a fixed point on a straight line. Let P be the position of a particle at any time 't'.

Let $\quad OP = x,\ v = \frac{dx}{dt}$

$$\therefore \qquad \frac{d^2x}{dt^2} = -kx$$

Let $\frac{dx}{dt} = v$, $\frac{vdv}{dx} = -kx$

i.e. $$\frac{v^2}{2} = \frac{-kx^2}{2} + \frac{c}{2} \qquad \text{... (1)}$$

Let at $x = a,\ v = 0$

$$\Rightarrow \qquad 0 = -ka^2 + c \Rightarrow c = ka^2$$

Substituting equation (1) we get,

$$v^2 = k(a^2 - x^2)$$

$$\therefore \qquad v^2 = k(a^2 - x^2)$$

$$\therefore \qquad v = \pm\sqrt{k}\sqrt{a^2 - x^2}$$

i.e. $$\frac{dx}{dt} = -\sqrt{k}\sqrt{a^2 - x^2},$$

Negative sign is attached because as t increases x decreases.

$$\therefore \qquad \frac{dx}{\sqrt{a^2 - x^2}} = -\sqrt{k}\,dt$$

Integrating $\sin^{-1}\frac{x}{a} = -\sqrt{k}\,t + c$

Again at $t = 0,\ x = a$

$$\therefore \qquad c = \frac{\pi}{2}$$

i.e. $$\sin^{-1}\frac{x}{a} = -\sqrt{k}\,t + \frac{\pi}{2}$$

i.e. $$x = a\sin\left(-\sqrt{k}\,t + \frac{\pi}{2}\right) = a\cos\left(\sqrt{k}\,t\right)$$

Note :

i) Maximum displacement on either sides of '0' is a called amplitude = a.

ii) SHM is a periodic motion with period $\frac{2\pi}{\sqrt{k}}$.

iii) Frequency $= \frac{\sqrt{k}}{2\pi}$

iv) $\frac{d^2x}{dt^2} = -\omega^2 x$ represents a differential equation of simple harmonic motion.

v) We have maximum acceleration when velocity = 0.

Example 2.34 : *A particle executes SHM when it is 2 cm from mid path its velocity is 10 cm/sec, and when it is 6 cm from centre of its velocity is 2 cm/sec. Find the period and its greatest acceleration.* **(Dec.-2004)**

Solution : The differential equation for S.H.M. is

$$\frac{d^2x}{dt^2} = -\omega^2 x \quad \text{... (1)}$$

Equation (1) becomes,

$$v\frac{dv}{dx} = -\omega^2 x$$

$$vdv = -\omega^2 x\, dx$$

Integrating,

$$\frac{v^2}{2} = \frac{\omega^2 x^2}{2} + \frac{c}{2}$$

i.e. $$v^2 = -\omega^2 x^2 + c$$

We have at x = 2 cm, v = 10 cm/sec

$$\therefore \quad 100 = -\omega^2 4 + c \quad \text{... (2)}$$

and at $x = 6$ cm, $v = 2$ cm/sec

$$4 = -36\omega^2 + c \quad \text{... (3)}$$

Solving equation (2) and equation (3) we get

$$96 = 32\omega^2 \Rightarrow \omega^2 = \frac{96}{32} = 3$$

and $$c = 112$$

$$v^2 = -3x^2 + 112$$

At the maximum acceleration velocity v = 0

$$\therefore \quad 0 = -3x^2 + 112 \Rightarrow x^2 = \frac{112}{13} \Rightarrow x = \frac{\sqrt{112}}{\sqrt{3}}$$

$$\text{Maximum acceleration} = 3\frac{\sqrt{112}}{\sqrt{3}} = \sqrt{3}\sqrt{112} = \sqrt{336}$$

and $$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{3}}$$

Example 2.35 : *A point executing simple harmonic motion has velocity v_1 and v_2 and acceleration a_1 and a_2 in two position respectively. Show that the distance between the two position is $\left|\dfrac{v_1^2 - v_2^2}{a_1 + a_2}\right|$.* **(May-2003, May-2004, May-2005, May-2006)**

Solution : Differential equation of SHM is

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

We have

Position	Velocity	acceleration
$x = x_1$	$v = v_1$	$a = a_1$
$x = x_2$	$v = v_2$	$a = a_2$

$$\left.\begin{aligned} a_1 &= -\omega^2 x_1 \\ a_2 &= -\omega^2 x_2 \end{aligned}\right\} \quad \text{... (1)}$$

Adding we get, $$-\omega^2 (x_1 + x_2) = a_1 + a_2 \quad \text{... (2)}$$

Next let $$v = \frac{dx}{dt}$$

$$\therefore \quad v\frac{dx}{dt} = \frac{d^2x}{dt^2} = -\omega^2 x$$

i.e. $$vdv = -\omega^2 x\, dx$$

Integrating,

$$\therefore \quad \frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + \frac{c}{2}$$

$$\therefore \quad v^2 = -\omega^2 x^2 + c$$

$$\therefore \quad v_1^2 = -\omega^2 x_1^2 + c$$

$$v_2^2 = -\omega^2 x_2^2 + c$$

Subtracting,

$$v_1^2 - v_2^2 = -\omega^2 (x_1^2 - x_2^2)$$

$$= (a_1 + a_2)(x_1 - x_2) \quad \text{using equation (2)}$$

i.e. $$x_1 - x_2 = \left|\frac{v_1^2 - v_2^2}{a_1 + a_2}\right|$$

Example 2.36 : *The motion of a particle moving along a straight line is given by* $\frac{d^2x}{dt^2} + 16x = 0$ *with initial condtions* $x = 5$, $\frac{dx}{dt} = 0$ *when* $t = 0$. *Find the displacement at time t.*

Solution : $\frac{d^2x}{dt^2} + 16x = 0$

Put $\frac{dx}{dt} = v$

$$\therefore \quad v\frac{dv}{dx} + 16x = 0$$

$$vdv + 16x\,dx = 0$$

$$\frac{v^2}{2} + 16\frac{x^2}{2} = \frac{c}{2}$$

$$v^2 + 16x^2 = c$$

We have at $t = 0$, $\frac{dx}{dt} = v = 0$, $x = 5$

$$\therefore \quad c = 400.$$

Thus $v^2 + 16x^2 = 400$

i.e. $\frac{dx}{dt} = 4\sqrt{25 - x^2} \Rightarrow \frac{dx}{\sqrt{25 - x^2}} = 4dt$ (V.S. form)

Integrating

$$\sin^{-1}\left(\frac{x}{5}\right) = 4t + c_1$$

i.e. $x = 5\sin(4t + c_1)$

We have at $t = 0$, $c = 5 \Rightarrow c_1 = \frac{\pi}{2}$

$$x = 5\sin\left(4t + \frac{\pi}{2}\right) = 5\cos 4t$$

Example 2.37 : *A spring of neglegible weight hangs vertically. A mass m is attached to the other end. If the mass is moving with velocity* v_0 *when the spring is unstretched, find the velocity v as a function of the stretch x. (Take* λ *as Young's modulus of the spring).*

Solution : Step 1 :

Equation of motion of spring

$$m\frac{d^2x}{dt^2} = mg - T$$

The tension of an elastic string or spring is proportional to the extension of the string beyond its natural length. If x is the increse in length of the spring then tension is λ x if velocity of the mass m is v; then the equation of motion is,

$$mv\frac{dv}{dx} = mg - \lambda x$$

Step 2 :

$$mv\, dv = (mg - \lambda x)\, dx$$

This is V.S. form hence integrating

$$\int mv\, dv = \int (mg - \lambda x)\, dx + c$$

$$m\frac{v^2}{2} = mgx - \lambda\frac{x^2}{2} + c$$

Step 3 :

To find the value of c given that at $x = 0$, $v = v_0$

$$\therefore \quad \frac{mv_0^2}{2} = c$$

$$\therefore \quad mv^2 = 2mgx - \lambda x^2 + mv_0^2$$

Example 2.38 : *A particle is oscillating in a straight line about a centre of force towards which when at a distance x, the force is mn^2x and a is the amplitude of the oscillation. When at a distance $\frac{a\sqrt{3}}{2}$ from O, the particle receives a blow in the direction of motion which generates a velocity $\frac{3\,na}{2}$. If this velocity be away from O, show that the new amplitude is $a\sqrt{3}$.*

Solution : Step 1 :

Let the particle at time t be at a distance x from O. Then by D' Alembert's rule,

$$m\ddot{x} = -mn^2x$$

$$\therefore \quad \ddot{x} = -n^2x$$

$$\therefore \quad v\frac{dv}{dx} = -n^2x \qquad \left(\because \ddot{x} = \frac{d^2x}{dt^2} = v\frac{dv}{dx}\right)$$

Step 2 :

$$\therefore \quad v\,dv = -n^2 x dx$$

Integrating,

$$\frac{v^2}{2} = -n^2 \frac{x^2}{2} + A \qquad \text{A is constant}$$

Step 3 :

To find A, let $x = \frac{a\sqrt{3}}{2},\ v = \frac{3na}{2}$

$$\therefore \quad A = \frac{3}{2} n^2 a^2$$

$$\therefore \quad \frac{v^2}{2} = \frac{n^2x^2}{2} + \frac{3n^2a^2}{2}$$

$$\therefore \quad v^2 = n^2\left(3a^2 - x^2\right)$$

Now to find new amplitude, we put v = 0.

$$\therefore \quad x = \sqrt{3}\,a \text{ is new amplitude.}$$

Example 2.39 : ***A particle rests in equilibrium under the attraction of two centres of force which attracts directly as the distance, their attractions per unit of mass at unit of distance being μ and μ' the particle is slightly displaced towards one of them; show that the time of a small oscillation is $\frac{2\pi}{\sqrt{\mu + \mu'}}$.***

Solution : Step 1 :

Let O to O′ be the centres of attraction and A be the position of equibrium of the particle.

Let $OA = d$ and $O'A = d'$

The forces of attraction are μd and $\mu' d'$ at A they are equal to balance.

$$\therefore \quad \mu d = \mu' d' \qquad \text{... (1)}$$

Let the particle be displaced through distance x.

So that AP = x

Hence, the equation of motion is

$$\ddot{x} = \text{Acceleration in the positive direction.}$$

$$= \mu' PO' - \mu PO = \mu(d' - x) - \mu(d + x)$$

$$= (\mu + \mu')\,x \qquad \text{... by (1)}$$

Step 2 :

Thus the motion of P is S.H.M. and the period of the oscillation is $\frac{2\pi}{\sqrt{\mu + \mu'}}$.

Exercise 2.5 :

1) *A particle executes SHM of amplitude 4 cm if its acceleration at a distance of 1 cm from mean position in 3 cm/sec^2. Find its velocity at a distance of 2 cm from the mean position.*
[Ans. : 6 cm/sec]

2) *The velocity of a particle moving in a straight line with SHM at a distance x from a fixed point is given by $v^2 = \pi^2(4 - x^2)$. Find the periodic time, the greatest acceleration. Find also the least time taken by the particle to move from $x_1 = \sqrt{2}$ to $x_1 = 1$* **[Ans. : a = 2, $\omega = \pi$, T = 2]**

3) *A body moves in SHM accelerating to the law $\frac{d^2x}{dt^2} = -8x$, if $\frac{dx}{dt} = 0$ at $x = 5$, find the period and amplitude.* **[Ans. : $\left[\frac{\pi}{\sqrt{2}}, \frac{5}{\sqrt{3}}\right]$]**

4) *A particle is oscillating in a straight line about a centre of force 'O', towards which when at a distance x the force is mn^2x and a is the amplitude of oscillation. When at a distance $\frac{a\sqrt{3}}{2}$ from O, a particle receives a blow in direction of motion which genearates the velocity na. If the velocity is away from O, show that new amplitude is $a\sqrt{3}$.*

Hint : $v\frac{dv}{dx} = -n^2x$ at $x = \frac{a\sqrt{3}}{2}$, $v = \frac{3na}{2}$

We get new amplitude by putting velocity = 0.

2.8 Mass-Spring System

Hooke's law :

Force is proportional to displacement.

Let s be the extended length of the spring (may be stretched or compressed) then the magnitude of the force F exerted on the spring is

F = ks where k is constant called as Modulus of the springs. Dimension of k is given by $k = \frac{\lambda}{l}$ = restoring force per unit length (where l – natural length of spring)

Refer Fig. 2.13 on next page.

In equilibrium position the weight mg balances $\frac{\lambda}{l}e$

$$\therefore \qquad mg = \frac{\lambda}{l}e \qquad \ldots (1)$$

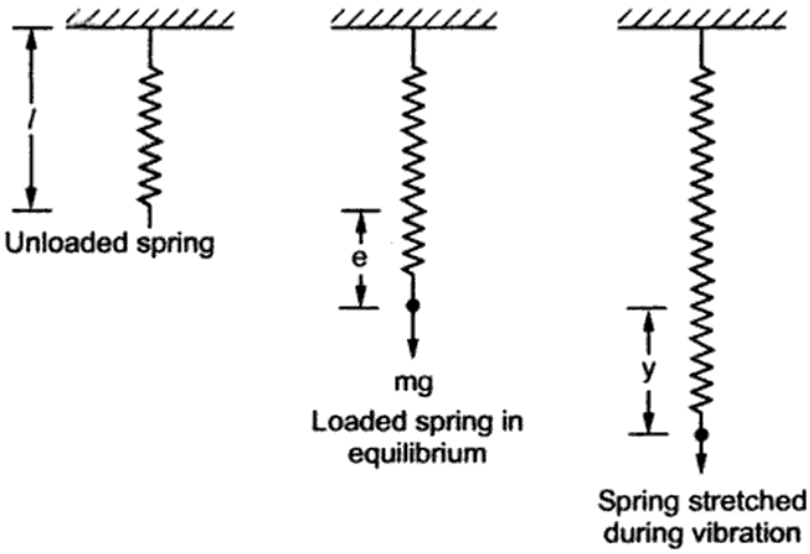


Fig. 2.13

Next the weight is set to motion in such a way that it moves in a purely vertical direction. Let y be the position of mass below equilibrium at any time 't'

$$\therefore \quad m\frac{d^2y}{dt^2} = mg - \frac{\lambda}{l}(y + e)$$

$$= -\frac{\lambda}{l}y \qquad \text{using equation (1)}$$

$$\therefore \quad \frac{d^2y}{dt^2} = -\frac{\lambda}{lm}y = -\omega^2 y$$

Where $\omega^2 = \dfrac{\lambda}{lm} = \dfrac{g}{e}$ using (1)

i.e. $$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

This is a differential equation of SHM.

Its complete solution is

$$y = A\cos(\omega t + \alpha) \qquad \text{... (3)}$$

$\therefore$ Equation (3) describes a periodic motion with period.

$$= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{e}{g}} = T$$

and its frequency $= \dfrac{1}{T} = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{e}}$

Note : A spring which obey's Hooke's law are called linear spring.

Example 2.40 : *A spring is loaded with a mass of 2 kg. It produces a static deflection $\frac{1}{4}$ m. A mass of 2 kg is suddenly added to the original mass show that the maximum deflection produced is 0.75 m.*

Solution : Let k = restoring force per unit length. In equilibrium position, mg = ke

Where e – changes in length in equilibrium position.

$$2g = \frac{1}{4}k \Rightarrow k = 8g \qquad \text{... (1)}$$

Let x be the position of mass at any time t when a weight of 4 kg is suddenly added to the original mass.

$\therefore$ By Newton's law,

$$m\frac{d^2x}{dt^2} = mg - k\left(x + \frac{1}{4}\right)$$

$$4v\frac{dv}{dx} = 4g - 8gx - 2g \qquad \text{... by using (1)}$$

$$= 2g - 8gx$$

$$\therefore \qquad 4vdv = 4\left(\frac{1}{2} - 2gx\right) dx$$

Integrating, we get,

$$\frac{v^2}{2} = \left(\frac{1}{2}gx - gx^2\right) + c$$

Now at $x = 0, \; v = 0 \Rightarrow c = 0$

$$\frac{v^2}{2} = \left(\frac{1}{2}gx - gx^2\right)$$

i.e. $$v^2 = (gx - 2gx^2)$$

The elongation will be maximum when v = 0

$$gx - 2gx^2 = 0 \Rightarrow x = 0 \text{ or } x = \frac{1}{2}$$

Hence maximum elongation produced = 0.5 + 0.25

= 0.75 m

[Where 0.25 is elongated length in equilibrium position]

Example 2.41 : *A mass hangs from a fixed point by means of a tight elastic spring given a small vertical dispacement if n is the number of oscillations per second in ensuing SHM and L is the length of a spring when the system is in equilibrium show that natural length of spring is* $L - \frac{g}{4\pi^2 n^2}$. **(Dec.-2000)**

Solution : Let l_0 be the natural length.

$\therefore L - l_0$ is the exrended length in equilibrium position when a mass m is attached.

$$\therefore \qquad mg = k(L - l_0) \qquad \text{... (1)}$$

where 'x' be the position of a particle at any time 't'

$\therefore$ Equation of motion is

$$m\frac{d^2x}{dt^2} = mg - k[L - l_0 + x]$$

$$= -kx \qquad \text{[using equation (1)]}$$

$$\text{i.e.} \qquad \frac{d^2x}{dt^2} = \frac{g}{L - l_0}\,x$$

$$= -\omega^2 x \qquad \text{where } \omega^2 = \frac{g}{L - l}$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

This is a differential equation of SHM.

$$\text{Period} = T = \frac{2\pi}{\omega}$$

$$\text{We have } n = \frac{1}{T} = \text{number of oscillation's per second.}$$

$$n^2 = \frac{\omega^2}{4\pi^2}$$

$$\text{i.e.} \qquad 4n^2\pi^2 = \omega^2 = \frac{g}{L - l_0}$$

$$\text{i.e.} \qquad L - l_0 = \frac{g}{4\pi^2 n^2}$$

$$\text{i.e.} \qquad l_0 = L - \frac{g}{4\pi^2 n^2}$$

Example 2.42 : *In the case of stretched elastic spring which has one end fixed and a particle of mass m at the other end, equation of motion is* $m\dfrac{d^2x}{dt^2} = -\dfrac{mg}{e}(x-1)$ *where l is the natural length of the spring and e is the elongation due to weight mg. Find x and v under the condition that at t = 0; x =* x_0 *and v = 0.*

Solution : We have,

$$\frac{d^2x}{dt^2} = \frac{-g}{e}(x-l)$$

Let $$\frac{dx}{dt} = v$$

$$v\frac{dv}{dx} = -\frac{g}{e}(x-l)$$

$$vdv = -g\,(x-l)\,dx$$

Integrating,

$$\frac{v^2}{2} = \frac{-g}{e}\frac{(x-l)^2}{2} + \frac{c}{2}$$

We have at $x = x_0,\ v = 0$

$\Rightarrow$ $$0 = c \Rightarrow c = \frac{g}{e}(x_0 - l)^2$$

$$v^2 = \frac{g}{e}[(x_0 - l)^2 - (x-l)^2]$$

$$\frac{dx}{dt} = v = \sqrt{\frac{g}{e}}\sqrt{(x_0-l)^2 - (x-l)^2}$$

Separating variables

$$\frac{dx}{\sqrt{(x_0-l)^2 - (x-l)^2}} = \sqrt{\frac{g}{e}}\,dt$$

i.e. $$\sin^{-1}\left(\frac{x-l}{x_0-l}\right) = \sqrt{\frac{g}{e}}\,t + c_1$$

Again at $t = 0,\ x = x_0$

$\therefore$ $$c_1 = \frac{\pi}{2}$$

$\therefore$ $$\sin^{-1}\left(\frac{x-l}{x_0-l}\right) = \sqrt{\frac{g}{e}}\,t + \frac{\pi}{2}$$

i.e.
$$\frac{x - l}{x_0 - l} = \sin\left(\sqrt{\frac{g}{e}}\, t + \frac{\pi}{2}\right) = \cos\left(\sqrt{\frac{g}{e}}\, t\right)$$

$$\lambda = l + (x_0 - l)\cos\left(\sqrt{\frac{g}{e}}\, t\right)$$

Example 2.43 : *Two idential loads are suspended from the end of a spring. Find the motion imparted to one end if the other breakes loose in case the increase in length of spring under action of one load at rest is a.*

Solution : Let l be the natural length of spring

Let
$$\frac{\lambda}{l} = \text{restoring force per unit length.}$$

We have In equilibrium (static) the elongation 'a' due to weight mg.

$$\because \quad mg = \frac{\lambda}{l} a \Rightarrow \lambda = \frac{l\,mg}{a} \qquad \dots (1)$$

Let x be the position of mass of any time 't' after one of the load breaks loose.

Equation of motion is

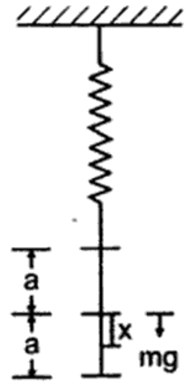


Fig. 2.14

$$m\frac{d^2x}{dt^2} = mg - \frac{\lambda}{l}(a + x)$$

$$= -\frac{\lambda}{l}x = -\frac{mg}{a}x \qquad \text{by equation (1)}$$

i.e.
$$\frac{d^2x}{dt^2} = -\frac{g}{e}x$$

Let
$$\frac{dx}{dt} = v$$

$$\therefore \quad v\frac{dv}{dx} = \frac{-g}{a}xdx$$

Integrating,

$$\frac{v^2}{2} = \frac{-g}{a}\frac{x^2}{2} + \frac{c}{2}$$

We have at $x = a, \; v = 0$

$\therefore \quad c = ag$

$$v^2 = \frac{-g}{a}x^2 + ag$$

$$= \frac{g}{a}\left(a^2 - x^2\right)$$

$$\therefore \quad v = \frac{dx}{dt} = \sqrt{\frac{g}{a}}\sqrt{a^2 - x^2}$$

i.e.

$$\frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\frac{g}{a}}\, dt$$

Integrating,

$$\sin^{-1}\frac{x}{a} = \sqrt{\frac{g}{a}}\, t + c_1$$

Again at, $t = 0, \; x = a$

$$\therefore \quad \sin^{-1}(1) = c_1 = \frac{\pi}{2}$$

$$\therefore \quad \sin^{-1}\frac{x}{a} = \sqrt{\frac{g}{a}}\, t + \frac{\pi}{2}$$

$$\therefore \quad x = a\sin\left(\sqrt{\frac{g}{a}}\, t + \frac{\pi}{2}\right)$$

$$= a\cos\left(\sqrt{\frac{g}{a}}\, t\right)$$

Example 2.44 : *A particle of mass m is attached to one end of a light elastic string of natural length a and modulus $\frac{mg}{k}$. The other end of the string is fixed to a point O and the particle is allowed to fall from rest O. Obtain velocity of a particle and show that its highest magnitude is $\sqrt{ag\,(2+k)}$.* **(Dec.-2003)**

Solution :

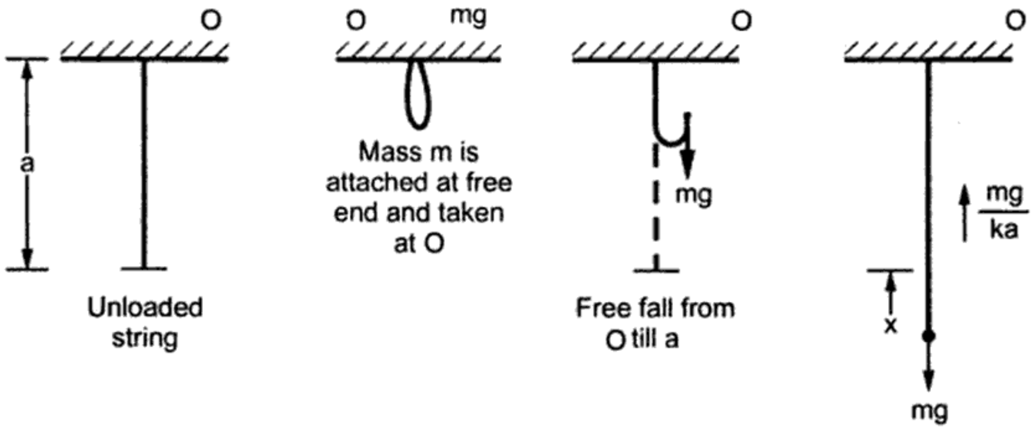


Fig. 2.15

First we set an initial conditions the particle is allowed to fall from rest at 'a'.

Velocity at this point = $\sqrt{2ga}$

[Using the standard equation of physics $u^2 = v^2 + 2gs$ here initial velocity $v = 0$ at $s = O$]

$\therefore$ The final velocity $u = \sqrt{2ga}$

Let 'x' be the position of mass below 'a' at any time. Equation of motion is

$$m\frac{d^2x}{dt^2} = mg - \frac{mg}{ka}x$$

Let $$v = \frac{dx}{dt}$$

$$v\frac{dv}{dx} = g - \frac{g}{ka}x$$

Integrating,

$$\frac{v^2}{2} = gx - \frac{g}{ka}\frac{x^2}{2} + c \qquad \dots (1)$$

Again at $x = 0,\ v = \sqrt{2ga}$

Substituting in equation (1) we get

$$c = ga$$

$$\frac{v^2}{2} = gx - \frac{g}{ka}\frac{x^2}{2} + ga$$

$$v^2 = 2gx - \frac{g}{ka}x^2 + 2ga = v(s)$$

For max/min value of v, $\frac{dv}{dx} = 0$

$$\frac{dv}{dx} = 2g - \frac{2g}{ka}x = 0$$

$$\Rightarrow \quad x = ka$$

$$\frac{d^2v}{dx^2} = \frac{-2g}{ka}x < 0$$

$\therefore$ v is max when $x = ka$

$$\therefore \quad v_{max}^2 = 2gka - \frac{g}{ka}k^2a^2 + 2ga$$

$$= gka + 2ag$$

$$= ag(2 + k)$$

$$\therefore \quad v_{max} = \sqrt{ag(2+k)}$$

Exercise 2.6 :

1) *An elastic string without weight of natural length 'l' and modulus of elasticity being weight of n-grams is suspended by one end, and a mass m is attached to the other, show that the time of oscillation is* $2\pi\sqrt{\frac{ml}{ng}}$

Hint : *Here* $\frac{\lambda}{l} = \frac{ng}{l}$ *Equation of motion is* $m\frac{d^2x}{dt^2} = \frac{-ng}{l}x$

$\frac{d^2x}{dt^2} = \frac{-ng}{lm}x = \omega^2 x$ *This is a equation of SHM*

$\therefore$ *Period* $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{lm}{ng}}$

2) *An elastic spring of natural length l is fixed at a point A at the other end a particle of mass m is attached so that spring elongates by a length 21. If the particle is dropped from A. Show that it descends ds a distance* $l(2+\sqrt{3})$ *before coming to rest.* **(Dec.-2005)**

Hint : *Let* $\frac{\lambda}{l}$ *be the restoring force per unit length.*

$\therefore \frac{\lambda}{l}2l = mg \quad x = \frac{mg}{2}$ *equation of motion* $m\frac{d^2x}{dt^2} = mg - \frac{\lambda}{l}(l + x)$

i.e. $\frac{d^2x}{dt^2} = mg - \frac{mg}{2l}x$ *Integrate and use the initial condition that at* $x = 0$.

$v = \sqrt{2gl}$ *that at* $v = 0$ *we get descent.*

3) *A spring of negligible weight hangs vertically. A mass m is attached to the other end. If the mass is moving with velocity* v_0 *when the string is unstretched find the velocity as a function of x (take* λ *as Young's modulus of the spring).*

Hint : $mv\dfrac{dv}{dx} = mg - \lambda x$ *(No equilibrium is set)*

$$v = \sqrt{2gx - \frac{\lambda}{m}x^2 + v_0^2}$$

4) *Let 'a' be the natural length of a spring which is fixed at one end and at the other end two masses M and M' are attached. M' cut loose, show that the distance of M from the upper end of the spring at time 't' is* $a + b + c\cos\left(\sqrt{\frac{g}{b}}t\right)$. *Where b and c are elongated length of a spring in static equilibrium when masses M M' are attached to the spring.*

5) *A weight of 7 kg is suspended from a spring of modulus* $\frac{36}{35}$ *kg/cm. at t = 0. While the weight is hanging in static equilibrium. It is suddenly given an initial velocity 48 cm/s in the downward direction taking the acceleration due to gravity 980* cm/sec^2.

   i) *Find the verticle displacement as a function of time t.*

   ii) *What is the period and frequency of subsequent motion?* **[Ans. :** $T = \frac{\pi}{6}$ **sec;** $\eta = \frac{6}{\pi}$ **Hz]**

   iii) *Through what amplitude does the weight moves?* **[Ans. : v = 12 B cos 12 t, y = – 4 sin 12 t]**

   iv) *At what time does the weights reach its extreme displacement above and below its equilibrium position?* **[Ans. : The extreme occurs when** $t = \frac{\pi}{24} + \frac{n\pi}{12}$, **n = 0, 2,...]**

6) *Find the modulus of a spring for which a spring mass system in which w = 5 kg will have a circular frequency of 14 rad/s.*

   Hint : $\dfrac{d^2x}{dt^2} + \omega^2 x = 0$ *we have* $\omega = 14$

   $$14^2 = \frac{kg}{5} \quad k = \frac{14^2 \times 5}{g} = \frac{980}{9800} = 1 \text{ kg/cm}$$

7) *Find the displacement ds a function of time for*

   i) *W = 15 g, k = 3 g/cm,* $y_0 = -1$ *cm,* $\eta_0 = 7$ *cm/s* **[Ans. : y = – cos 14t +** $\frac{1}{2}$ **sin 14t]**

   ii) *W = 96 l b, k = 16 lb/m,* $y_0 = 10$ *in* $v_0 = 0$ **[Ans. : y = 10 cos 8t]**

   Note : *In FPs stystem g = 32* ft/s^2.

8) *A mass* m_1 *is attached to a spring and allowed to vibrate with undamped motion having period 'P'. At some later time, a second mass* m_2 *instantaneously fused with ml. Prove that the new object having the mass* $m_1 + m_2$, *exists simple harmonic motion with period* $\dfrac{P}{\sqrt{1 + m_2 m_1}}$

9) *When a 8 pound weight is suspended from a spring, it stretches a spring 2 inches. Determine the equation of motion when an object with a mass of 7 kg is suspended from the spring and the system is set to motion by striking the object an upward blow.*

   1 kg = 2.2046 lb

   1 min = 2.541 cms

2.9 Heat Flow

The basic principles involved in conduction of heat are :

a) Heat flows from higher temperature to lower temperature (i.e. hot to cold)

b) The quantity of heat in a body is proportional to mass, temperature and specific heat.

c) Fourier's law of heat conduction :

The Fourier's law of heat conduction states that the heat flux q (calories per second) through an area element A (cm^2) is proportional temperature gradient normal to the area.

$$q = \text{Thermal conductivity} \times \text{Area} \times \text{Temperature gradient.}$$

Use same notations

i.e.
$$q = -kA\frac{\partial T}{\partial x} \qquad ...(1)$$

Or
$$q = -kA\frac{dT}{dx} \text{ (one dimensional heat flow)}$$

Where k is the proportionality constant which is called co-efficient of thermal conductivity depends on material of body.

Note : The negative sign is attached in (1) because if q is positive (in the positive x-direction) then T (a) > T(b), heat flows from hot to cold and $\frac{dT}{dx}$ is negative from Fig. 2.16.

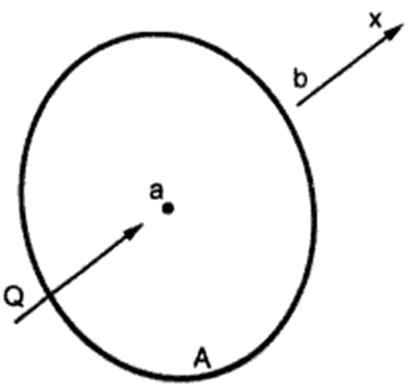


Fig. 2.16

Example 2.45 : ***A steam pipe 20 cm in diameter is protected with a covering 6 cm thick for which the co-efficient of thermal conductivity is k = 0.0003 cal/cm in steady state. Find the heat lost per hour through a meter length of the pipe, if the inner surface of the pipe is at 200 °C and the outer surface of the covering is at 30 °C.*** **(May-2006)**

Solution : By the Fourier's law of heat conduction :

$$q = -kA\frac{dT}{dx}$$

Or $$q = -k(2\pi x)\frac{dT}{dx}$$

Or $$q\frac{dx}{x} = -2\pi k\, dT \qquad ...(1)$$

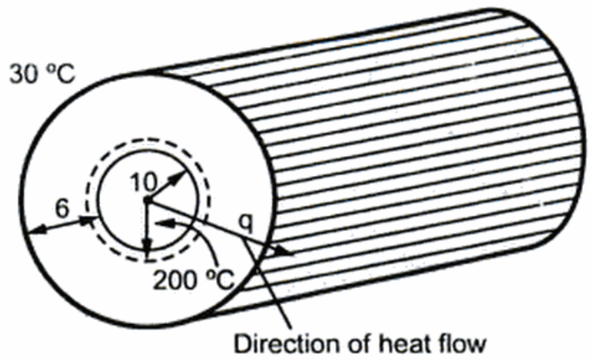


Fig. 2.17

Given $x = 16$ cm, when $T = 30$ °C

and $x = 10$ cm, when $T = 200$ °C

From equation (1)

$$\frac{q}{2\pi k}\int_{10}^{16}\frac{dx}{x} = -\int_{200}^{30} dT$$

Or $$\frac{q}{2\pi k}\log\left(\frac{16}{10}\right) = 200 - 30 = 170$$

$$\Rightarrow \qquad q = \frac{170 \times (2\pi k)}{\log(1.6)} \text{ cal/sec}$$

$$\text{Required heat loss} = \frac{340 \times (3.14) \times 0.0003}{\log_e 1.6} \times 100 \times 60 \times 60$$

$$= 245443.386/\text{cal.}$$

Example 2.46 : *A pipe 20 cm is diameter contains steam at 150 °C and is protected with a covering 5 cm thick for which k = 0.0025. If the temperature of the outer surface of the covering is 40 °C. Find the temperature half-way through the covering under steady state conditions.*

Solution : Let q cal/sec be quantity of heat. Heat flowing across the surface area $A = 2\pi x$ sq.cm. By Fourier's law of heat conduction, we have

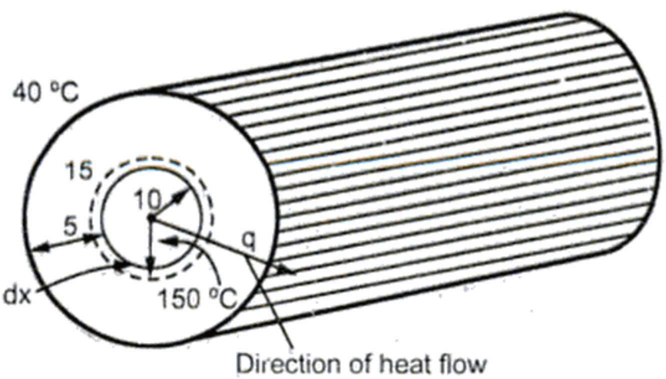


Fig. 2.18

$$q = -kA\frac{dT}{dx}$$

$$= -k(2\pi x)\frac{dT}{dx}$$

$$\Rightarrow \qquad dT = -\frac{q}{2\pi k}\frac{dx}{x}$$

On integrating we get,

$$T = -\frac{q}{2\pi k}\log_e x + c \qquad ...(1)$$

Given $T = 150\,°C$, When $x = 10$ cm

Also $T = 40\,°C$, When $x = (10 + 5) = 15$ cm

$\therefore$ From equation (1),

$$150 = -\frac{q}{2\pi k}\log_e 10 + c \qquad ...(2)$$

and

$$40 = -\frac{q}{2\pi k}\log_e 15 + c \qquad ...(3)$$

Subtracting equation (3) from equation (2), we have

$$110 = -\frac{q}{2\pi k}\left(\log_e 10 - \log_e 15\right)$$

$$= -\frac{q}{2\pi k}\log_e 1.5 \qquad ...(4)$$

Now, let $t = T$, When $x = 12.5$ cm

Equation (1) becomes,

$$t = -\frac{q}{2\pi k}\log_e 12.5 + c \qquad ...(5)$$

Subtracting equation (2) from equation (5), we get

$$t - 150 = -\frac{q}{2\pi k}\left(\log_e 12.5 - \log_e 10\right)$$

$$= -\frac{q}{2\pi k}\log_e 1.25 \qquad ...(6)$$

Dividing equation (6) by equation (4), we have,

$$\frac{t - 150}{110} = -\frac{\log_e 1.25}{\log_e 1.5}$$

Or $\quad t = 150 - (110 \times 0.55)$

$$= 150 - 60.5 = 89.5$$

$\therefore$ When $\quad x = 12.5$ cm, $T = 89.5$ °C

Example 2.47 : *Obtain a formula for the steady state heat loss per unit time from a unit length of pipe of radius r_0 carrying steam of temperature T_0 if the pipe is covered with insulation of thickness W. The outer surface of which remains at constant temperature T_1. What is the temperature distribution through the insulation ?* **(Dec.-2003)**

Solution : By Fourier law of coduction of heat

$$q = -kA\frac{dT}{dx} \qquad \text{... (1)}$$

Consider a circular layer of insulation of radius 'r'.

Let T be the temperature distribution on this layers.

Surface area $\quad A = (2\pi r)1 = 2\pi r$

x = r direction normal to area

k = constant (Thermal conductivity)

q = constant

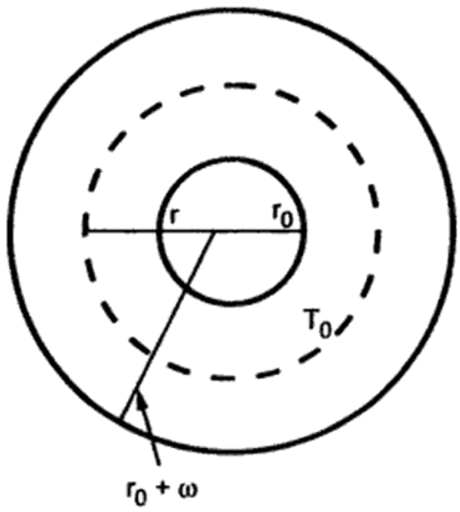


Fig. 2.19

Equation (1) becomes,

$$q = -k(2\pi r)\frac{dT}{dr}$$

Separating variables

$$q\,\frac{dr}{r} = -2\pi\, k dT \qquad \text{... (2)}$$

We have as r changes from r_0 to $r_0 + \omega$, T changes from T_0 to T_1.

$$q \int_{r_0}^{r_0+\omega} \frac{dr}{r} = -2\pi k \int_{T_0}^{T_1} dT$$

i.e. $$q\left[\log r\right]_{r_0}^{r_0+\omega} = -2\pi k\left[T\right]_{T_0}^{T_1}$$

i.e. $$q\,\log\left(\frac{r_0+\omega}{r_0}\right) = -2\pi k\left[T_1 - T_0\right] \qquad \text{... (3)}$$

Equation (1) gives us value of q.

To find the temperature distribution T. Integrate equation (1) with limits as r - changes from r_0 to r. T - changes from T_0 to T (say).

$$q\int_{r_0}^{r} \frac{dr}{r} = -2\pi k \int_{T_0}^{T} dT$$

i.e. $$q\log\frac{r}{r_0} = -2\pi k(T - T_0) \qquad \text{... (4)}$$

Dividing equation (4) by equation (3), we get

$$\frac{\log\left(\frac{r}{r_0}\right)}{\log\left(\frac{r_0+\omega}{r_0}\right)} = \left(\frac{T - T_0}{T_1 - T_0}\right)$$

$$\Rightarrow \qquad T - T_0 = \frac{(T_1 - T_0)\log\left(\frac{r}{r_0}\right)}{\log\left(\frac{r_0+\omega}{r_0}\right)}$$

Or $$T = T_0 + \frac{(T_1 - T_0)\log\left(\frac{r}{r_0}\right)}{\log\left(\frac{r_0+\omega}{r_0}\right)}$$

Example 2.48 : ***A long hollow pipe has an inner diameter of 10 cm and outer diameter of 20 cm the inner surface is kept at 200 °C and outer surface at 50°C. The thermal conductivity is 0.12. How much heat is lost per minute from a portion of the pipe 20 m long?*** **(Dec.-2005)**

Solution : Consider a layer of insulation of radius 'r' ($5 \leq r \leq 10$)

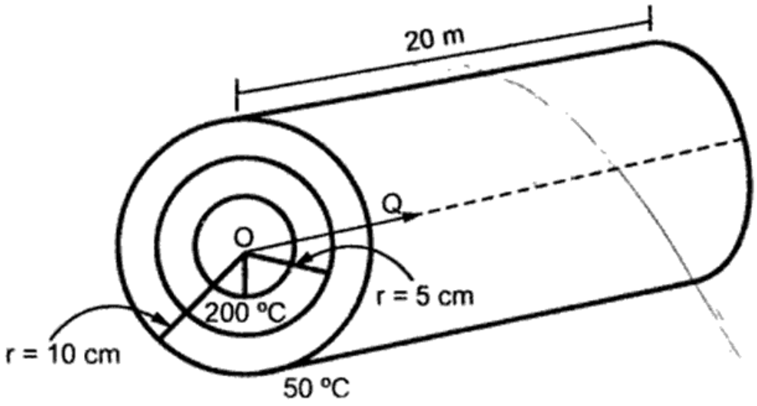


Fig. 2.20

Here the isothermal surfaces are cylinders, consider one such cylinder of radius x cm and length 1 cm. The surface area of this cylinder is $A = 2\pi x$ sq.cm.

Let Q cal./sec. be the quantity of heat flowing across this area.

Then $$Q = -kA\frac{dT}{dx}$$

$$= -k\,2\pi x\frac{dT}{dx}$$

Or $$dT = -\frac{Q}{2\pi k}\frac{dx}{x}$$

On integrating, we get,

$$T = -\frac{Q}{2\pi k}\log_e x + c \qquad \text{... (1)}$$

Given $T = 200$ when $x = 5$

$$\therefore \quad 200 = -\frac{Q}{2\pi k}\log_e 5 + c \qquad \text{... (2)}$$

Also $T = 50$, when $x = 10$

$$\therefore \quad 50 = -\frac{Q}{2\pi k}\log_e 10 + c \qquad \text{... (3)}$$

From equation (2) and equation (3)

$$150 = \frac{Q}{2\pi k}\left(\log_e 10 - \log_e 5\right)$$

Or $$150 = \frac{Q}{2\pi k}\log_e 2$$

$$\therefore \quad Q = \frac{2\pi k \times 150}{\log_e 2} = \frac{300\pi \times 0.12}{\log_e 2}$$

$$= 163 \text{ cal/sec.}$$

Now, heat lost per minute through 20 metre length of pipe

$$= 60 \times 2000 \times Q$$

$$= 120000 \times 163 = 1956000 \text{ cal.}$$

Example 2.49 : *The inner and outer surfaces of a spherical shell are maintained at T_0 and T_1 temperatures respectively. If the inner and outer radii of the shell are r_0 and r_1 respectively and the thermal conductivity of the shell is k, find the amount of heat lost from the shell per unit time. Find also the temperature distribution through the shell.*

Solution : Consider a spherical shell of radius r such that $r_0 < r < r_1$

Let the temperature distribution on this shell be T. Then conduction through this shell is.

$$q = -kA\frac{dT}{dr}$$

Or
$$q = -k(4\pi r^2)\left(\frac{dT}{dr}\right) \quad \ldots (1)$$

i.e.
$$q\int_{r_0}^{r_1}\frac{dr}{r^2} = -4\pi k\int_{T_0}^{T_1} dT$$

i.e.
$$\left(\frac{-q}{r}\right)_{r_0}^{r_1} = -4\pi k[T]_{T_0}^{T_1}$$

$$-q\left(\frac{1}{r_1}-\frac{1}{r_0}\right) = -4\pi k(T_1 - T_0)$$

$$\Rightarrow \quad q = 4\pi k r_0 r_1\frac{(T_1 - T_0)}{(r_0 - r_1)} \quad \ldots (2)$$

Next integrating equation (1), we get,

$$q\int_{r_0}^{r}\frac{dr}{r^2} = -4\pi k\int_{T_0}^{T} dT$$

$$-q\left(\frac{1}{r}-\frac{1}{r_0}\right) = -4\pi k(T - T_0)$$

i.e.
$$q = 4\pi k\, r_0 r\frac{(T - T_0)}{r_0 - r} \quad \ldots (3)$$

From equation (2) and (3), we get,

$$r_0 r_1\left(\frac{T_1 - T_0}{r_0 - r_1}\right) = r_0 r\frac{(T - T_0)}{(r_0 - r)}$$

i.e.
$$\frac{(T - T_0)}{(r_0 - r)} = \frac{r_1}{r}\,\frac{T_1 - T_0}{r_0 - r_1} \quad \Rightarrow \quad T = T_0 + \frac{r_1}{r}\,\frac{(T_1 - T_0)(r_0 - r)}{r_0 - r_1}$$

Example 2.50 : *For steady heat flow through the wall of a hollow sphere of inner and outer radii r_1 and r_2 respectively, the temperature u at a distance r $(r_1 < r < r_2)$ from the centre of the sphere is given by* $r\dfrac{d^2u}{dr^2} + 2\dfrac{du}{dr} = 0$

If u_1 and u_2 are the temperatures at inner and outer surfaces of the sphere respectively. Find u in terms of r. **(Dec.-2001)**

Solution : Put $\dfrac{du}{dr} = v$ in $r\dfrac{d^2u}{dr^2}+2\dfrac{du}{dr} = 0$

$$\frac{d^2u}{dr^2} = \frac{dv}{dr}$$

$\therefore$ Equation (1) becomes,

$$r\frac{dv}{dr} + 2v = 0$$

$$\frac{dv}{v} + 2\frac{dr}{r} = 0 \quad \text{(V.S. form)}$$

Integrating, we get,

$$\log v + 2\log r = \log c_1$$

i.e. $$vr^2 = c_1$$

i.e. $$v = \frac{du}{dr} = \frac{c_1}{r^2}$$

i.e. $$du = \frac{c_1}{r^2}dr \quad \text{(V.S. form)}$$

Integrating, we get,

$$u = -\frac{c_1}{r} + c_2 \quad \text{... (1)}$$

Now given $r = r_1$, at $u = u_1$

and $r = r_2$, at $u = u_2$

From equatoin (1) $$u_1 = \frac{-c_1}{r_1} + c_2 \quad \text{... (2)}$$

$$u_2 = \frac{-c_1}{r_2} + c_2 \quad \text{... (3)}$$

Solving equatoin (2) and equatoin (3) for c_1 and c_2 we get,

$$c_1 = \frac{r_1 r_2 (u_2 - u_1)}{r_2 - r_1} \quad \text{... (4)}$$

$$c_2 = \frac{u_2 r_2 - u_1 r_1}{r_2 - r_1} \quad \text{... (5)}$$

Substituting the values of c_1 and c_2 in equatoin (1) we get,

$$u = \frac{r_1 r_2}{r} \frac{(u_1 - u_2)}{r_2 - r_1} + \frac{u_2 r_2 - u_1 r_1}{r_2 - r_1}$$

$$= \frac{1}{r_2 - r_1}\left[(u_2 r_2 - u_1 r_1) - \frac{r_1 r_2 (u_2 - u_1)}{r}\right]$$

➠ **Example 2.51 :** *In one dimensional steady state heat conduction for a hollow cylinder with constant thermal conductivity k in the region $r_1 < r < r_2$. The temperature at a distance r is given by $\frac{d}{dr}\left[r\frac{dT}{dr}\right] = 0$ with $T = T_1$ at $r = r_1$ and $T = T_2$ at $r = r_2$. Determine steady state temperature distribution in the cylinder in terms of r.*

Solution : We have $$\frac{d}{dr}\left[r\frac{dT}{dr}\right] = 0$$

Integrating, we get,

$$r\frac{dT}{dr} = c_1$$

i.e. $$dT = c_1 \frac{dr}{r} \qquad \text{(V.S. form)}$$

Integrating, we get,

$$T = c_1 \log r + c_2 \quad \text{... (1)}$$

Now given at $r = r_1,\ T = T_1$

and at $r = r_2,\ T = T_2$

From equatoin (1), $$T_1 = c_1 \log r_1 + c_2 \quad \text{... (2)}$$

$$T_2 = c_1 \log r_2 + c_2 \quad \text{... (3)}$$

Solving equatoin (2) and equatoin (3) for c_1 and c_2 we get,

$$c_1 = \frac{T_1 - T_2}{\log\left(\frac{r_1}{r_2}\right)}$$

$$c_2 = T_1 - \frac{(T_1 - T_2)\log r_1}{\log\left(\frac{r_1}{r_2}\right)}$$

Substituting in equatoin (1) we get the required temperature distribution.

$$T = \frac{T_1 - T_2}{\log\left(\frac{r_1}{r_2}\right)}\log r + \frac{(-T_1 \log r_2 + T_2 \log r_1)}{\log\left(\frac{r_1}{r_2}\right)}$$

Exercise 2.7 :

1) *Using Fourier law of heat conduction, obtain a formula for the amount of heat lost under steady state condition from a furnace of 1 ft^2 wall and h feet thick if the temperature in the furnace is T_0. The temperature of the air outside the furnace is T_1 and thermal conductivity of material of wall is k, what is the temperature distribution through the walls.*

   a) $Q = \frac{k}{h}(T_1 - T_0)$

   b) $T = T_0 - (T_0 - T_1)\left(\frac{x}{h}\right)$

2) *A steam at 100 °C runs through the pipe 10 cms in diameter. It is insulated by a material of 5 cms thick, for which k = 0.06 and the temperature outside the pipe 30°. Find the amount of heat lost per hour from a portion of the pipe of 1 meter length.* **[Dec.-1999] [Ans. : 136989 cal]**

3) *Finds the amount of heat passing through 1 sq.cm of a refrigerator wall. If the thickness of the wall is 6 cm and the temperature inside the refrigerator is 0 °C while outside is 20 °C (k = 0.0002)* **[Ans : 0.000667 cal/sec.]**

2.10 Chemical Engineering Problems

Mixing of a solutions

Let Q be the amount of salt present in the tank at any time 't'. Then $\frac{dQ}{dt}$ = rate at which Q is changing is equal to the rate at which the salt is entering the tank minus rate at which the salt is leaving the tank.

Example 2.52 : ***A tank is initially filled with 100 gallons of salt solution containing 1 lb of salt per gallon. Fresh brine containing 2 lb of salt per gallon runs into the tank at the rate 5 gal/min. and the mixture, assumed to be kept uniform by stirring runs out at the same rate. Find the amount of salt in the tank at any time 't' and determine how long it will take for this amount to reach 150 lb ?***

Solution : First we calculate rate at which salt enters the tank

$$= \frac{5 \text{ gal}}{\text{min}} \times \frac{2\ lb}{\text{gal}} = 10\ lb/\text{min}$$

At any time 't', the amount of salt per gallon is $\frac{Q}{100}$ pounds per gallon. Since the concentration of a salt in the mixture running out of the tank is the same as the concentration $\frac{Q}{100}$ in the tank itself.

The rate at which salt leaves the tank

$$= \frac{5 \text{ gal}}{\text{min}} \times \frac{Q}{100} \frac{lb}{\text{gal}} = \frac{Q}{20} \frac{lb}{\text{min}}$$

$$\frac{dQ}{dt} = 10 - \frac{Q}{20}$$

i.e. $$\frac{dQ}{dt} + \frac{Q}{20} = 10$$

This is a L.D.E.

$\therefore$ $$\text{I.F.} = e^{\int \frac{1}{20} dt} = e^{\frac{1}{20} t}$$

$\therefore$ $$Qe^{\frac{1}{20} t} = \int 10 \cdot e^{\frac{1}{20} t}\, dt + c$$

$$= 200\, e^{\frac{1}{20} t} + c \qquad \ldots (1)$$

Initially at $t = 0, Q = 100$

$\Rightarrow$ $100 = 200 + c \Rightarrow c = -100$

Substituting in equation (1)

$$Q = 200 - 100\, e^{-\frac{1}{20} t}$$

$$= 100\left(2 - e^{-\frac{1}{20} t}\right)$$

Let at $t = T, Q = 150$

$\therefore$ $$150 = 100\left(2 - e^{-\frac{1}{20} T}\right)$$

$$e^{-\frac{1}{20} T} = \frac{50}{100}$$

i.e. $$e^{\frac{1}{20} T} = 2$$

$$T = 20 \log 2 = 13.9 \text{ min.}$$

Example 2.53 : *A tank contains 100 gal of brine in which 50 lb of salt is dissolved. Brine containing 2lb/gal of salt runs into the tank at the rate 3 gal/min, and the mixture, assumed to be kept uniform by stirring, runs out at the rate of 2 gal/min. Assuming that the tank is sufficiently large to avoid overflow. Find amount of salt in the tank on a function of time 't' when will be the concentration of salt in the tank reach $\frac{3}{2}$ lb/gal? How much salt will be in the tank at the end of 30 min?*

Solution : Let Q be the amount of salt at any time 't'

$$\frac{dQ}{dt} = \begin{Bmatrix}\text{The rate at which} \\ \text{salt is entering}\end{Bmatrix} - \begin{Bmatrix}\text{The rate at which} \\ \text{salt is leaving}\end{Bmatrix}$$

$$= 3 \times 2 - 2 \times C$$

Where C = concentration at any time t

Now initial volume of brine 100 gal, in one minute 3 gallons of brine enters the tank and 2 gallons leaves the tank so that volume of the tank increases at the rate $3-2=1$ gal/min.

Volume of liquid at time t = 100 + t gal

$$\therefore \quad C = \frac{Q}{100+t}$$

$$\therefore \quad \frac{dQ}{dt} = 6 - \frac{2Q}{100+t}$$

i.e. $$\frac{dQ}{dt} + \frac{2}{100+t} Q = 6$$

This is linear differential equation

$$\therefore \quad \text{I.F.} = e^{\int \frac{2}{100+t} dt} = e^{2\log(100+t)} = (100+t)^2$$

It's solution is,

$$Q(100+t)^2 = \int 6(100+t)^2 \, dt + C_1$$

$$= \frac{6(100+t)^3}{3} + C$$

i.e. $$Q = 2(100+t) + \frac{C}{(100+t)^2} \ lb$$

We have at $t = 0, \ Q = 50$

$$50 \times 100^2 = 2 \times 100^3 + C$$

$$\therefore \quad C = -2 \times 100^3 + 50 \times 100^2$$

$$= 50 \times 100^2 \left[1 - 4\right]$$

$$= -150 \times 100^2$$

$$\therefore \quad Q = 2(100 + t) - 150\left[\frac{100}{100 + t}\right]^2$$

i) At $T = T, \ Q = \frac{3}{2}$

$$\frac{3}{2} = 2(100 + T) - 150\left(\frac{100}{100 + T}\right)^2$$

$$\therefore \quad T = 44 \text{ min}$$

Next at $t = 30 \text{ min} \quad Q = 171.2\ lb$

2.11 Dissolving a Solid in a Liquid

Example 2.54 : ***Under certain conditions the rate at which a solid substance dissolves varies directly as the product or undissolved solid present in the solvent and the difference between the saturation concentration of the substance. If 20 lb of solute is dumped into a tank containing 120 lb solvent at the end of 12 min the concentration is observed to be 1 part in 30, find the amount of solute in solution at any time t if the saturation concentration is 1 part solute in 3 parts of solvent.***

Solution : Let Q be the amount of a material in solution at time t then 20 − Q is the amount of undissolved material present at that time and $\frac{Q}{120}$ is the concentration at that time.

∴ By the given law,

$$\frac{dQ}{dt} = k(20 - Q)\left(\frac{1}{3} - \frac{Q}{120}\right)$$

$$= \frac{k}{120}(20 - Q)(40 - Q)$$

Separating the variables,

$$\frac{dQ}{(Q - 20)(Q - 40)} = \frac{k}{120}\, dt$$

By partial fraction,

$$\left(\frac{1}{Q-40} - \frac{1}{Q-20}\right) dQ = \frac{k}{6}\, dt$$

Integrating,

$$\log (Q - 40) - \log (Q - 20) = \frac{k}{6} t + c$$

i.e. $$\log \frac{Q - 40}{Q - 20} = \frac{k}{6} t + c$$

Initially at $t = 0,\ Q = 0$ we get

$$c = \log \frac{40}{20} = \log 2$$

$\therefore$ $$\log \frac{Q - 40}{Q - 20} = \frac{k}{6} t + \log 2$$

i.e. $$\frac{k}{6} t = \log \left[\frac{40 - Q}{2(20 - Q)} \right]$$

i.e. $$\frac{40 - Q}{40 - 2Q} = e^{\frac{k}{6} t} \qquad \ldots (1)$$

When $t = 12,\ \frac{Q}{120} = \frac{1}{20}$ or $Q = 4$

Substituting in equation (1) we get,

$$\log \frac{36}{32} = 2k \text{ or } k = \frac{1}{2} \log \frac{9}{8} = 0.05889$$

$$\frac{40 - Q}{40 - 2Q} = e^{\frac{0.0589\, t}{6}}$$

Or $$Q = \frac{40 - 40e^{0.0098\, t}}{1 - 2e^{0.0098\, t}}$$

Example 2.55 : ***A chemical dissolves in water at a rate proportional to the product of the amount undissolved and the difference between the concentration in a saturated solution and the concentration in the actual solution. In 100 gms of a saturated solution it is known that 50 grams of the substance are dissolved. If when 30 gms of chemical are agitated with 100 gms of water, 10 gms are dissolved in 2 hours. How much will be it dissolved in 5 hours.***

Solution : Let Q be the grams of chemical undissolved time 't' hours.

$\therefore$ $$\frac{30 - Q}{100} = \left\{ \begin{array}{l} \text{Actual concentration} \\ \text{of the solution of time } t \end{array} \right\}$$

and $$\frac{50}{100} = \{\text{concentration of saturated solution}\}$$

Then by the given law

$$\frac{dQ}{dt} = -kQ\left(\frac{30-Q}{100} - \frac{50}{100}\right)$$

$$= \frac{kQ}{100}(Q+20)$$

i.e. $$\frac{dQ}{Q(Q+20)} = \frac{k}{100}\,dt$$

i.e. $$\left(\frac{1}{Q} - \frac{1}{Q+20}\right)dQ = \frac{k}{5}\,dt$$

Integrating, we get,

$$\log Q - \log(Q+20) = \frac{k}{5}t + c_1$$

We have at $t = 0,\ Q = 30$

$$\log 30 - \log 50 = c_1 \Rightarrow c_1 = \log\frac{3}{5}$$

Also when

$$t = 2,\ Q = 20$$

$$\log\frac{20}{20+20} = \frac{k}{5}2 + \log\frac{3}{5}$$

$$\Rightarrow \quad \frac{2k}{5} = \log\frac{1}{2} - \log\frac{3}{5}$$

$$= \log\left(\frac{5}{6}\right)$$

$$\Rightarrow \quad k = \frac{5}{2}\log\left(\frac{5}{6}\right) = -0.46$$

Substituting in equation (1) we get,

$$\log\left(\frac{Q}{Q+20}\right) = \frac{t}{2}\log\left(\frac{5}{6}\right) + \log\frac{3}{5}$$

$\therefore$ at $t = 5 \qquad Q = ?$

$$\log\frac{Q}{Q+20} = \frac{5}{2}\log\frac{5}{6} + \log\frac{3}{5}$$

$$\frac{Q}{Q+20} = \left(\frac{5}{6}\right)^{\frac{5}{2}}\frac{3}{5} = \frac{3}{5}e^{-0.46} = 0.38$$

Solving we get, $Q = 12$

$\therefore$ Amount dissolved in 5 hours $= 30 - 12$

$= 18$ gms.

Exercise 2.8 :

1) *A tank is initially filled with 100 liters of fresh water. Two liters of brine, each containing 1 gram of dissolved salt, run into the tank per minute, and mixture kept uniform by stirring runs out at the rate of 1 litre per minute. Find the amount of salt present when the tank contains 150 liters of salt present when the tank contains 150 liters of brine.*

   Hint : *Q gm of salt present in brine at any time 't'*

   the $\frac{dQ}{dt} = 2 - C = 2 - \frac{Q}{100+t}$

   [Ans. : Q = 83.3 gm]

2) *A tank contains 5000 liters of fresh water salt water which contains 100 gm of salt pert litre flows into at the rate of 10 lit. per minute and the mixture kept uniform by stirring, runs out the same rate. When will the tank contains 200000 gm of salt? How long will it take for the quantity of salt in the tank to increases from 150000 gm to 250000 gm?*

   Hint : *Let Q gm of salt present at any time the*

   $$\frac{dQ}{dt} = 100 \times 10 - \frac{Q}{5000} \times 10 = 1000 - \frac{Q}{500}$$

   [Ans. : $t = 500 \ \log \frac{50000}{500000 - Q}$

   $T_{Q = 200000} = 255.52 \text{ min} = 4 \text{ hours } 15.52 \text{ min}$

   $T_{1Q = 150000} = 500 \log \frac{10}{7}$

   $T_{2Q = 200000} = 500 \log 2$

   $T_1 - T_2 = 2 \text{ hours } 48.23 \text{ min}$**]**

3) *A tank initially contains 50 gallons of fresh water. Brine, containing 2 pounds per gallons of salt, flows into the tank at the rate of 2 gallons per minute and the mixture kept uniform by stirring, runs out at the same rate. How long will take for the quantity of salt in the tank to increases from 40 to 80 pounds.* **[Ans. : 27 min 28 sec.]**

4) *A tank contains 1000 litre of brine in which 20 kg of salt is dissolved. Brine containing 0.1 kg per litre of salt is runs into the tank at the rate of 40 litre per minute and mixture is kept uniform by stirring, runs out at the rate 30 litre per minute. Assuming that tank is sufficiently large to avoid overflow, find the amount of salt in the tank as a function of time. When will be the concentration of salt in the tank reach 0.05 kg per litre? How much salt will be in the tank after 30 minutes?*

   Hint : *Refer example 2.53.*

   $$\frac{dQ}{dt} = 4 - \frac{30Q}{1000 + 10t}$$

   $$Q = 100 + t - 80 \times (100)^3 \, (100 + t)^{-3}$$

   at $C = 0.05$, $Q = 12.47$ *min*

5) *Under certain conditions it is observed that the rate at which a solid substance dissolves varies directly as a product of the amount of undissolved solid present in the solvent and the difference between saturation concentration and the instantaneous concentration of the substance. If 20 kg of solute is dumped into a tank containing 500 litre of solvent and at the end of 10 minutes the concentration observed in one part is fifty. find the solute in solution at any time t, if the saturation, concentration is 1 part solute in 10 parts of solvent,*

Hint : $\frac{dQ}{dt} = k(20 \times Q)\left(\frac{1}{10} - \frac{Q}{500}\right)$

$k = 0.752$

$Q = 20\,\frac{1 - e^{-0.045\,t}}{1 - 0.4e^{-0.045\,t}}$

6) *A tank contains 10,000 litre of brine in which 200 kg. of salt are dissolved. Fresh water runs into the tank at the rate 100 liter per minute and the mixture, kept uniform by stirring, runs out at the same rate. How long will it be before only 20 kg. of salt are left in the tank.* **[Ans. : 230 minute]**

2.12 Miscellaneous Examples

Example 2.56 : ***Uranium disintegration at a rate proportional to the amount then present at any instant. If M_1 and M_2 grams of uranium are present at times T_1 and T_2 respectively. Show that the half–life of uranium is $\frac{(T_2 - T_1)\,\log 2}{\log(M_1\,/\,M_2)}$.***

Solution : Step 1 :

The differential equation of disintegration of uranium is

$$\frac{dm}{dt} = -\,km \qquad \text{(where m be the mass of time 't')}$$

Step 2 :

$$\frac{dm}{m} = -\,kdt$$

This is V.S. form as M_1 and M_2 grams are present at time T_1 and T_2, integrating,

$$\int_{M_1}^{M_2} \frac{dm}{m} = -\int_{T_1}^{T_2} kdt \Rightarrow (\log M)_{M_1}^{M_2} = -\,k(t)_{T_1}^{T_2}$$

$$\log\frac{M_1}{M_2} = k(T_2 - T_1) \qquad \text{... (1)}$$

Step 3 :

Also for its half-life $M_2 = M/2$

We have,

$$\int_{M}^{M/2} \frac{dm}{m} = -\,k\int_{0}^{T} dt \Rightarrow (\log m)_{M}^{M/2} = -\,k(t)_{0}^{T}$$

$$\log\frac{M}{2} - \log M = -\,kt$$

$$\Rightarrow \qquad kt = \log 2 \qquad k = \frac{\log 2}{T}$$

Substituting in equation (1)

$$\therefore \qquad \log\frac{M_1}{M_2} = \frac{\log 2}{T}(T_2 - T_1)$$

$$T = \frac{(T_2 - T_1)\log 2}{\log\frac{M_1}{M_2}}.$$

Example 2.57 : ***In a certain chemical reaction, a molecule of one substance 'A' combines with the molecule of second substance B to form the molecule of 'C'. if a and b are the amounts of A and B respectively at time t = 0, the amount x of 'C' at the time 't' is given by*** $\frac{dx}{dt} = k(a - x)(b - x)$ ***where k is constant and x = 0 when t = 0. Supposing that x = 2 when t = 10 minutes, find x at the end of 20 minutes a = 4, b = 3.***

Solution : Step 1 :

Given $$\frac{dx}{dt} = k(a - x)(b - x)$$

Step 2 :

$$\frac{dx}{(a-x)(b-x)} = k\,dt \qquad \text{(V.S. form)}$$

Integrating we get,

$$\int \frac{dx}{(a-x)(b-x)} = \int k\,dt + c$$

$$\frac{1}{a-b}\int\left\{\frac{1}{(b-x)} - \frac{1}{(a-x)}\right\}dx = k\int dt + c$$

$$\frac{1}{a-b}\log\frac{a-x}{b-x} = kt + c \qquad \text{... (1)}$$

Step 3 :

Here we have to find unknown c, for that we are given that x = 0, at t = 0

$$\therefore \qquad c = \frac{1}{a-b}\log\frac{a}{b}$$

Step 4 :

Substituting c in equation (1),

$$\frac{1}{a-b}\log\frac{a-x}{b-x} = kt + \frac{1}{a-b}\log\frac{a}{b}$$

$$kt = \frac{1}{a-b}\log\frac{b(a-x)}{a(b-x)} \qquad \text{... (2)}$$

Again, at x = 2, t = 10, we have from

$$k = \frac{1}{10(a-b)}\log\frac{b(a-2)}{a(b-2)}$$

Substituting in equation (2)

$$\therefore \qquad t = \frac{1}{10(a-b)}\log\frac{b(a-2)}{a(b-2)} = \frac{1}{(a-b)}\log\frac{b(a-x)}{a(b-x)} \qquad \text{... (3)}$$

at t = 20, we have from equation (3)

$$\frac{2}{a-b}\log\frac{b(a-x)}{a(b-2)} = \frac{1}{a-b}\log\frac{b(a-x)}{a(b-x)}$$

$$\log\left\{\frac{b(a-2)}{a(b-2)}\right\}^2 = \log\left\{\frac{b(a-x)}{a(b-x)}\right\}$$

$$\frac{(a-x)}{(b-x)} = \frac{b(a-2)^2}{a(b-2)^2}$$

When a = 4, b = 3, we have from equation (4)

$$\frac{4-x}{3-x} = \frac{3(4)}{4(1)} \quad x = \frac{5}{2}$$

Example 2.58 : ***The rate at which a mass of radium decomposes is at any instant proportional to the mass then present and the mass decrease to half its original value in 1600 years. If the initial mass be 100 milligrams. In how many years will it be 80 milligrams?***

Solution : Step 1 :

Let x is the mass of radium, which decomposes at any time 't'. Then

$$\frac{dx}{dt} \propto -x$$

$$\therefore \qquad \frac{dx}{dt} = -kx$$

Step 2 :

$$\frac{dx}{x} = -k\,dt \qquad \text{(V.S. form)}$$

Integrating we get,

$$\int \frac{dx}{x} = -k\int dt + c$$

$$\log x = -kt + c$$

Step 3 :

To find the unknown c Initially t = 0, x = 100 $\therefore$ c = log 100

Substituting in equation (1), we have,

$$\log \frac{x}{100} = -kt \qquad \text{... (2)}$$

At t = 1600, x is x/2, we have from (2)

$$k = -\frac{1}{1600} \log \frac{x}{200}$$

Substituting k in equation (2), we have,

$$\log \frac{x}{100} = t\left(\frac{1}{1600} \log \frac{x}{200}\right)$$

At $x = 80$, we have

$$t = \frac{1600 \log 80/100}{\log 80/200} = \frac{1600 \log 4/5}{\log 2/5} \text{ years.}$$

Example 2.59 : *For a thick cylinder under integral pressure, if p is the compressive stress and 'f' the tensile stress at a distance 'r' from the axis of the cylinder, the differential equation is* $r\frac{dp}{dr} + p + f = 0$. *Assuming* $f + ap = b$ *(where a, b are constants) and* $p = p_1$ *and* $p = 0$ *when* $r = r_2$ *show that* $\left(\frac{r_1}{r_2}\right)^{a-1} = \left(\frac{1-a}{b}\right)p_1 + 1$

Solution : Step 1 :

Putting f = b − ap in given equation, we get,

$$r\frac{dp}{dr} + p + b - ap = 0$$

$$r\frac{dp}{dr} = p(a-1) - b$$

Step 2 :

$$\frac{dr}{r} + \frac{dp}{p(1-a)+b} = 0$$

Integrating we get,

$$\int \frac{dr}{r} + \int \frac{dp}{p(1-a)+b} = \log c$$

$$\log r + \frac{1}{1-a} \log\{p(1-a)+b\} = \log c$$

$$(1-a) \log r + \log \{p(1-a)+b\} = \log c_1$$

$$\log(r)^{1-a} \{p(1-a)+b\} = \log c_1$$

$$r^{1-a} \{p(1-a)+b\} = c_1$$

$$p(1-a)+b = c_1 r^{a-1} \qquad \text{... (1)}$$

Step 3 :

To find the unknown c_1 we have when $r = r_1$, $p = p_1$ and when $r = r_1$, $p = 0$

From equation (1) $$p_1(1-a)+b = c_1 r_1^{a-1} \qquad \text{... (2)}$$

$$b = c_1 r_2^{a-1} \qquad \text{... (3)}$$

$$\therefore \quad c_1 = \frac{b}{r_2^{a-1}}$$

From equation (2) we get,

$$\frac{p_1(1-a)}{b} + 1 = \left(\frac{r_1}{r_2}\right)^{a-1}$$

Example 2.60 : *The amount x of a substance present in a certain reaction at time t is given by* $\frac{dx}{dt} + \frac{x}{10} = 2 - 1.5^{-t/10}$. *If at t = 0, x = 0.5, find x at t = 10.*

Solution : Step 1 :

Given $$\frac{dx}{dt} + \frac{x}{10} = 2 - 1.5\, e^{-t/10}$$ which is Linear equation.

$$\therefore \quad \text{I.F.} = e^{\int \frac{1}{10} dt} = e^{t/10}$$

Step 2 :

G.S. is given by,

$$x e^{t/10} = \int e^{t/10}\left[2 - 1.5\, e^{-t/10}\right] dt + c$$

Or $$x e^{t/10} = 2 \times 10 \times e^{t/10} - \frac{3}{2} t + c$$

$$x = 20 - \frac{3}{2} t e^{-t/10} + c e^{-t/10} \qquad \text{... (1)}$$

Step 3 :

To find the unknown c we are given that at $t = 0$, $x = 0.5$

$\therefore$ from equation (1), $\frac{1}{2} = 20 + c$

$$\Rightarrow \qquad c = -\frac{39}{2}$$

Substituting in equation (1), we get

$$x = 20 - \frac{3}{2} te^{-t/10} - \frac{39}{2} e^{-t/10} \qquad \text{... (2)}$$

Step 4 :

Put $t = 10$ in (2), we get

$$x = 20 - 15e^{-1} - \frac{39}{2} e^{-1}$$

$$x = 20 - \frac{69}{2e}$$

Example 2.61 : ***A boat is rowed with velocity u directly across a stream of width a. If the velocity of the current is directly proportional to the product of the distances from the two banks, find the path of the boat and the distance down stream to the point where it lands.***

Solution : Step 1 :

Taking the origin at the point from where the boat starts. At any time t after its start from O, let the boat be at P (x, y), so that

$$\frac{dx}{dt} = \text{Velocity of the current} = ky\,(a - y)$$

$$\frac{dy}{dt} = \text{Velocity with which the boat is being rowed} = u$$

$$\frac{dy}{dx} = \frac{dy}{dt} + \frac{dx}{dt} = \frac{u}{ky(a-y)} \qquad \text{... (1)}$$

This gives the direction of the resultant velocity of the boat, which is also the direction of the tangent to the path of the boat.

Step 2 :

Now equation (1) is of the variables separable form and we can write is as

$$y\,(a - y)\,dy = \frac{u}{k}\,dx$$

Integrating, we get,

$$\frac{ay^2}{2} - \frac{y^3}{3} = \frac{u}{k} x + c$$

Step 3 :

To find the value of c, we imply given conditions. Since y = 0 when x = 0,

$$\therefore \qquad c = 0$$

Hence the equation of the path of the boat is

$$x = \frac{k}{6u} y^2 (3a - 2y)$$

Putting y = a, we get distance AB, down stream where the lands = $ka^3/6u$.

Example 2.62 : ***If the population of a country doubles in 50 years, in how many years will it treble under the assumption that the rate of increase is proportional to the number of inhabitants?***

Solution : Step 1 :

Let y denote the population at time t years and y_0 the population at time t = 0. Then

$$\frac{dy}{dt} = ky \qquad \text{... (1)}$$

Step 2 :

$$\frac{dy}{y} = k\,dt, \text{ Where k is the proportionality factor.}$$

V.S. form. Integrating, we have,

$$\log y = kt + \log c\,, \qquad \text{... (2)}$$

Or $\qquad y = ce^{kt}$

Step 3 :

To find the value of c_1, we know that at time t = 0, $y = y_0$ from equation (2)

$$c_1 = y_0$$

Thus, $\qquad y = y_0 e^{kt} \qquad$ ... (3)

At $\qquad t = 50, \; y = 2y_0.$

From equation (3), $2y_0 = y_0 e^{50k} e^{50k} = 2.$

When $\qquad y = 3y_0$, equation (3) gives $3 = e^{kt}$.

Then $\qquad 3^{50} = e^{50kt} = \left(e^{50k}\right)^t = 2^t$

$$t = 79 \text{ years}$$

Example 2.63 : ***In a certain culture of bacteria, the rate of increases is proportional to the number doubles in 4 hours, how many may be expected at the of 12 hours?***

Solution : Step 1 :

Let x denotes the number of bacteria at time t hours.

Then $$\frac{dx}{dt} = kx$$

Step 2 :

$$\frac{dx}{x} = k\,dt$$

This is V.S. form. Integrating, we have,

$$\log x = kt + \log c \Rightarrow x = ce^{kt} \qquad \text{... (2)}$$

Step 3 :

To find value of c assuming that

$x = x_0$ at time $t = 0$, $c = x_0$ and $x = x_0 e^{kt}$

At time $t = 4,\ x = 2x_0.$

Then $2x_0 = x_0\, e^{4k}$ and $e^{4k} = 2.$

When $t = 12,\ x = x_0\, e^{12k} = x_0 \left(e^{4k}\right)^3 = x_0\,(2)^3 = 8x_0$

that is, there are 8 times the original number.

University Questions

May - 2003

1. *A constant e.m.f. 'E' volts is applied to a circuit containing constant resistance 'R' ohms in series with a constant inductance 'L' henries. If the initial current is zero, show that the current builds upto half of its theoretical maximum in* $\left(\frac{L \log 2}{R}\right)$ *seconds.*
2. *A particle is projected vertically upwards with velocity* v_1 *and resistance of air produces a retardation* kv^2 *where 'v' is the velocity. Find the velocity* v_2 *with which the particle will return to point of protection.*
3. *A pipe 20 cm in diameter contains steam at 150°C and is protected with a covering of 5 cm thick for which K = 0.0025. If the temperature of the outer surface of the covering is 40°C, find the temperature half-way through the covering under steady state conditions.* **[6 Marks]**

4. *A particle executing SMH has velocities v_1 and v_2 and accelerations a_1 and a_2 in two positions respectively. Show that the distance between the two position is* $\left|\dfrac{v_1^2-v_2^2}{a_1+a_2}\right|$ **[6 Marks]**

5. *The temperature of a body decreases at a rate '$k\theta$' where $\theta°$ is the amount the body is hotter than surrounding air. The body is heated by a source which makes body's temperature increased at a rate 'αt' where 't' is the time and 'α' is a constant. If this source is applied at $t = 0$, the body is then at the temperature of surrounding air. Show that* $\theta = \dfrac{\alpha}{k}\left(t - \dfrac{1}{k} + \dfrac{1}{k}e^{-kt}\right)$. **[6 Marks]**

Dec. - 2003

1. *A particle of mass m is attached to one end of light elastic spring of natural length a and modules $\frac{mg}{K}$. The other end of the spring is fixed to point O. The particle is allowed to fall from the point O. Find the velocity of the particle and show that its highest magnitude is* $\sqrt{ag(K+2)}$ **[6 Marks]**

2. *A body at 80 °C. Cools down to 60 °C in 20 minutes. The temperature of the air being 40 °C. What will be the temperature of the body after 40 minutes from the original ?* **[5 Marks]**

3. *The equation of emf in terms of current i for an electric circuit having resistance R and condenser of capacity C in series* $E = Ri + \int \frac{i}{C}dt$. *Find current i at any time t, when* $E = E_0 \sin \omega t$. **[5 Marks]**

4. *Develop a formula for the steady state heat loss per unit time from a unit length of pipe of radius r_0 carrying steam at temperature T_0. If the pipe is covered with insulation of thickness w, the outer surface of which remains at the constant temperature T_1. Find the temperature distribution T as a function of the radius.* **[6 Marks]**

May - 2004

1. *A body of mass m falling from rest is subject to the force of gravity and an air resistance proportional to the square of the velocity (kv^2). If it falls through a distance 'x' and possesses a velocity 'v' at that instant. Prove that,* $\dfrac{2kx}{m} = \log\left(\dfrac{a^2}{a^2-v^2}\right)$ *where* $mg = ka^2$. **[5 Marks]**

2. *In a certain containing inductance L, resistance R and voltage E, the current I is given by* $E = RI + L\frac{dI}{dt}$ *where I (0) = 0. If L = 640 H, R = 250 Ω and E = 500 units, find the times that elapses before the current reaches 90 % of its maximum value.* **[6 Marks]**

3. *A point executing SHM has velocity v_1 and v_2 and acceleration a_1 and a_2 in two positions respectively. Show that the distance between the two positions is* $\left|\dfrac{v_1^2-v_2^2}{a_1+a_2}\right|$ **[6 Marks]**

4. *A body of temperature 100 °C is placed in a room whose temperature is 25 °C and cools to 50 °C in minutes. What will be it's temperature after a further interval of 5 minutes.* **[5 Marks]**

5. *A steam pipe 20 cm in diameter is protected with a covering 6 cm thick for which the coefficient of thermal conductivity is K = 0.0003 cal/cm deg sec steady state. Find the heat loss per hour through a meter length of the pipe, if the surface of the pipe is at 200 °C and the outer surface of the covering is at 30 °C.* **[5 Marks]**
6. *A particle is moving in a straight line with acceleration* $k\left(x+\frac{a^4}{x^3}\right)$ *directed towards origin. If it starts from rest at a distance from the origin. Prove that it will arrive at the origin at the end of time* $\frac{\pi}{4\sqrt{k}}$. **[6 Marks]**
7. *The charge Q on the plate of a condenser of capacity 'C' charged through a resistance R by steady voltage V satisfies the differential equation,* $R\frac{dQ}{dt}+\frac{Q}{C}=V$. *If Q = 0 at t = 0 then show that* $Q=CV[1-e^{-t/RC}]$. *Find the current flowing into the plate.* **[5 Marks]**
8. *Uranium disintegrates at a rate proportional to the amount that present at any instant. If* M_1 *and* M_2 *grams of uranium are present at times* T_1 *and* T_2 *respectively. Show that the half life of uranium is,* $\frac{(T_2-T_1)\log 2}{\log(M_1/M_2)}$. **[6 Marks]**

Dec. - 2004

1. *A particle of mass m moves in a horizontal straight line OA with an acceleration* $\frac{mk}{r^3}$ *at a distance r and directed towards O. If initially the particle was at rest at a distance 'a' from O, find the distance of particle from O at the end of time* $\frac{a^2}{2}\sqrt{\frac{3}{k}}$. **[6 Marks]**
2. *The equation of an L-R circuit is given by* $L\frac{di}{dt}+Ri=\sin 10t$. *If i = 0 at t= 0 find the expression for i in terms of t.* **[5 Marks]**
3. *A particle executes SHM. When it is 2 cm from mid path its velocity is 10 cm/sec. and when it is 6 cm from the centre of its path, its velocity is 2 cm/sec. Find the period and its greatest acceleration.* **[6 Marks]**
4. *If 30 % of a ratio active substance disappeared in 10 days, how long will it take for 90 % of it to disappear.* **[5 Marks]**
5. *A pipe 30 cm in diameter contains at 150 °C and is protected with a covering 5 cm thick for which k = 0.0025. If the temperature of the outer surface of the covering is 40 °C, find the temperature at a point which is at 16.5 cm from the axis of the pipe.* **[6 Marks]**
6. *A constant electromotive force E volts is applied to a circuit containing inductance L henries. If the initial current is zero, find the time that lapses before the current reaches 25 % of its maximum value.* **[5 Marks]**
7. *A body of mass m falls from rest under the influence of gravity and a retarding force due to air resistance proportional to the square of the velocity. Find the velocity and distance described as function of time.* **[5 Marks]**

8. *Water at temperature 100 °C cools in 10 minutes to 80 °C in a room having temperature 25 °C. Find a) temperature of water after 20 minutes, ii) the time required to lower the temperature of water to 40 °C.*

May - 2005

1. *If the temperature of the body drops from 100 °C to 60 °C in one minute. When the temperature of the surrounding is 20 °C, what will be the temperature of the body at the end of the second minute ?* **[5 Marks]**
2. *A resistance of 100 ohms, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in a circuit as a function of time.* **[5 Marks]**
3. *A body of mass m falling from rest is subjected to the force of gravity and an air resistance proportional to the square of the velocity kv^2. If it falls through a distance x and possesses a velocity v at that instant, prove that* $\frac{2kx}{m} = \log\left(\frac{a^2}{(a^2 - v^2)}\right)$, *where* $mg = ka^2$. **[6 Marks]**
4. *A point executing simple harmonic motion has velocities v_1 and v_2 acceleration a_1 and a_2 in two positions respectively. Show that the distance between the two positions is* $\left|\frac{(v_1^2 - v_2^2)}{(a_1 + a_2)}\right|$. **[6 Marks]**
5. *A pipe 20 cm in diameter contains steam at 150 °C and is protected with a covering 5 cm thick for which K = 0.0025. If the temperature of the outer of the covering is 40 °C, find the temperature half-way through the covering under steady state conditions.* **[6 Marks]**
6. *A circuit consist of resistance R ohms and condenser of c farads connected to constant e.m.f. If $\frac{q}{c}$ is the voltage of the condenser at time t after closing the circuit, show that the voltage at time t is* $E\left(1 - e^{-t/RC}\right)$ **[5 Marks]**
7. *The distance descended by a parachuter satisfies the differential equation* $\frac{dv}{dt} = g\left(1 - \frac{v^2}{k^2}\right)$ *where v is velocity, k and g are constants. If v = 0 and x = 0 at t = 0, show that* $x = \frac{k^2}{g} \log \cosh (gt / k)$. **[6 Marks]**
8. *Find the orthogonal trajectories of the circle defined by $r = a \cos\theta$ which all pass through the origin and have their centres on the initial line, 'a' being the variable diameter.* **[5 Marks]**

Dec. - 2005

1. *If the temperature of the body drops from 100 °C to 60 °C in one minute when the temperature of the surrounding is 20 °C, what will be the temperature of the body at the end of the second minute?* **[6 Marks]**
2. *The equation of an L-R circuit is given by*

   $$L\frac{dI}{dt} + RI = 10 \sin t \qquad \text{If } I = 0 \text{ at } t = 0$$

   Express I as a function of t. **[5 Marks]**

3. *A body starts moving from rest. It is opposed by a force per unit mass of value cx and resistance per unit mass of value bv^2, where x and v are the displacement and velocity of the particle at that instant. Show that the velocity of the particle is given by*

$$V^2 = \frac{c}{2b^2}\left(1 - e^{-2bx}\right) - \frac{cx}{b}$$

[5 Marks]

4. *An elastic spring of natural length l is fixed at a point A. To the lower end is attached a particle of mass m so that the spring stretches to a length 2l. If the particle is dropped from A, show that it descends a distance $l\left(2+\sqrt{3}\right)$ before coming to rest.* **[6 Marks]**

5. *A long hollow pipe has an inner diameter of 10 cm and outer diameter of 20 cm. The inner surface is kept at 200 °C and the outer surface at 50 °C. The thermal conductivity is 0.12. How much heat is lost per minute from a portion of the pipe 20 meters long ? Find the temperature at a distance $x = 7.5$ cm from the centre of the pipe.* **[6 Marks]**

6. *The charge Q on the plate of a condenser of capacity C charged through a resistance R by a steady voltage V satisfies the D.E.*

$R\dfrac{dQ}{dt} + \dfrac{Q}{C} = V$. *If* $Q = 0$ *at* $t = 0$, *show that* $Q = CV\left[1 - e^{-t/RC}\right]$. *Find the current flowing into the plate.* **[5 Marks]**

7. *A particle of unit mass is projected vertically upward with velocity u. Assuming that the air resistance is k times the instantaneous velocity of the particle, show that the particle will return to point of projection with velocity V given by*

$$V + u = \frac{g}{k}\log\left(\frac{g+ku}{g-kV}\right).$$

[5 Marks]

8. *According to Newton's Law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is 30 °C and the substance cools from 100 °C to 70 °C in 15 minutes, find when the temperature will be 40 °C.* **[6 Marks]**

May - 2006

1. *A body originally at 80 °C cools down to 60 °C in 20 minutes, the temperature of air being 40 °C. What will be the temperature of the body after 40 minutes from the original ?* **[5 Marks]**

2. *In a circuit containing inductance L, resistance R and voltage E, the current I is given by* $E = RI + L\dfrac{dI}{dt}$. *Given L = 640 H, R = 200 Ω and E = 500 V, I being zero when t = 0, find the time elapses before it reaches 90% of its maximum value.* **[6 Marks]**

3. *The distance x descended by a parachuter satisfies the differential equation :*

$\dfrac{dv}{dx} = g\left(1 - \dfrac{v^2}{k^2}\right)$ *where v is the velocity, k and g are constants.*

If v = 0 and x = 0 at t = 0, show that $x = \dfrac{k^2}{g}\log\cosh\left(\dfrac{gt}{k}\right)$.

[5 Marks]

4. Find the current I in the circuit having resistance R and condenser of capacity C in series with e.m.f. $E \sin \omega t$. **[5 Marks]**

5. A point executing simple harmonic motion has velocity v_1 and v_2 and acceleration a_1 and a_2 in two positions respectively. Show that the distance between the two positions is

$$\left|\frac{v_1^2 - v_2^2}{a_1 + a_2}\right|$$

[6 Marks]

6. A steam pipe 20 cm in diameter is protected with a covering 6 cm thick for which the coefficient of thermal conductivity $k = 0.0003$. Assuming steady state conditions, find the heat lost per hour through a meter length of the pipe if the surface of the pipe is at 200 °C and the outer surface of the covering is at 30 °C. **[6 Marks]**

7. A metal ball is heated to a temperature of 100 °C and at time $t = 0$ it is placed in a water which is maintained at 40 °C if, the temperature of the ball is reduced to 60 °C in 4 minutes, find the time at which the temperature of the ball is 50 °C. **[5 Marks]**

8. A body of mass m, falling from rest is subjected to the force of gravity and an air resistance, proportional to the square of the velocity (kv^2). If it falls through a distance x and possesses a velocity v at that instant, prove that :

$$\frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right), \text{ where } mg = ka^2$$

[6 Marks]

Dec. - 2006

1. Solve any three :

a) A body at temperature 100 °C is placed in a room where temperature is 20 °C and cools to 60 °C in 5 minutes. Find its temperature after a further interval of 3 minutes. **[5 Marks]**

b) A constant e.m.f. E volts is applied to a circuit containing a constant resistance R ohms in series and a constant inductance L henries. If the initial current is zero, show that the current builds upto half its theoretical maximum in $\frac{L \log 2}{R}$ seconds. **[5 Marks]**

c) A particle of unit mass is projected vertically upward with velocity u. Assuming that the air resistance is k times the instantaneous velocity of the particle, show that the particle will return to point of projection with velocity V given by

$$V + u = \frac{g}{k} \log\left[\frac{g + ku}{g - kV}\right]$$

[6 Marks]

d) A long hollow pipe has an inner diameter of 10 cm and outer diameter of 20 cm. The inner surface is kept at 200 °C and the outer surface at 50 °C. The thermal conductivity is 0.12. How much heat is lost per minute from a portion of the pipe 20 meters long ? Find the temperature at a distance $x = 7.5$ cm. **[6 Marks]**

2. Solve any three :

a) The temperature of the air is 30 °C and the substance cools from 100 °C to 70 °C in 15 minutes. Find when the temperature will be 40 °C ? **[5 Marks]**

b) An electrical circuit contains an inductance of 5 henries and a resistance of 12 ohms in series with an e.m.f. 120 sin 20t volts. Find the current at t = 0.01, if it is zero when t = 0. **[5 Marks]**

c) A particle is moving in a straight line with an accleration $k\left[x+\frac{a^4}{x^3}\right]$ *directed towards origin. If it starts from rest at a distance a from the origin, prove that it will arrive at origin at the end of time* $\frac{\pi}{4\sqrt{k}}$. **[6 Marks]**

d) An elastic spring of natural length l is fixed at a point A. To the lower end is attached a particle of mass m so that the spring stretches to a length 2 l. If the particle is dropped from A, show that it descends a distance $l\left(2+\sqrt{3}\right)$ *before coming to rest.* **[6 Marks]**

May - 2007

1. *Solve any three :*

a) A body originally at 80 °C cools down to 60 °C in 20 minutes, the temperature of the air being 40 °C, what will be the temperature of the body after 40 minutes from the origin ? **[5 Marks]**

b) A voltage Ee^{-at} *is applied at t = 0 to a circuit containing inductance L and resistance R. Show that the current at any time t is* $\frac{E}{R-aL}\left(e^{-at}-e^{-\frac{Rt}{L}}\right)$. **[5 Marks]**

c) A particle is moving in a straight line with an acceleration $k\left[x+\frac{a^4}{x^3}\right]$ *directed towards origin. If it starts from rest at a distance 'a' from the origin, prove that it will arrive at origin at the end of* $\frac{\pi}{4\sqrt{k}}$. **[6 Marks]**

d) A particle executes SHM when it is 2 cm from midpath, its velocity is 10 cm/sec and when it is 6 cm from the centre of its path, its velocity is 2 cm/sec. Find the period and its greatest acceleration. **[6 Marks]**

2. *Solve any three :*

a) A long hollow pipe has an inner diameter of 10 cm and outer diameter of 20 cm. The inner surface is kept at 200 °C and the outer surface at 50 °C. The thermal conductivity is 0.12. How much heat is lost per minute from a portion of the pipe 20 meters long ? Find the temperature at a distance x = 7.5 cm from the centre of the pipe. **[6 Marks]**

b) A circuit consists of resistance R ohms and condenser of 'C' farads connected to a constant e.m.f. If $\frac{q}{C}$ *is the voltage of the condenser at time t after closing the circuit, show that the voltage at time t is,* $E\left(1-e^{-t/CR}\right)$. **[5 Marks]**

c) The distance x descended by a person falling by means of parachute satisfies the differential equation $\left(\frac{dx}{dt}\right)^2 = k^2\left[1-e^{-\frac{2gx}{k^2}}\right]$ where k and g are constants, and $x = 0$ when $t = 0$. Show that $x = \frac{k^2}{g} \log \cosh\left(\frac{gt}{k}\right)$. **[6 Marks]**

d) Find the orthogonal trajectories of the family of $y^2 = 4ax$. **[5 Marks]**

Dec. - 2007

1. Solve any three :

a) When a thermometer is placed in a hot liquid bath at temperature T, the temperature θ indicated by the thermometer rises at the rate of T − θ. For a bath at 95 °C, the temperature reads 15 °C at t = 0 and 35 °C at t = 10 sec., what will be its temperature at t = 20 sec. ? **[5 Marks]**

b) Find the current in the circuit having resistance R and condenser of capacity C in series with e.m.f. E sin ωt. **[5 Marks]**

c) A body of mass m falls from rest under gravity in a fluid whose resistance to motion at any instant is mk times its velocity, where k is a constant. Find the terminal velocity of the body and also the time taken to acquire one-half of its limiting speed. **[6 Marks]**

d) The inner and outer surfaces of a spherical shell are maintained at T_0 and T_1 temperatures respectively. If the inner and outer radii of the shell are r_0 and r_1 respectively and thermal conductivity of the shell is k, find the amount of heat lost from the shell per unit time. Find also the temperature distribution through the shell. **[6 Marks]**

2. Solve any three :

a) A pipe 10 cms in diameter contains steam at 100 °C. It is covered with asbestos, 5 cms thick, for which k = 0.0006 and the outside surface is at 30 °C. Find the amount of heat lost per hour from a meter long pipe. **[5 Marks]**

b) A resistor of 20 ohms is connected in series with a capacitor of 0.01 farads and an e.m.f. E volts given by $40e^{-3t} + 20e^{-6t}$. If q = 0 at t = 0, find the maximum charge on the capacitor. **[5 Marks]**

c) The resistance to movement of a ship through water is of the form $a^2 + b^2v^2$, where v is instantaneous velocity and a, b are constants. Prove that the time in which the speed falls to one half of its original value u is given by $\frac{W}{abg}\tan^{-1}\frac{abu}{2a^2 + b^2u^2}$ where W is the weight of the ship. **[6 Marks]**

d) A particle of mass m is attached to one end of a light elastic string of natural length a and modulus $\frac{mg}{k}$. The other end of the string is fixed to a point O and the particle is allowed to fall from rest at O. Obtain velocity of the particle and show that the highest magnitude of velocity is $\sqrt{ag(2+k)}$. **[6 Marks]**

May - 2008

1. *Solve any three :*

a) *A particle moving in a straight line with an acceleration* $K\left[x+\frac{a^4}{x^3}\right]$ *is directed towards origin. It it starts from rest at a distance a from origin, prove that it will arrive at origin at the end of time* $\frac{\pi}{4\sqrt{K}}$. **[6 Marks]**

b) *The charge Q on a plate of condenser of capacity C is charged through a resistance R by steady voltage V. If Q = 0 at t = 0, show that* $Q = CV\,[1-e^{-t/RC}]$. **[5 Marks]**

c) *One dimensional steady state heat conduction for a hollow cylinder with constant thermal conductivity k in the region* $a \le r \le b$, *the temperature T at a distance r is given by* $\frac{d}{dr}\left[r\frac{dT}{dr}\right]=0$. *With* $T = T_1$, *when r = a and* $T = T_2$ *when r = b. Use this to determine steady state temperature T in a cylinder in terms of r.* **[6 Marks]**

d) *A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of surrounding air being 40°C.*

i) *What will be the temperature of the body after 40 minutes from the original ?*

ii) *Find the time required to cool down the body to 70°C.* **[5 Marks]**

2. *Solve any three of the following :*

a) *Find the orthogonal trajectories of xy = C.* **[5 Marks]**

b) *Determine the least velocity with which a particle must be projected vertically upwards so that it does not return to the earth. Assume that it its acted upon by the gravitational attraction of earth only.* **[6 Marks]**

c) *A particle executes SHM when it is 2 cm from the mid path, its velocity is 10 cm/sec. and when it is 6 cm from centre of its path, its velocity is 2 cm/sec. Find its period and greatest accelerations.* **[6 Marks]**

d) *In a circuit containing inductance L = 640 H, resistance R = 250 Ω and voltage E = 500 volts. Current i is being zero when t = 0. Find the time that elapses, before it reaches 50% of its maximum value.* **[5 Marks]**

Dec. - 2008

1. *Solve any three :*

a) *Find orthogonal trajectories of the family of cardioids* $r = a\,(1-\cos\theta)$ **[5 Marks]**

b) *If temperature of the air is 30 °C and substance cools from 100 °C to 70 °C in 15 minutes. Find when the temperature will be 40 °C.* **[5 Marks]**

c) *A Particle of mass m is suspended from one end of the spring whose other end is attached to a fixed point. If the extension in length due to the mass of the particle is e, find the period of oscillation.* **[6 Marks]**

d) A pipe 10 cm in diameter contains steam at $100\,°C$. *It is covered with asbestos sheet 5 cm thick for which k = 0.0006 and outer surface temperature is* $30\,°C$. *Find the amount of heat lost per hour through one meter length pipe.* **[6 Marks]**

2. *Solve any three.*

a) In an electric circuit containing inductance and resistance in series with constant e.m.f. E, if initial current is zero, show that the current builds up to half its theoretical maximum in $\frac{L \log 2}{R}$ *seconds.* **[5 Marks]**

b) Water at temperature $100\,°C$ *cools in 10 minutes to* $88\,°C$ *in a room of temperature* $25\,°C$. *Find the temperature of water after 20 minutes.* **[5 Marks]**

c) If 30% of a radioactive substance disappeared in 10 days, how long will it take for 90% of it to disappear ? **[6 Marks]**

d) A particle of mass m is projected upword with velocity V. Assuming the air resistance k times its velocity, show that it will reach maximum height in time $\frac{m}{k}\log\left(1+\frac{kV}{gm}\right)$ *and find the distance travelled at any time t.* **[6 Marks]**

❑❑❑

3 Fourier Series

3.1 Introduction

It is often required to represent general periodic functions as a series of sine and cosine terms. The series of sine and cosine terms is called as a Fourier series.

Periodic functions are of common occurrence in many physical and engineering problems for example, in conduction of heat and mechanical vibration, Fourier series is useful to express these functions in a series of sines and cosines.

The periodic functions having many discontinuities are of practical importance but can't be expressed in terms of Taylor's series but can be developed as a Fourier series.

3.2 Periodic Functions

A function is called periodic if it is defined for every real x and if there exists some positive number T such that $f(x + nT) = f(x)$. 'T' is called as a period of f(x) i.e. the graph of f(x) repeats after the interval T. $(T > 0)$ $n = 1, 2, \ldots$.

If T is the period of T then 2T, 3T, 4T are also periods of f(x). The smallest of all these is called as the premitive period of f(x) or fundamental period of f(x).

For example : The fundamental period of sin x or cos x is '2π' while that of tanx is 'π'.

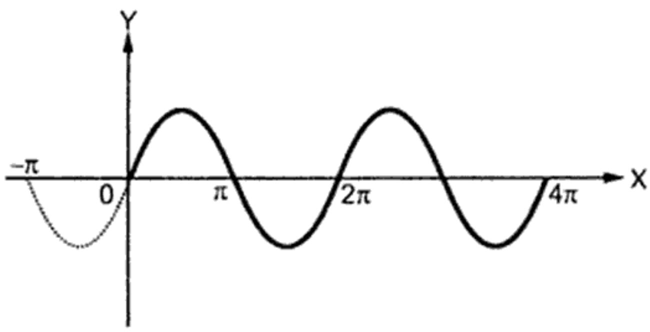
Note : Constant function is the only function having no premitive period.

Example 3.1 : *Graph the periodic functions i) $f(x) = \sin x$, ii) $f(x) = \cos x$*

Solution : We know that

$$\sin(x + 2\pi) = \sin x$$

$$\cos(x + 2\pi) = \cos x$$



f(x) = sin x

Fig. 3.1

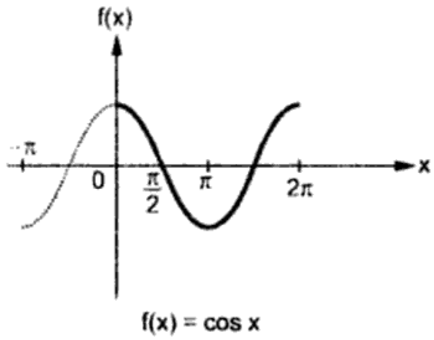


Fig. 3.2

Example 3.2 : *Sketch the following functions which are assumed to be periodic with period 2π.*

Solution : i) $f(x) = x, \ -\pi < x < \pi$

$$f(x + 2\pi) = f(x)$$

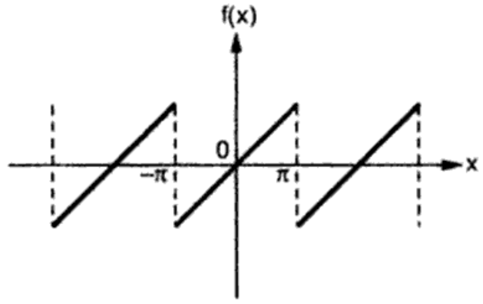


Fig. 3.3

ii) $f(x) = |x| \quad -\pi < x < \pi$

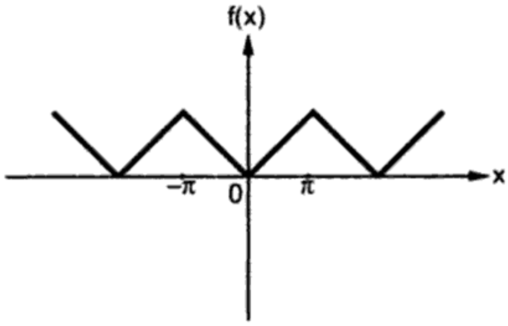


Fig. 3.4

iii) $f(x) = x^2 \quad -\pi < x < \pi$

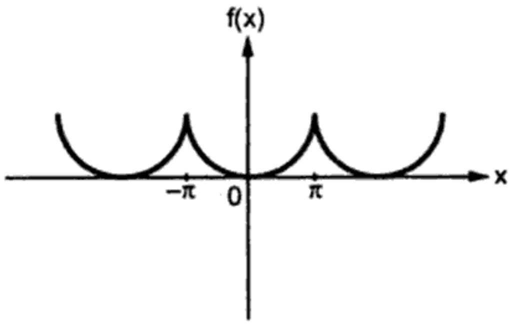


Fig. 3.5

iv) $$f(x) = \pi + x \quad \text{if} \quad -\pi < x < 0$$
$$= \pi - x \quad \text{if} \quad 0 < x < \pi$$

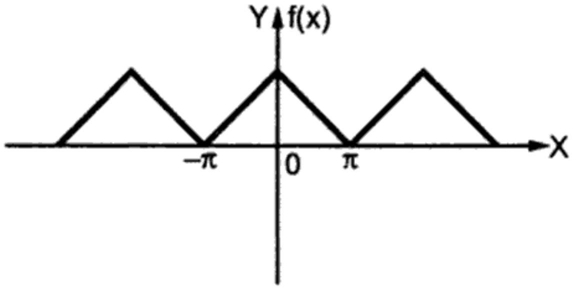


Fig. 3.6

v) $$f(x) = \sin x \qquad 0 < x < \pi$$
$$= 0 \qquad -\pi < x < 0$$

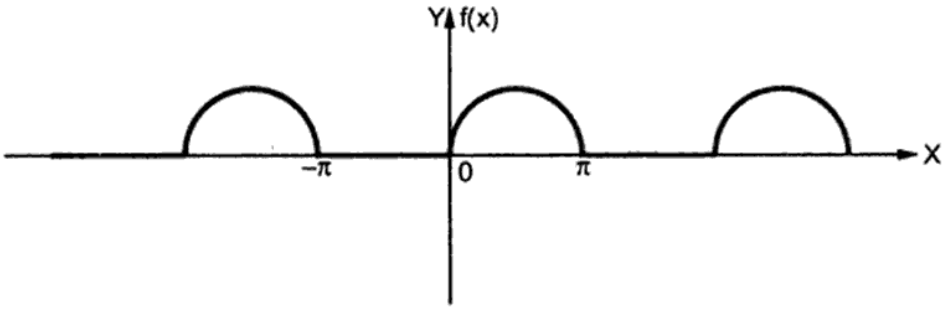


Fig. 3.7

vi) $$f(x) = x - x^2 \qquad -1 < x < 1$$
$$f(x+2) = f(x)$$

i.e. Periodic with period 2.

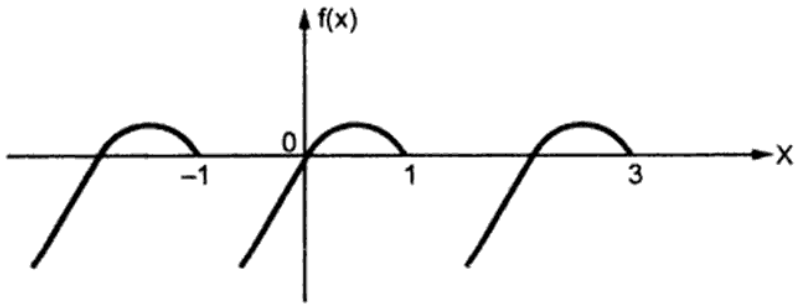


Fig. 3.8

vii) $$f(x) = \pi x \quad \text{if} \quad 0 \le x \le 1$$
$$= \pi(2 - x) \qquad 1 \le x \le 2$$

Periodic with period 2.

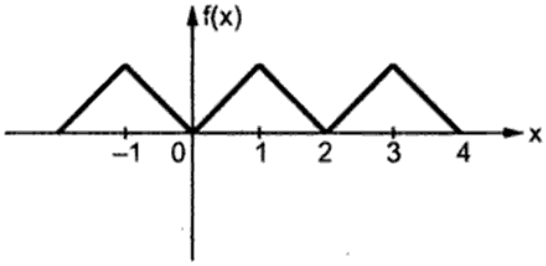


Fig. 3.9

viii) $f(x) = x - 1 \qquad -1 < x < 0$

$\quad = x + 1 \qquad 0 < x < 1$

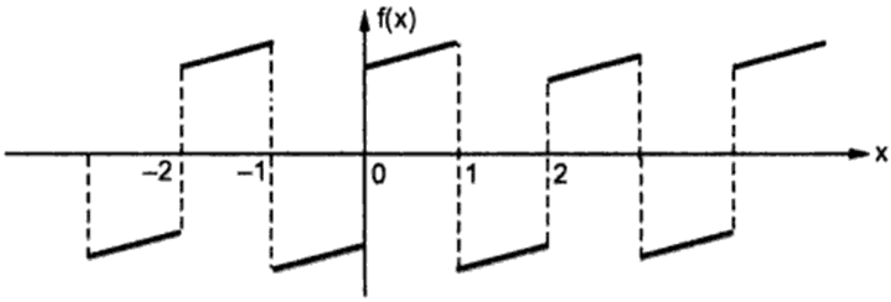


Fig. 3.10

3.3 Formulae Required to Evaluate Integrals

1) Integration by parts : If u, v are functions of x.

then $\int u\, v\, dx = u \int v\, dx - \int \left(\frac{du}{dx} \cdot \int v\, dx\right) dx$

2) Generalized rule of integration by parts (Leibnitz's Rule)

$$\int u\, v\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \text{-----}$$

where dashes indicate derivatives and suffixes indicates integrals.

Note : When the first function i.e. u is power of variable i.e. (x^n) or a polynomial in x (i.e. $ax^n + bx^{n-1} + ... + c$) and the second function is sin ax or cos ax or e^{ax} then it is very easy to integrate by using generalized rule.

For example : $\int (x^2 + x) \sin ax\, dx$

$$= \left\{(x^2 + x)\left(-\frac{\cos ax}{a}\right) - (2x + 1)\left(-\frac{\sin ax}{a^2}\right) + (2)\left(\frac{\cos ax}{a^3}\right)\right\}$$

3) $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

4) $2 \cos A \cos B = \cos(A - B) + \cos(A + B)$

5) $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

6) $\int e^{ax} \sin nx \, dx = \dfrac{e^{ax}}{a^2 + n^2} [a \sin nx - n \cos nx]$

7) $\int e^{ax} \cos nx \, dx = \dfrac{e^{ax}}{a^2 + n^2} [a \cos nx + n \sin nx]$

8) $\int x \sin nx \, dx = x\left(\dfrac{-\cos nx}{n}\right) - (1)\left(\dfrac{-\sin nx}{n^2}\right)$

9) $\int x \cos nx \, dx = x\left(\dfrac{\sin nx}{n}\right) - (1)\left(\dfrac{-\cos nx}{n^2}\right)$

10) $\int x^2 \sin nx \, dx = \left\{x^2\left(\dfrac{-\cos nx}{n}\right) - (2x)\left(\dfrac{-\sin nx}{n^2}\right) + 2\left(\dfrac{\cos nx}{n^3}\right)\right\}$

11) $\int x^2 \cos nx \, dx = \left\{x^2\left(\dfrac{\sin nx}{n}\right) - 2x\left(\dfrac{-\cos nx}{n^2}\right) + 2\left(\dfrac{-\sin nx}{n^3}\right)\right\}$

12) $\int \sin nx \, dx = \dfrac{-\cos nx}{n}$

13) $\int \cos nx \, dx = \dfrac{\sin nx}{n}$

14) $\int \sin x \sin nx \, dx = \dfrac{1}{2}\int [\cos (1 - n) x - \cos (1 + n) x] \, dx$

15) $\int \cos x \cos nx \, dx = \dfrac{1}{2}\int [\cos (1 - n) x + \cos (1 + n) x] \, dx$

16) $\int \sin x \cos nx \, dx = \dfrac{1}{2}\int [\sin (1 + n) x + \sin (1 - n) x] \, dx$

17) $\int \cos x \sin nx \, dx = \dfrac{1}{2}\int [\sin (n + 1) x + \sin (n - 1) x] \, dx$

18) $\cos n\pi = (-1)^n, \quad \sin n\pi = 0$

19) $\cos 2n\pi = (-1)^{2n} \quad \sin 2n\pi = 0$

20) $\cos(2n \pm 1)\pi = (-1)^{2n \pm 1} = -1, \sin (2n \pm 1)\pi = 0$

21) $\cos (n - 1)\pi = \cos(n + 1)\pi$

22) $(-1)^n = (-1)^{-n}$

23) $\sin \dfrac{n\pi}{2}$ and $\cos \dfrac{n\pi}{2}$ takes the values 1, 0, – 1 depending on the value of n.

Note : Substituting the limits $(0, 2\pi)$, $(-\pi, \pi)$, $(0, \pi)$ in above integrals (6) to (17) we get following table, students are requested to solve all these integrals for practice.

Integrals ↓ \ Limits →	$\int_0^{2\pi}$	$\int_0^{\pi}$	$\int_{-\pi}^{\pi}$
$\int x \sin nx\, dx$	$\frac{-2\pi}{n}$	$\frac{-\pi(-1)^n}{n}$	$\frac{-2\pi(-1)^n}{n}$
$\int x \cos nx\, dx$	0	$\frac{(-1)^n - 1}{n^2}$	0
$\int x^2 \sin nx\, dx$	$\frac{-4\pi^2}{n}$	$\frac{-\pi^2(-1)^n}{n} + \frac{2}{n^3}[(-1)^n - 1]$	0
$\int x^2 \cos nx\, dx$	$\frac{4\pi}{n^2}$	$\frac{2\pi(-1)^n}{n^2}$	$\frac{4\pi(-1)^n}{n^2}$
$\int e^{ax} \sin nx\, dx$	$\frac{n}{a^2+n^2}[1 - e^{2a\pi}]$	$\frac{n}{a^2+n^2}[1 - e^{a\pi}(-1)^n]$	$\frac{-n(-1)^n[e^{a\pi} - e^{-a\pi}]}{(a^2+n^2)}$
$\int e^{ax} \cos nx\, dx$	$\frac{a}{a^2+n^2}[e^{2a\pi} - 1]$	$\frac{a}{a^2+n^2}[e^{a\pi}(-1)^n - 1]$	$\frac{a(-1)^n[e^{a\pi} - e^{-a\pi}]}{a^2+n^2}$
$\int \sin nx\, dx$	0	$\frac{1-(-1)^n}{n}$	0
$\int \cos nx\, dx$	0	0	0
$\int \sin x \sin nx\, dx$ $n \neq 1$	0	0	0
$\int \cos x \cos nx\, dx$ $n \neq 1$	0	0	0
$\int \sin x \cos nx\, dx$ $n \neq 1$	0	$\frac{1+(-1)^n}{1-n^2}$	0
$\int \cos x \sin nx\, dx$ $n \neq 1$	0	$\frac{n[1+(-1)^n]}{n^2-1}$	0

3.4 Even and Odd Functions

The definitions of even and odd functions are strictly for the functions which are defined in both sided intervals, finite or infinite, which is centred at $x = 0$ i.e. the interval is of the type $-a < x < a$.

Note : If the function is defined in the interval $(-l, l)$ or $(-2\pi, 2\pi)$ or $(-\infty, \infty)$ or $(-\pi, \pi)$ ... etc then only check for even and odd.

Definition : If f(x) is defined in $-l < x < l$ is said to be even if $f(-x) = f(x)$ and is said to be odd if $f(-x) = -f(x)$.

Note :

i) $f(x) = x^2$, constant, x sin x, cos x in $(-l, l)$ are even functions of x.

ii) $f(x) = x$, sin x, tan x, in $(-l, l)$ are odd functions of x.

iii) If f(x) is even i.e. the values of y for $-x$ and x are same, therefore the graph of $y = f(x)$ is symmetric about y-axis.

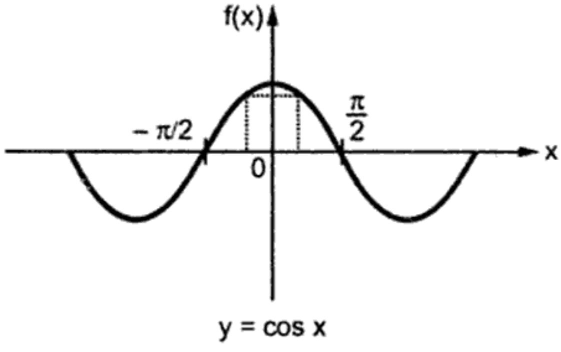


Fig. 3.11

For example : The graph of $y = \cos x$ is symmetric about y-axis.

iv) If f(x) is odd i.e. the values of y for $-x$ and x differ by sign only, therefore the graph of $y = f(x)$ is symmetric about origin (or in oppposite quadrants).

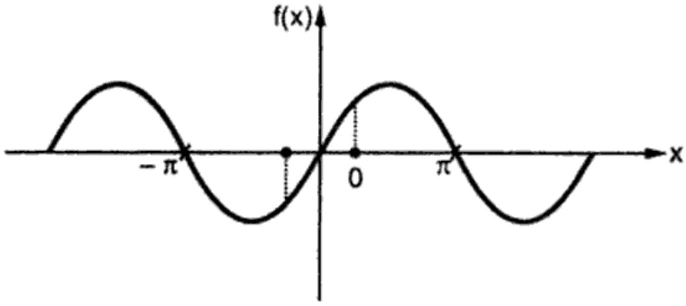


Fig. 3.12

For example : The graph of $y = \sin x$ is symmetric about opposite quadrants.

v) $$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

If f(x) is an even function of x.

vi) $$\int_{-a}^{a} f(x)\,dx = 0$$

If f(x) is an odd function of x.

vii) Algebraic properties of even and odd functions.

	f(x)	g(x)	f(x) ± g(x)	$f(x) \overset{\times}{\underset{\div}{}} g(x)$
1)	even	even	even	even
2)	odd	odd	odd	even
3)	even	odd	neither odd nor even	odd
4)	odd	even	neither odd nor even	odd

viii) Any function f(x) can be expressed as a sum of even and odd functions

i.e. $$f(x) = \left[\frac{f(x)+f(-x)}{2}\right] + \left[\frac{f(x)-f(-x)}{2}\right]$$

$$= F_1(x) + F_2(x)$$

where $F_1(x)$ is an even function of x and $F_2(x)$ is an odd function of x.

3.5 Important Definitions

1) Continuous function : f(x) is said to be continuous at x = a if $\lim_{x \to a} f(x) = f(a)$ otherwise function is said to be discontinuous at x = a.

The function at x = a is discontinuous if any one of the following holds.

i) $\lim_{x \to a} f(x)$ is not defined (or limit does not exist)

ii) $\lim_{x \to a+} f(x) \neq \lim_{x \to a-} f(x)$

iii) $\lim_{x \to a} f(x) \neq f(a)$

iv) f(a −) or f(a +) does not exist.

Next discontinuity can be further classified into two category.

i) Finite discontinuity

ii) Infinite discontinuity.

The Jump of f(x) at x = a is denoted as J or J (a) and defined as

$$J = f(a+) - f(a-)$$

i.e. $$J = \lim_{x \to a+} f(x) - \lim_{x \to a-} f(x)$$

Note :

i) If J is finite then the discontinuity at x = a is said to be finite discontinuity in such a case the value of the function at x = a can be found as

$$f(a) = \frac{f(a+) + f(a-)}{2}$$

$$f(a) = \frac{\lim\limits_{x \to a+} f(x) + \lim\limits_{x \to a-} f(x)}{2}$$

ii) If J is infinite then the discontinuity at x = a is said to be infinite discontinuity.

For example :

1) $$f(x) = x + a \qquad -a < x < 0$$

$$= x - a \qquad 0 < x < a$$

$$f(0-) = \lim_{x \to 0-} f(x) = \lim_{x \to 0} (x + a) = a$$

$$f(0+) = \lim_{x \to 0+} f(x) = \lim_{x \to 0} (x - a) = -a$$

$$f(0-) \neq f(0+)$$

$$J(0) = f(0+) - f(0-)$$

$$= -a - a$$

$$= -2a$$

$\therefore$ f(x) has a finite discontinuity at x = a.

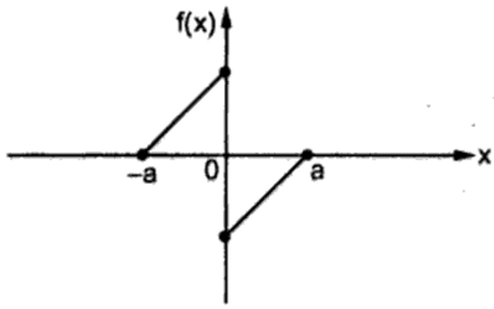


Fig. 3.13

Here $$f(0) = \frac{f(0-) + f(0+)}{2}$$

$$= \frac{a + (-a)}{2}$$

$$= 0$$

2)

$$f(x) = \tan x \qquad 0 < x < \pi$$

$$f(\pi/2-) = -\infty$$

$$f(\pi/2+) = \infty$$

Here J $\left(\frac{\pi}{2}\right)$

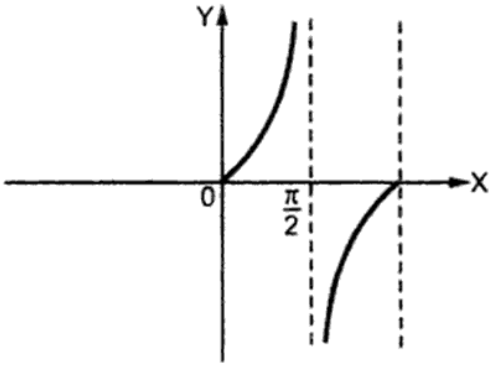


Fig. 3.14

$\therefore$ f(x) has infinite discontinuity at $x = \pi/2$.

3.6 Dirichlet's Conditions

Let f(x) be a function defined in $C < x < C + 2L$ such that

i) f(x) is defined and single valued in the given interval also $\int_{C}^{C+2L} f(x)\, dx$ exists.

ii) f(x) may have finite number of finite discontinuities in the interval.

iii) f(x) may have finite number of maxima or minima in the given interval.

3.7 Definition : Fourier Series

Let f(x) be a periodic function of period 2L defined in the interval (C, C + 2L) and satisfies Dirichlet's conditions, then f(x) can be expressed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

where a_0, a_n, b_n are called as Fourier constants or Fourier coefficients and are given by

$$a_0 = \frac{1}{L}\int_{C}^{C+2L} f(x)\, dx$$

$$a_n = \frac{1}{L}\int_{C}^{C+2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L}\int_{C}^{C+2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Depending upon the values of C and L we get the following table 3.1 and 3.2 of Fourier constants.

Name of the series	Interval	Fourier constants	Expression for the Fourier series
Fourier series	$0 \le x \le 2L$	$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$ $a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ $b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right)$
Fourier series	$-L \le x \le L$ f(x) neither even nor odd	$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$ $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right)$
Fourier series	$-L \le x \le L$ f(x) is an even function.	$a_0 = \frac{2}{L}\int_0^{L} f(x)\,dx$ $a_n = \frac{2}{L}\int_0^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ $b_n = 0$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right)\right)$
Half range cosine series	$0 \le x \le L$		
Fourier series	$-L \le x \le L$ f(x) is an odd function	$a_0 = 0$ $a_n = 0$ $b_n = \frac{2}{L}\int_0^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$	$f(x) = \sum_{n=1}^{\infty}\left(b_n\sin\left(\frac{n\pi x}{L}\right)\right)$
Half range sine series	$0 \le x \le L$		

Table 3.1

Name of the series	Interval	Fourier constants	Expression for the Fourier series
Fourier series general formula	$c \le x \le c + 2\pi$	$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$ $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$ $b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$

Fourier series	$0 \le x \le 2\pi$	$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$ $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$ $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$
Fourier series	$-\pi \le x \le \pi$ f(x) neither even nor odd	$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$ $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$
Fourier series	$-\pi \le x < \pi$ f(x) is an odd function	$a_0 = 0$ $a_n = 0$ $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$	$f(x) = \sum_{n=1}^{\infty}(b_n \sin nx)$
Half range sine series	$0 \le x \le \pi$		
Fourier series	$-\pi \le x \le \pi$ f(x) is an even function	$b_n = 0$ $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$ $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx)$
Half range cosine series	$0 \le x \le \pi$		

Table 3.2

3.8 Definition : Fourier Series in the Interval $(-\pi, \pi)$ or $(0, 2\pi)$

If f(x) is a periodic function with period 2π, defined in the interval $c \le x \le x + 2\pi$ and satisfies the Dirchlet's conditions then f(x) can be represented by a series.

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots..$$

i.e.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

This representation of f(x) is called Fourier series and a_0, a_n, b_n are called the Fourier coefficients.

Note 1 : $\sin^{-1}x$ cannot have a Fourier series expansion in any interval, since it is not a single-valued function.

Note 2 : tan x cannot have a Fourier series expansion in $(0, 2\pi)$, since it becomes infinite at $x = \pi/2$.

3.9 Fourier Convergence Theorem

Let f(x) be a periodic function of period '2L' and satisfies Dirichlet's conditions in (C, C + 2L) then the Fourier series converges to f(x) at every point x at which f is continuous, and to the mean value $\dfrac{f(x+)+f(x-)}{2}$ at every point x at which f is discontinuous.

Type 1 (a) : Examples involving exponential functions in the interval $(0,2\pi)$ or $(-\pi,\pi)$.

Example 3.3 : *Find the Fourier series to represent $f(x) = e^{ax}$ in the interval $-\pi < x < \pi$.*

Solution : As the interval is $-\pi < x < \pi$ we must check for even and odd.

As $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$

$f(x) = e^{ax}$ is neither odd nor even function.

$$\therefore \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{a\pi}\left[e^{a\pi} - e^{-a\pi}\right]$$

$$= \frac{2\sinh a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\frac{e^{ax}}{a^2+n^2}\left[a\cos nx + n\sin nx\right]\right\}_{-\pi}^{\pi}$$

$$= \frac{2a(-1)^n\left[e^{a\pi} - e^{-a\pi}\right]}{\pi(a^2+n^2)} \qquad \left\{\begin{array}{l}\text{As } \cos n\pi = (-1)^n \\ \quad\;\; \sin n\pi = 0\end{array}\right.$$

$$= \frac{2a(-1)^n \sinh a\pi}{\pi(a^2+n^2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\frac{e^{ax}}{a^2+n^2}[a\sin nx - n\cos nx]\right\}_{-\pi}^{\pi}$$

$$= \frac{-n(-1)^n[e^{a\pi} - e^{-a\pi}]}{\pi(a^2+n^2)} \qquad \begin{cases}\text{As} & \cos n\pi = (-1)^n \\ & \sin n\pi = 0\end{cases}$$

$$= \frac{-2n(-1)^n \sinh a\pi}{\pi(a^2+n^2)}$$

The Fourier series formula for f(x) in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

Substituting the values of a_0, a_n, b_n we get

$$e^{ax} = \frac{\sinh a\pi}{a\pi} + \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}[a\cos nx - n\sin nx]$$

Example 3.4 : *Find the Fourier series of* $f(x) = e^{-x}$ $\quad 0 \le x \le 2\pi$ *and* $f(x+2\pi) = f(x)$

Solution : As the interval is $0 \le x \le 2\pi$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

and the Fourier series formula in $(0, 2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$\therefore \quad a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}dx = \frac{1}{\pi}\left(\frac{e^{-x}}{-1}\right)_0^{2\pi}$$

$$= \frac{1}{\pi}(1 - e^{-2\pi})$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\frac{e^{-x}}{1+n^2}[-\cos nx + n\sin nx]\right\}_0^{2\pi}$$

$$= \frac{1-e^{-2\pi}}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\frac{e^{-x}}{1+n^2}[(-1)\sin nx - n\cos nx]\right\}_0^{2\pi}$$

$$= \frac{n(1-e^{-2\pi})}{\pi(1+n^2)}$$

Substituting a_0, a_n, b_n in Fourier series formula we get.

$$e^{-x} = \frac{1-e^{-2\pi}}{2\pi} + \left(\frac{1-e^{-2\pi}}{\pi}\right)\sum_{n=1}^{\infty}\frac{1}{1+n^2}(\cos nx + n\sin nx)$$

Example 3.5 : *Prove that if* $-\pi < x < \pi$, $f(x+2\pi) = f(x)$.

i) $\cosh ax = \dfrac{2a\sinh a\pi}{\pi}\left[\dfrac{1}{2a^2} + \displaystyle\sum_{n=1}^{\infty}\dfrac{[(-1)^n]}{(n^2+a^2)}\cos nx\right]$

ii) $\sinh ax = \dfrac{2\sinh a\pi}{\pi}\left[\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}n\sin nx}{n^2+a^2}\right]$

Solution : As the interval is $-\pi < x < \pi$ we must check whether f(x) is even or odd function.

i) $f(x) = \cosh ax$, As $\cosh(ax) = \cosh(-ax)$

$\therefore$ f(x) is an even function of x.

Thus $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = 0$$

$\therefore$ The Fourier series formula becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \text{... (1)}$$

Now
$$a_0 = \frac{2}{\pi}\int_0^{\pi} \cosh ax\,dx$$

$$= \frac{2}{\pi}\left[\frac{\sinh ax}{a}\right]_0^{\pi}$$

$$= \frac{2\sinh a\pi}{a\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cosh ax \cdot \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{e^{ax} + e^{-ax}}{2}\right)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} e^{ax}\cos nx\,dx + \int_0^{\pi} e^{-ax}\cos nx\,dx\right\}$$

Consider

$$\int_0^{\pi} e^{ax}\cos nx\,dx = \left\{\frac{e^{ax}}{a^2+n^2}[a\cos nx + n\sin nx]\right\}_0^{\pi}$$

$$= \frac{a}{a^2+n^2}[e^{a\pi}(-1)^n - 1]$$

Replacing a by – a we get

$$\int_0^{\pi} e^{-ax}\cos nx\,dx = \frac{-a}{a^2+n^2}[e^{-a\pi}(-1)^n - 1]$$

$$\therefore \qquad a_n = \frac{1}{\pi}\left\{\frac{a}{a^2+n^2}[e^{a\pi}(-1)^n - 1] - \frac{a}{a^2+n^2}[e^{-a\pi}(-1)^n - 1]\right\}$$

$$= \frac{a(-1)^n}{\pi(a^2 + n^2)} [e^{a\pi} - e^{-a\pi}]$$

$$a_n = \frac{2a(-1)^n \sinh a\pi}{\pi(a^2 + n^2)}$$

Substituting in (1) the Fourier series is

$$\cosh ax = \frac{\sinh a\pi}{a\pi} + \frac{2a \sinh a\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 + n^2} \cos nx$$

$$= \frac{2a \sinh a\pi}{\pi} \left[\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 + n^2} \cos nx \right]$$

ii) As $f(x) = \sinh(ax)$ and $\sinh(-ax) = -\sinh(ax)$

$\therefore f(x)$ is an odd function of x in $-\pi < x < \pi$

$\therefore$ $a_0 = 0$

$$a_n = 0$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

and the Fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

Now $\quad b_n = \frac{2}{\pi} \int_0^{\pi} \sinh ax \cdot \sin nx \, dx$

$$= \frac{2}{\pi} \int_0^{\pi} \left(\frac{e^{ax} - e^{-ax}}{2} \right) \sin nx \, dx$$

$$= \frac{1}{\pi} \left\{ \int_0^{\pi} e^{ax} \sin nx \, dx - \int_0^{\pi} e^{-ax} \sin nx \, dx \right\}$$

Consider

$$\int_0^{\pi} e^{ax} \sin nx \, dx = \left\{ \frac{e^{ax}}{a^2 + n^2} [a \sin nx - n \cos nx] \right\}_0^{\pi}$$

$$= \frac{n}{a^2 + n^2} [1 - e^{a\pi} (-1)^n]$$

Replacing a by – a we get

$$\int_0^{\pi} e^{-ax} \sin nx \, dx = \frac{n}{a^2 + n^2} [1 - e^{-a\pi} (-1)^n]$$

$$\therefore \quad b_n = \frac{1}{\pi} \cdot \frac{n}{a^2 + n^2} \{[1 - e^{a\pi} (-1)^n] - [1 - e^{-a\pi} (-1)^n]\}$$

$$b_n = \frac{n}{\pi(a^2 + n^2)} \cdot (-1)(-1)^n [e^{a\pi} - e^{-a\pi}]$$

$$= \frac{2n(-1)^{n+1} \sinh a\pi}{\pi(a^2 + n^2)}$$

Substituting in (1) we get

$$\sinh ax = \frac{2 \sinh a\pi}{\pi} \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{a^2 + n^2} \cdot \sin nx$$

Example 3.6 : *Prove that in the interval $0 < x < \pi$* **[May-94, Dec.-94, May-97]**

$$\frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi}\left[\frac{\sin x}{a^2 + 1} - \frac{2\sin 2x}{a^2 + 4} + \frac{3\sin 3x}{a^2 + 9} \; .....\right]$$

Solution : As the interval is $0 < x < \pi$ and the series involves only sine terms. $\therefore$ It is asked to find Fourier half range sine series for the given function $\frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}}$.

i.e. $\frac{2 \sinh ax}{2 \sin h\, a\pi}$

$\therefore$ To show that

$$\frac{\sinh ax}{\sinh a\pi} = \frac{2}{\pi}\left[\frac{\sin x}{a^2 + 1} - \frac{2 \sin 2x}{a^2 + 4} + \frac{3 \sin 3x}{a^2 + 9} - ....+....\right]$$

It is enough to show that

$$\sinh ax = \frac{2 \sinh a\pi}{\pi}\left[\frac{\sin x}{a^2 + 1} - \frac{2\sin 2x}{a^2 + 4} + \frac{3 \sin 3x}{a^2 + 9} - ....+....\right]$$

i.e To expand $f(x) = \sinh ax$ as a half range Fourier sine series in $0 < x < \pi$.

$$\therefore \quad a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$\therefore \qquad b_n = \frac{2}{\pi}\int_0^{\pi} \sinh ax \cdot \sin nx \, dx$$

$$= \frac{2n(-1)^{n+1}\sinh a\pi}{\pi(a^2+n^2)} \qquad \text{from previous problem.}$$

$\therefore$ The Fourier series becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

i.e.
$$\sinh ax = \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{a^2+n^2}\sin nx$$

$$\therefore \qquad \frac{\sinh ax}{\sinh a\pi} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{a^2+n^2}\sin nx$$

Substituting n = 1, 2, 3, ... we get,

$$\frac{\sinh ax}{\sinh a\pi} = \frac{2}{\pi}\left[\frac{\sin x}{a^2+1} - \frac{2\sin 2x}{a^2+4} + \frac{3\sin 3x}{a^2+9} - \ldots + \ldots\right]$$

Type 1 (b) : Example involving exponential functions in an arbitrary interval [i.e. (0, 2*l*) or (– *l*, *l*)].

Example 3.7 : *Obtain the Fourier series of* $f(x) = e^{|x|}, -2 \le x \le 2$

Solution : As the interval is (– 2, 2) we must check for even and odd.

As $\qquad f(x) = e^{|x|}$

and $\qquad f(-x) = e^{|-x|} = f(x) \qquad \therefore f(x)$ is an even function of x.

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

b_n = 0 and the Fourier series formula is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Here the interval is (– 2, 2) i.e. (– L, L) $\Rightarrow$ L = 2

Now
$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_0 = \frac{2}{2}\int_0^2 e^x\,dx$$

$$= (e^2 - 1)$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{2}\int_0^2 e^x \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \left\{\frac{e^x}{1+\left(\frac{n\pi}{2}\right)^2}\left[\cos\frac{n\pi x}{2} + \frac{n\pi}{2}\sin\frac{n\pi x}{2}\right]\right\}_0^2$$

$$= \frac{4}{(4+n^2\pi^2)}\{e^2[(-1)^n + 0] - e^0[1+0]\}$$

$$= \frac{4}{4+n^2\pi^2}[(-1)^n e^2 - 1] \qquad \left\{\begin{array}{ll}\text{As} & \cos n\pi = (-1)^n \\ & \sin n\pi = 0\end{array}\right\}$$

Substituting in Fourier series

$$e^{|x|} = \frac{e^2-1}{2} + \sum_{n=1}^{\infty} \frac{4[(-1)^n e^2 - 1]}{4+n^2\pi^2} \cdot \cos\left(\frac{n\pi x}{2}\right)$$

Exercise 3.1 : Problems on Type 1

1) *Obtain the Fourier series of following f (x)*

$f(x) = e^x \qquad -1 \le x \le 1$

$$\left[\textbf{Ans.}: e^x = \frac{(e-e^{-1})}{2} + (e-e^{-1})\sum_{n=1}^{\infty}\left[\frac{(-1)^n \cos n\pi x}{1+n^2\pi^2} - \frac{n\pi(-1)^n \sin n\pi x}{1+n^2\pi^2}\right]\right]$$

2) $f(x) = e^{-x}$ in $-l \le x \le l$

$$\left[\text{Ans.}: e^{-x} = \sinh hl\left\{\frac{1}{l} + \sum_{n=1}^{\infty}\left[\frac{2l(-1)^n}{l^2+n^2\pi^2}\cos\frac{n\pi x}{l} + \frac{2\pi n(-1)^n}{l^2+n^2\pi^2}\sin\frac{n\pi x}{l}\right]\right\}\right]$$

3) $f(x) = \begin{cases} e^x & -1 \le x \le 0 \\ e^{-x} & 0 \le x \le 1 \end{cases}$ period 2

Hint : f(x) is an even function of x.

$$\left[\text{Ans.}: f(x) = \frac{e-1}{e} + 2\left(\frac{e-1}{e}\right)\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2\pi^2}\cos n\pi x\right]$$

4) $f(x) = \sinh ax \quad 0 < x < \pi$ *find half range cosine series.*

$$\left[\text{Ans.}: a_0 = \frac{2}{\pi}[\cosh a\pi - 1],\ a_n = \frac{2a}{\pi(a^2+n^2)}[(-1)^n\cosh a\pi - 1]\right]$$

5) $f(x) = \cosh ax \quad 0 < x < \pi$ *find half range sine series.*

$$\left[\text{Ans.}: b_n = \frac{2n}{\pi(a^2+n^2)}[1-(-1)^n\cosh a\pi]\right]$$

Type 2 (a) : Examples involving sin x or cos x in the interval $(0, 2\pi)$ or $(-\pi, \pi)$.

Note : We need the following integral

$$I = \int_0^{\pi} \sin x \cos nx \, dx$$

$$= \frac{1}{2}\int_0^{\pi}[\sin(1+n)x + \sin(1-n)x]\,dx$$

$$= \frac{1}{2}\left[\frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= \frac{1}{2}\left[\frac{(-1)^{1+n}}{1+n}(-1) - \frac{(-1)^{1-n}}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= \frac{1}{2}\left[\frac{(-1)^n+1}{1+n} + \frac{(-1)^n+1}{1-n}\right] \qquad \text{As } (-1)^n = (-1)^{-n}$$

$$= \frac{1+(-1)^n}{1-n^2} \qquad n \neq 1 \qquad \dots (1)$$

Note : If f(x) involves sin x or cos x then find a_1, b_1.

Example 3.8 : *An alternating current I after passing the rectifier has the form* **[May-98, May-03]**

$$I = \begin{cases} I_0 \sin x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$$

where I_0 is the maximum current and the period 2π. Express I as a Fourier series. Also graph the function and deduce that

$$\frac{1}{2} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} \ \ldots\ldots$$

Solution : As the interval is $(0, 2\pi)$ we have to find a_0, a_n, b_n.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} I_0 \sin x\,dx + \int_{\pi}^{2\pi} 0\,dx\right\}$$

$$= \frac{I_0}{\pi}[-\cos x]_0^{\pi} = \frac{2 I_0}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} I_0 \sin x \cos nx\,dx + \int_{\pi}^{2\pi} 0\,dx\right\}$$

$$= \frac{I_0}{\pi}\left[\frac{1 + (-1)^n}{1 - n^2}\right] + 0 \qquad \text{.... From integral (1)}$$

As the above integral is not valid if n = 1 $\therefore$ at n = 1

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} I_0 \sin x \cos x\,dx + \int_{\pi}^{2\pi} 0\,dx\right\}$$

$$= \frac{I_0}{\pi}\left[\frac{\sin^2 x}{2}\right]_0^{\pi} + 0 \qquad \left\{\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right\}$$

$$= 0$$

Now
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} I_0 \sin x \sin nx\,dx + \int_{\pi}^{2\pi} 0\,dx$$

$$= \frac{I_0}{2\pi}\int_0^{\pi} [\cos(1-n)x - \cos(1+n)x]\,dx$$

$$= \frac{I_0}{2\pi}\left[\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right]_0^{\pi}$$

$$= 0 \qquad \text{if } n \neq 1.$$

$\therefore$ At $n = 1$.

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} I_0 \sin x \cdot \sin x\,dx + \int_{\pi}^{2\pi} 0$$

$$= \frac{I_0}{\pi}\cdot 2\int_0^{2\pi} \sin^2 x\,dx \qquad \{\text{Using conversion formula}\}$$

$$= \frac{I_0}{\pi}\cdot 2\cdot\left(\frac{1}{2}\cdot\frac{\pi}{2}\right) \qquad \{\text{Using reduction formula}\}$$

$$= \frac{I_0}{2}$$

The formula for Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \sum_{n=2}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Substituting all the Fourier constants we get

$$I = \frac{I_0}{\pi} + \frac{I_0}{2}\sin x + \sum_{n=2}^{\infty}\frac{I_0}{\pi}\left[\frac{1+(-1)^n}{1-n^2}\right]\cos nx$$

is the required Fourier series.

To find the deduction put $x = 0$.

$$0 = \frac{I_0}{\pi} + 0 + \sum_{n=2}^{\infty} \frac{I_0(1 + (-1)^n)}{\pi(1 - n^2)} \cdot 1$$

$$-\frac{I_0}{\pi} = \frac{I_0}{\pi} \sum_{n=2}^{\infty} \frac{1 + (-1)^n}{(1-n)(1+n)}$$

i.e.
$$1 = \sum_{n=2}^{\infty} \frac{1 + (-1)^n}{(n-1)(n+1)}$$

Put n = 2, 3, 4, 5 ......

$$1 = \frac{2}{1.3} + 0 + \frac{2}{3.5} + 0 + \frac{2}{5.7} + \ldots\ldots$$

i.e.
$$\frac{1}{2} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots\ldots$$

is the required deduction.

The graph of f(x) is

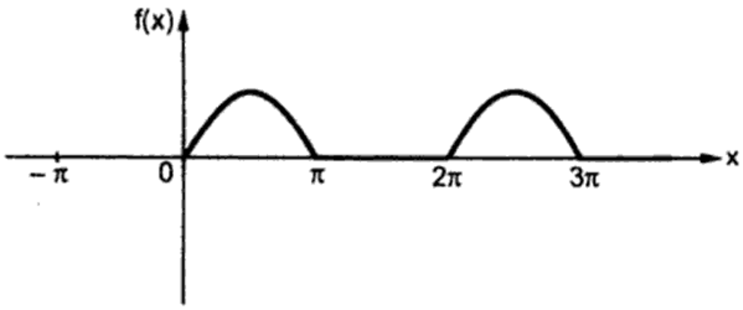


Fig. 3.15

Example 3.9 : ***Find cosine series for sin x in $0 < x < \pi$ hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots\ldots$ and graph the function for cosine series.***

[Dec.-92, May-93, May-96, Dec.- 99]

Solution : The Fourier cosine series formula is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx$$

$$b_n = 0$$

Now
$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x \, dx = \frac{2}{\pi}(-\cos x)_0^{\pi}$$

$$= -\frac{2}{\pi}(\cos\pi - \cos 0) = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\frac{1+(-1)^n}{1-n^2}\right] \qquad n \neq 1 \text{ from integral (1)}$$

$\therefore$ At $n = 1$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx$$

$$= \frac{2}{\pi}\left[\frac{\sin^2 x}{2}\right]_0^{\pi} = 0$$

$$b_n = 0$$

Thus the Fourier cosine series is given by

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$f(x) = \frac{2}{\pi} + 0 + \sum_{n=2}^{\infty} \frac{2}{\pi} \frac{1+(-1)^n}{(1-n^2)} \cos nx$$

$$f(x) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{1+(-1)^n}{(n-1)(n+1)} \cos nx$$

To find the deduction put $x = \pi/2$.

$$f\left(\frac{\pi}{2}\right) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{1+(-1)^n}{2}\left[\frac{1}{n-1} - \frac{1}{n+1}\right]\cos\frac{n\pi}{2}$$

As
$$f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

and
$$\frac{1}{(n-1)(n+1)} = \frac{1}{2}\left[\frac{1}{n-1} - \frac{1}{n+1}\right]$$

$$(1)\frac{\pi}{2} = 1 - \sum_{n=2}^{\infty} \frac{1+(-1)^n}{2}\left[\frac{1}{n-1} - \frac{1}{n+1}\right]\cos\frac{n\pi}{2}$$

Put n = 2, 3, 4 ....

$$\frac{\pi}{2} = 1 - \left[-\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) - \left(\frac{1}{5} - \frac{1}{7}\right) + \ldots\ldots \right]$$

$$= 2 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} \ldots\ldots$$

i.e.

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots\ldots$$

We know that graph of cosine series must be symmetric about y-axis (As the function is even for cosine series).

We know the actual graph for

y = sin x

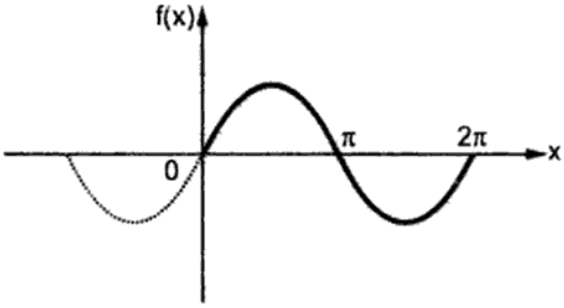


Fig. 3.16

∴ The graph for

y = sin x as a cosine series will be symmetric about y-axis.

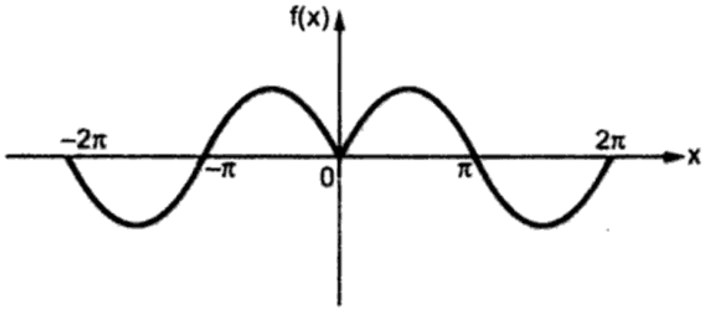


Fig. 3.17

➡ **Example 3.10 :** *Show that if $0 < x < \pi$* **[Dec.-91, May-2003, Dec.-2003]**

$$\cos x = \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{m}{4m^2 - 1} \sin 2mx$$

Also graph the function $y = f(x)$ as a sine series.

Solution : As the series involves all sine terms we have to find half range sine series for $y = \cos x$ in $0 < x < \pi$.

$$\therefore \quad a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx$$

and Fourier series formula for sine series is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x + \sin(n-1)x]\, dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^{n+1}+1}{n+1} + \frac{-(-1)^{n-1}+1}{n-1}\right]$$

$$= \frac{1}{\pi}[1+(-1)^n]\left[\frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1+(-1)^n}{\pi}\cdot\left(\frac{2n}{n^2-1}\right), \qquad (n \neq 1)$$

$\because$ At $n = 1$, we have

$$b_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\sin x\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos x \sin x\, dx$$

$$= \frac{2}{\pi}\left[\frac{\sin^2 x}{2}\right]_0^{\pi} = 0$$

$\therefore$ The Fourier series formula becomes

$$f(x) = b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

Substituting Fourier constants we get

$$\cos x = 0 + \sum_{n=2}^{\infty} \frac{2n}{\pi} \cdot \frac{1+(-1)^n}{n^2-1} \cdot \sin nx$$

The term $1+(-1)^n = 0$ if n odd

$= 2$ if n even

$\therefore$ Replacing n by 2m we get

$$\cos x = \sum_{2m=2}^{\infty} \frac{2(2m)}{\pi} \cdot \frac{2}{(2m)^2-1} \cdot \sin 2mx$$

$$= \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{m}{4m^2-1} \cdot \sin 2mx$$

We know that the graph for sine series must be symmetric about opposite quadrants. (As the function is odd for sine series)

We know the actual graph for

$y = \cos x$

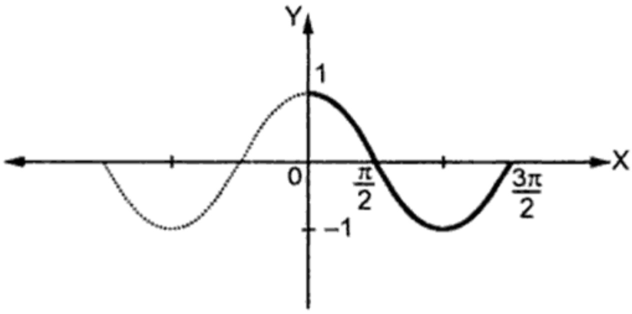


Fig. 3.18

$\therefore$ The graph for $y = \cos x$ as a sine series must be symmetric about the opposite quadrants.

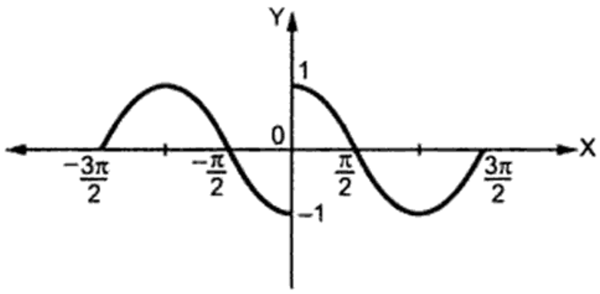


Fig. 3.19

Example 3.11 : *Obtain Fourier series expansion of f(x) = cos ax in $(0, 2\pi)$, where a is not an integer.*

Deduce that $\pi \cot 2\pi a = \frac{1}{2a} + \sum_{n=1}^{\infty} \frac{a}{a^2-n^2}$

[May-2000, Dec.-92]

Solution : As 'a' is not an integer.

$\therefore \qquad \sin a\pi \neq 0$

and $\qquad \cos a\pi \neq (-1)^a$.

Given interval is $(0, 2\pi)$

$$\therefore \qquad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \cos ax\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin ax}{a}\right]_0^{2\pi}$$

$$= \frac{\sin 2a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \cos ax \cdot \cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} [\cos(a+n)x + \cos(a-n)x]\,dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(a+n)x}{a+n} + \frac{\sin(a-n)x}{a-n}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\sin 2(a+n)\pi}{a+n} + \frac{\sin 2(a-n)\pi}{a-n}\right]$$

Now $\quad \sin 2(a+n)\pi = \sin(2a\pi + 2n\pi)$

$$= \sin 2a\pi \cdot \cos 2n\pi + \sin 2n\pi \cdot \cos 2a\pi$$

$$= \sin 2a\pi\,(-1)^{2n} + 0$$

$$= \sin 2a\pi$$

Similarly

$$\sin 2(a-n)\pi = \sin 2a\pi$$

$$\therefore \qquad a_n = \frac{1}{2\pi}\left[\frac{\sin 2a\pi}{a+n} + \frac{\sin 2a\pi}{a-n}\right]$$

$$= \frac{a \sin 2a\pi}{\pi(a^2 - n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \cos ax \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} [\sin(n+a)x + \sin(n-a)x] \, dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(n+a)x}{n+a} - \frac{\cos(n-a)x}{n-a}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-\cos 2(n+a)\pi + 1}{n+a} + \frac{-\cos 2(n-a)\pi + 1}{n-a}\right]$$

$$\cos 2(n+a)\pi = \cos(2n\pi + 2a\pi)$$

$$= \cos 2n\pi \cos 2a\pi - \sin 2n\pi \sin 2a\pi$$

$$= (-1)^{2n} \cdot \cos 2a\pi - 0$$

$$= \cos 2a\pi$$

Similarly

$$\cos 2(n-a)\pi = \cos 2a\pi$$

$$\therefore \qquad b_n = \frac{1}{2\pi}\left[\frac{-\cos 2a\pi + 1}{n+a} + \frac{-\cos 2a\pi + 1}{n-a}\right]$$

$$= \frac{(1 - \cos 2a\pi)}{2\pi}\left[\frac{1}{n+a} + \frac{1}{n-a}\right]$$

$$= \frac{n(1 - \cos 2a\pi)}{\pi(n^2 - a^2)}$$

The formula for Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Substituting the Fourier coefficients we get

$$\cos ax = \frac{\sin 2a\pi}{2\pi a} + \frac{a \sin 2a\pi}{\pi} \sum_{n=1}^{\infty} \frac{\cos nx}{a^2 - n^2}$$

$$+ \frac{n(1 - \cos 2a\pi)}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n^2 - a^2}$$

To find the deduction put x = 2π.

$$\cos 2\pi a = \frac{\sin 2\pi a}{2\pi a} + \frac{a \sin 2\pi a}{\pi} \sum_{n=1}^{\infty} \frac{1}{a^2 - n^2}$$

$$\therefore \quad \pi \cot 2\pi a = \frac{1}{2a} + a \sum_{n=1}^{\infty} \frac{1}{a^2 - n^2}$$

Example 3.12 : *Prove that if $-\pi < x < \pi$ and a is not an integer*

i) $\sin ax = \frac{2\sin a\pi}{-\pi}\left[\frac{\sin x}{1^2 - a^2} - \frac{2\sin 2x}{2^2 - a^2} + \frac{3\sin 3x}{3^2 - a^2} \cdots\cdots\right]$

ii) $\cos ax = \frac{2a \sin a\pi}{\pi}\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 - n^2} \cos nx\right]$

Solution : As the interval is $-\pi < x < \pi$

∴ We must check for even and odd.

i) $f(x) = \sin ax \qquad -\pi < x < \pi$

is an odd function of x.

$$\therefore \quad a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin ax \cdot \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} [\cos(a - n)x - \cos(a + n)x] \, dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(a - n)\pi}{a - n} - \frac{\sin(a + n)\pi}{a + n}\right]$$

Now $\sin(a+n)\pi$

$$= \sin a\pi \cos n\pi + \cos a\pi \sin n\pi$$

$$= \sin a\pi(-1)^n + 0$$

Similarly

$$\sin(a-n)\pi = \sin a\pi\ (-1)^n$$

$$\therefore \quad b_n = \frac{1}{\pi}\left[\frac{\sin a\pi(-1)^n}{a-n} - \frac{\sin a\pi(-1)^n}{a+n}\right]$$

$$b_n = \frac{\sin a\pi(-1)^n}{\pi}\left[\frac{1}{a-n} - \frac{1}{a+n}\right]$$

$$= \frac{\sin a\pi(-1)^n}{\pi}\left[\frac{2n}{a^2-n^2}\right]$$

$\therefore$ The Fourier series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{becomes}$$

$$\sin ax = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2-n^2}\sin nx$$

$$= \frac{2\sin a\pi}{-\pi}\left[\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2}\cdots\right]$$

ii) $\quad f(x) = \cos ax \qquad -\pi < x < \pi$

is an even function of x.

$$\therefore \quad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos ax\ dx$$

$$= \frac{2}{\pi}\left[\frac{\sin ax}{a}\right]_0^{\pi}$$

$$= \frac{2\sin a\pi}{a\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos ax \; \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(a-n)x + \cos(a+n)x \;\; dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(a-n)x}{a-n} + \frac{\sin(a+n)x}{a+n}\right]_0^{\pi}$$

Now $\sin(a-n)\pi = \sin a\pi \cos n\pi - \cos a\pi \sin n\pi$

$$= \sin a\pi(-1)^n - 0$$

Similarly

$$\sin(a+n)\pi = \sin a\pi(-1)^n$$

$$\therefore \quad a_n = \frac{1}{\pi}\left[\frac{\sin a\pi(-1)^n}{a-n} + \frac{\sin a\pi(-1)^n}{a+n}\right]$$

$$= \frac{\sin a\pi(-1)^n}{\pi} \cdot \frac{2a}{a^2 - n^2}$$

$$b_n = 0$$

$\therefore$ The Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \text{becomes}$$

$$\cos ax = \frac{\sin a\pi}{a\pi} + \sum_{n=1}^{\infty} \frac{2 \sin a\pi \cdot a(-1)^n}{\pi(a^2 - n^2)} \cos nx$$

$$= \frac{2a \sin a\pi}{\pi}\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 - n^2} \cos nx\right]$$

Example 3.13 : ***Obtain the Fourier series for*** **[May-99]**
$f(x) = x \sin x$ *in* $-\pi < x < \pi$

Solution : As the interval is $-\pi < x < \pi$ we must check whether the function is odd or even.

Now $f(-x) = (-x) \sin(-x)$

$$= x \sin x$$

$$= f(x)$$

⇒ Even function

$$\therefore \quad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin x\,dx$$

$$= \frac{2}{\pi}[x(-\cos x) - (1)(-\sin x)]_0^{\pi}$$

$$= \frac{2}{\pi}[(\pi + 0) - (0 + 0)]$$

$$= 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(1+n)x + \sin(1-n)x]\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \sin(1+n)x\,dx + \int_0^{\pi} x \sin(1-n)x\,dx\right]$$

Consider $\int_0^{\pi} x \sin(1+n)x\,dx$

$$= \left[x\left(\frac{-\cos(1+n)x}{1+n}\right) - (1)\left(\frac{-\sin(1+n)x}{(1+n)^2}\right)\right]_0^{\pi}$$

$$= \frac{-\pi(-1)^{1+n}}{1+n} = \frac{\pi(-1)^n}{1+n}$$

Similarly

$$\int_0^{\pi} x \sin(1-n)x\,dx = \frac{\pi(-1)^n}{1-n}$$

$$\therefore \quad a_n = \frac{1}{\pi}\left[\frac{\pi(-1)^n}{1+n} + \frac{\pi(-1)^n}{1-n}\right]$$

$$= (-1)^n \cdot \frac{2}{1-n^2} \qquad n \neq 1$$

$$= \frac{2(-1)^n}{1-n^2} \qquad n \neq 1$$

When n = 1

$$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x) \cos x \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi(1)}{2} - 0\right]$$

$$= \frac{-1}{2}$$

$$b_n = 0$$

$\therefore$ The Fourier series formula is

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

Substituting Fourier coefficients we get.

$$x \sin x = 1 + \left(\frac{-1}{2}\right)\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^n}{1-n^2} \cos nx$$

Example 3.14 : *Expand $f(x) = x \sin x$ in the interval* $0 \leq x \leq 2\pi$.
[May-2004, Dec.-2005, Dec.-2007]

Solution : As the interval is $(0, 2\pi)$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \, dx$$

$$= \frac{1}{\pi}[x(-\cos x) - (1)(-\sin x)]_0^{2\pi}$$

$$= \frac{1}{\pi}[(-2\pi\cos 2\pi + 0) - (0 + 0)] = \frac{1}{\pi}[-2\pi]$$

$$= -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x \cos nx)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,[\sin(n+1)x - \sin(n-1)x]\; dx$$

$$= \frac{1}{2\pi}\Bigg[x\left\{-\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right\}$$

$$-(1)\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\Bigg]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[(2\pi)\left\{-\frac{\cos 2(n+1)\pi}{(n+1)} + \frac{\cos 2(n-1)\pi}{(n-1)}\right\} - 0\right]$$

$$= -\frac{1}{(n+1)} + \frac{1}{(n-1)} \qquad n \neq 1$$

As $\cos 2\,(n \pm 1)\,\pi = 1$

$$= \frac{2}{n^2 - 1}, \quad (n \neq 1)$$

When $n = 1$,

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x\, dx = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x\, dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{-2\pi\left(\frac{\cos 4\pi}{2}\right) + 0\right\} - 0\right] = \frac{1}{2\pi}[-\pi]$$

$$= -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x\sin nx)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\cos(n-1)x - \cos(n+1)x]\,dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{(n-1)} - \frac{\sin(n+1)x}{(n+1)}\right\}\right.$$

$$\left. - (1)\left\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{0 + \frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2}\right\}\right.$$

$$\left. - \left\{0 + \frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right\}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

As $\cos 2(n \pm 1)\pi = 1$

$$= 0, \text{ for } n > 1$$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1-\cos 2x}{2}\right)dx = \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\,dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - (1)\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{2\pi(2\pi) - \frac{4\pi^2}{2} - \frac{1}{4}\right\} - \left\{0 - \frac{1}{4}\right\}\right] = \frac{1}{2\pi}[2\pi^2]$$

$$= \pi$$

The Fourier series formula is

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \sum_{n=2}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Substituting Fourier coefficients we get

$$x \sin x = -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx$$

Type 2 (b) : Examples involving sine and cosine terms in arbitrary interval.

Example 3.15 : *Obtain Fourier expansion for sin ax, in the interval $-l < x < l$, where a is not an integer.* **[Dec.-94]**

Solution : Here f(x) = sin ax, and f(– x) = – sin ax = – f(x) is an odd function of x. Hence $a_0 = a_n = 0$, Fourier series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l \sin ax \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^l \left[\cos\left(\frac{n\pi}{l} - a\right)x - \cos\left(\frac{n\pi}{l} + a\right)x\right]dx$$

$$= \frac{1}{l}\left[\frac{l}{(n\pi - al)}\sin\left(\frac{n\pi - al}{l}\right)x - \frac{l}{(n\pi + al)}\sin\left(\frac{n\pi + al}{l}\right)x\right]_0^l$$

$$= \left[\frac{\sin(n\pi - al)}{(n\pi - al)} - \frac{\sin(n\pi + al)}{(n\pi + al)}\right] = \left[-\frac{\cos n\pi \sin al}{(n\pi - al)} - \frac{\cos n\pi \sin al}{(n\pi + al)}\right]$$

$$= -\frac{2n\pi}{(n^2\pi^2 - a^2l^2)}\cos n\pi \sin al = \frac{(-1)^{n+1}2n\pi \sin al}{(n^2\pi^2 - a^2l^2)}$$

Thus the Fourier series becomes

$$\sin ax = 2\pi \sin al \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{(n^2\pi^2 - a^2l^2)}\sin\frac{n\pi x}{l}$$

Example 3.16 : *A sinusoidal voltage E sin ωt is passed through a half-wave rectifier which clips the negative portion of the wave as given. Find the Fourier series of the resulting periodic function.*

$$f(t) = \begin{cases} 0 & -\frac{T}{2} < t < 0 \\ E \sin \omega t & 0 < t < \frac{T}{2} \end{cases} \;; \text{ where } T = \frac{2\pi}{\omega}$$

[Dec.-91]

Solution : Given f(t) is

$$f(t) = \begin{cases} 0 & -\frac{\pi}{\omega} < t < 0 \\ E \sin \omega t & 0 < t < \frac{\pi}{\omega} \end{cases}$$

Here $$L = \frac{\pi}{\omega}$$

The function is neither odd nor even.

$$\therefore \quad a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\,dt$$

$$= \frac{\omega}{\pi}\left[\int_{-\pi/\omega}^{0} (0)\,dt + \int_{0}^{\pi/\omega} E\sin\omega t\,dt\right] = \frac{\omega}{\pi}\left[-\frac{E\cos\omega t}{\omega}\right]_0^{\pi/\omega}$$

$$= \frac{2E}{\pi}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\cos(n\,\omega t)\,dt$$

$$= \frac{\omega E}{2\pi}\int_{0}^{\pi/\omega} 2\sin\omega t\cos n\omega t\,dt$$

$$\frac{\omega E}{2\pi}\int_{0}^{\pi/\omega} [\sin(1+n)\omega t + \sin(1-n)\omega t]\,dt$$

$$= \frac{\omega E}{2\pi}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$$

$$= \frac{E}{2\pi}\left[\left\{-\frac{\cos(1+n)\pi}{(1+n)} - \frac{\cos(1-n)\pi}{(1-n)}\right\} - \left\{-\frac{1}{(1+n)} - \frac{1}{(1-n)}\right\}\right]$$

$$= \frac{E}{2\pi}\left[\frac{(-1)^n}{(1+n)} + \frac{(-1)^n}{(1-n)} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

As $\{-\cos(1 \pm n)\pi = -(-1)^{n \pm 1}$

$= (-1)^n\}$

$$= \frac{E}{2\pi}\frac{2}{(1-n^2)}\,[1 + (-1)^n]$$

$$= \frac{E}{\pi} \cdot \frac{1+(-1)^n}{1-n^2}, \qquad n \neq 1$$

When n = 1,

$$a_1 = \frac{1}{L}\int_0^L f(x)\cos\frac{\pi t}{L}\,dt = \frac{\omega}{\pi}\int_0^{\pi/\omega} E\sin\omega t\cos\omega t\,dt$$

$$= \frac{\omega E}{2\pi}\int_0^{\pi/\omega}\sin 2\omega t\,dt = \frac{\omega E}{2\pi}\left[-\frac{\cos 2\omega t}{2\omega}\right]_0^{\pi/\omega} = 0$$

Now

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\sin n\omega t\,dt$$

$$= \frac{\omega E}{2\pi}\int_0^{\pi/\omega} 2\sin\omega t\sin\omega t\,dt$$

$$= \frac{\omega E}{2\pi}\int_0^{\pi/\omega}[\cos(1-n)\omega t - \cos(1+n)\omega t]\,dt$$

$$= \frac{\omega E}{2\pi}\left[\frac{\sin(1-n)\,\omega t}{(1-n)\omega} - \frac{\sin(1+n)\omega t}{(1+n)\omega}\right]_0^{\pi/\omega} \quad \text{for } n \neq 1$$

$$= 0 \qquad \text{except } n = 1$$

When n = 1

$$b_1 = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{\pi t}{L}\,dt = \frac{\omega}{\pi}\int_0^{\pi/\omega}(E\sin\omega t)\sin\omega t\,dt$$

$$= \frac{E\omega}{\pi}\int_0^{\pi}\left(\frac{1-\cos 2\omega t}{2}\right)dt = \frac{E\omega}{2\pi}\left[t - \frac{\sin 2\omega t}{2\omega}\right]_0^{\pi/\omega} = \frac{E\omega}{2\pi}\cdot\frac{\pi}{\omega}$$

$$= \frac{E}{2}$$

The Fourier series formula is

$$f(t) = \frac{a_0}{2} + a_1\cos\omega t + b_1\sin\omega t + \sum_{n=2}^{\infty}(a_n\cos n\omega t + b_n\sin n\omega t)$$

Substituting all Fourier coefficients we get

$$f(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t + \sum_{n=2}^{\infty}\frac{E}{\pi}\frac{[1+(-1)^n]}{(1-n^2)}\cdot\cos n\omega t$$

The graph of f(t) is given by

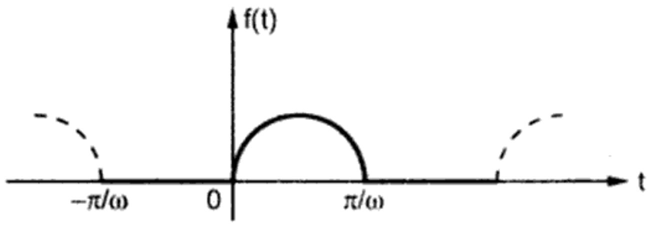


Fig. 3.20 Half-wave rectifier function

Exercise 3.2 : Problems on Type 2

1) *Express f(x) as a Fourier series and graph the function.*

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$$

Hence deduce that $\frac{1}{2} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} \ldots\ldots$

Hint : f(x) is neither even nor odd. Refer Example 3.8. **[May-2006]**

2) *Find the Fourier series for* $f(x) = |\sin x| \quad -\pi < x < \pi$

Hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots..$

Hint : f (x) is an even function of x. Refer Example 3.9.

3) *Obtain Fourier expansion for*

$$f(x) = \begin{cases} \cos x & -\pi < x < 0 \\ -\cos x & 0 < x < \pi \end{cases}$$ *and graph the function.*

Hint : f(x) is an odd function of x. Refer Example 3.10. $\left[\text{Ans.} : b_n = \frac{-2n}{\pi(n^2-1)} [1 + (-1)^n]\right]$

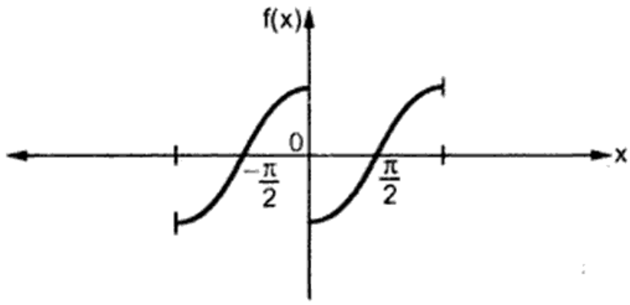


Fig. 3.21

4) *Determine the Fourier series for the following function*

$$f(x) = \begin{cases} \cos x & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$$

$\left[\text{Ans.} : a_0 = \frac{2}{\pi},\ a_1 = \frac{1}{2},\ b_1 = \frac{1}{2},\ a_n = \frac{1 + (-1)^n}{\pi(1-n^2)},\ b_n = \frac{n[1 + (-1)^n]}{\pi(1-n^2)}\right]$

5) *Prove that in the interval* $-\pi < x < \pi$

$$x \cos x = \frac{1}{2} \sin x + \sum_{n=2}^{\infty} \frac{2(-1)^n}{n^2 - 1} \sin nx$$

6) *Prove that in the interval* $-\pi < x < \pi$

$$x \sin x = 1 - \frac{1}{2} \cos x - \sum_{n=2}^{\infty} \frac{2(-1)^n}{n^2 - 1} \cos nx$$

7) *Prove that in the interval* $-\pi < x < \pi$

$$\frac{1}{2}(\pi - x) \sin x = \frac{1}{2} + \frac{1}{4} \cos x - \frac{1}{1.3} \cos 2x + \frac{1}{2.4} \cos 3x$$

8) *Determine the Fourier coefficients of the function*

$$f(x) = \begin{cases} 1 & -1 < x < 0 \\ \cos \pi x & 0 < x < 1 \end{cases}$$

what is the sum when x = 1.

$$\textbf{[Ans. :} a_0 = 1, a_n = 0, b_n = \frac{(-1)^n - 1}{n\pi} + \frac{n[1 + (-1)^n]}{\pi(n^2 - 1)}, f(1) = \frac{f(1+) + f(1-)}{2} = 0]$$

9) *Determine Fourier expansion for f(x) defined by*

$$f(x) = \begin{cases} 0 & -3 < x < -1 \\ 1 + \cos \pi x & -1 < x < 1 \\ 0 & 1 < x < 3 \end{cases}$$

Hint : f(x) is an even function of x.

$$\textbf{[Ans. :} a_0 = \frac{1}{3}, a_n = \frac{-18}{n\pi(n^2 - 9)} \sin \frac{n\pi}{3}, n \neq 3, a_3 = \frac{1}{3}]$$

10) *Expand* $f(x) = \sin^2 x$, $0 < x < \pi$ *in a half-range*

i) cosine series ii) sine series

$$\textbf{[Ans. : i)} a_0 = 1, a_2 = \frac{1}{2}, a_n = 0 \text{ if } n \neq 2$$

$$\textbf{ii)} b_n = \frac{4[(-1)^n - 1]}{n\pi(n^2 - 4)} \; n \neq 2, b_2 = 0]$$

11) $f(x) = \sin x \qquad -1 \leq x \leq 1$

$$\textbf{[Ans. :} f(x) = \sum_{n=1}^{\infty} \frac{2n\pi \sin(1)(-1)^{n+1}}{n^2\pi^2 - 1} \cdot \sin n\pi x]$$

12) $f(x) = \cos \lambda x$, $0 < x < \pi$, λ *is not an integer. Find half-range cosine series.*

$$\textbf{[Ans. :} \cos \lambda x = \frac{\sin \lambda\pi}{\lambda\pi} + \frac{2\lambda \sin \lambda\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{\lambda^2 - n^2} \cos nx]$$

13) *Expand* $\cos^2 x$ *in* $0 \leq x \leq \pi$ *as a series of sines and explicitly determine first three non-zero coefficients.* **[May-2002]**

$$\textbf{[Ans. :} b_n = \frac{4[(-1)^n - 1]}{\pi n(n^2 - 4)}, \; b_1 = \frac{8}{3\pi}, \; b_2 = 0]$$

Type 3 (a) : Examples involving polynomial in x in $(0, 2\pi)$ and $(-\pi, \pi)$.

Example 3.17 : *Find a Fourier series for $f(x) = x^2$ in $(-\pi, \pi)$ and hence deduce that*

i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \ldots\ldots = \frac{\pi^2}{12}$

ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \ldots\ldots = \frac{\pi^2}{6}$

iii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{5^2} \ldots\ldots = \frac{\pi^2}{8}$

Solution : $f(x) = x^2$ is an even function of x in $(-\pi, \pi)$. Thus $b_n = 0$.

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x^2\,dx$$

$$= \frac{2}{\pi}\cdot\frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right\}_0^\pi$$

$$= \frac{2}{\pi}\left\{\left(0 + \frac{2\pi(-1)^n}{n^2} + 0\right) - (0 - 0 + 0)\right\}$$

$$= \frac{4(-1)^n}{n^2}$$

$$b_n = 0$$

$\therefore$ The Fourier series formula

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{becomes}$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx \qquad \ldots (1)$$

For the deduction put x = 0 in (1).

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cdot 1$$

$$-\frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\therefore \quad \frac{\pi^2}{12} = \sum_{n=1}^{\infty} -\frac{(-1)^n}{n^2}$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \text{........} \qquad \text{... (A)}$$

Put $x = \pi$ in (1)

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cdot (-1)^n \qquad \{\text{As } \cos n\pi = (-1)^n$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\left(\frac{2\pi^2}{3}\right) \cdot \frac{1}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} \text{......} \qquad \text{... (B)}$$

Adding (A) and (B)

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} \text{.....}$$

$$\therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} \text{......} \qquad \text{... (C)}$$

Example 3.18 : ***Find Fourier series for f(x) = x in the interval (– π, π). Hence deduce that*** $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots$

Solution : f(x) = x is an odd function of x

$$\therefore \; a_0 = 0,\; a_n = 0,\; b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi(-1)^n}{n} - 0\right]$$

$$= \frac{-2(-1)^n}{n}$$

$\therefore$ The Fourier series formula

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{becomes}$$

$$x = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin nx$$

Put $x = \pi/2$

$$\frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin\frac{n\pi}{2}$$

$$\frac{\pi}{2} = \left[\frac{2}{1} + 0 + \frac{2}{3}(-1) + 0 + \frac{2}{5}(1) \ldots\ldots\right]$$

$$\therefore \qquad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots\ldots$$

Example 3.19 : *Find Fourier series for $f(x) = x + \frac{x^2}{4}$ in the interval $-\pi < x < \pi$ and $f(x) = f(x + 2\pi)$ hence show that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \ldots\ldots$* **[Dec.-99, Dec.-2004]**

Solution : Let $f(x) = f_1(x) + f_2(x)$ where

$$f_1(x) = x$$

and $$f_2(x) = \frac{x^2}{4} \quad \text{in} \quad -\pi < x < \pi$$

Note : The Fourier series of a sum $f(x) = f_1(x) + f_2(x)$ is the sum of the corresponding Fourier series.

We know the Fourier series for x, in $-\pi < x < \pi$. Using Example 3.18.

$$x = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin nx$$

Also we know Fourier series for x^2 in $-\pi < x < \pi$. Using Example 3.17.

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

Thus using above two series we can write

$$x + \frac{1}{4}x^2 = \left(\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx\right) + \frac{1}{4}\left(\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx\right)$$

$$x + \frac{x^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos nx + \frac{2(-1)^{n+1}}{n}\sin nx\right]$$

For the deduction put x = 0

$$0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\therefore \quad \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{-(-1)^n}{n^2}$$

$$= \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots\ldots$$

Example 3.20 : *Obtain the Fourier expansion for*

$$f(x) = \begin{cases} \pi + x & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$$

Solution :

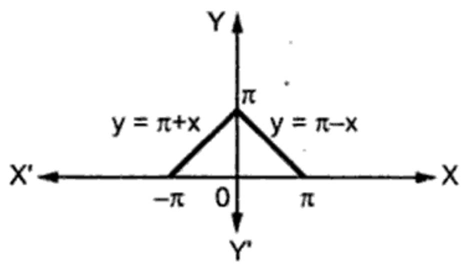


Fig. 3.22

Here the graph is symmetric about y axis ∴ Function is an even function of x.

Also

$$f(-x) = \begin{cases} \pi - x & -\pi < -x < 0 \\ \pi + x & 0 < -x < \pi \end{cases}$$

i.e.
$$f(-x) = \begin{cases} \pi - x & 0 < x < \pi \\ \pi + x & -\pi < x < 0 \end{cases}$$

$$= f(x) \Rightarrow \text{even function} \quad \therefore\ b_n = 0\ \forall n$$

$$\therefore \qquad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\,dx$$

$$= \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2 - \frac{\pi^2}{2}\right] = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0 - \frac{(-1)^n}{n^2}\right) - \left(0 - \frac{1}{n^2}\right)\right]$$

$$= \frac{2}{\pi n^2}\,[1 - (-1)^n]$$

The Fourier series formula

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \text{becomes}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\,[1 - (-1)^n]\cos nx$$

Example 3.21 : ***Find the Fourier series for periodic function***

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

[May-95, May-97, May-2001]

State the value of the series at x = 0 and hence deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

Solution :

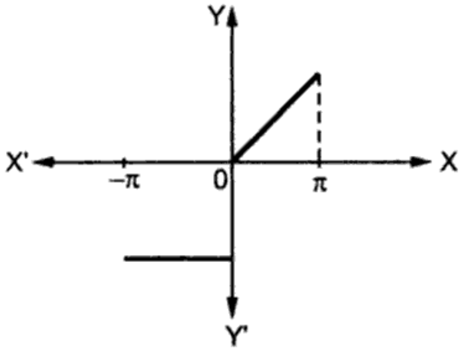


Fig. 3.23

Graph is not symmetric $\therefore$ The function is neither odd nor even.

$$\therefore \qquad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{0} (-\pi)\,dx + \int_{0}^{\pi} x\,dx\right\}$$

$$= \frac{1}{\pi}\left\{(-\pi x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = \frac{-\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{\pi}^{0} -\pi\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0}\right] + \left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{(0-0) + \left[\left(0 + \frac{(-1)^n}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{(-1)^n - 1}{n^2}\right\}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-\pi\left(\frac{-\cos nx}{n}\right)_{-\pi}^{0} + \left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\pi\frac{(1-(-1)^n)}{n} + \left[\left(\frac{-\pi(-1)^n}{n} + 0\right) - (0-0)\right]\right\}$$

$$= \frac{1}{n}[1 - (-1)^n - (-1)^n]$$

$$= \frac{1}{n}[1 - 2(-1)^n]$$

$\therefore$ The formula for Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx \quad \text{becomes}$$

$$f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{\pi n^2}\right)\cos nx + \left(\frac{1-2(-1)^n}{n}\right)\sin nx \qquad \dots (1)$$

As f(x) is discontinuous at x = 0.

$$\therefore \qquad f(0) = \frac{f(0-) + f(0+)}{2}$$

$$= \frac{\lim_{x\to 0-} f(x) + \lim_{x\to 0+} f(x)}{2}$$

$$= \frac{-\pi + 0}{2} = \frac{-\pi}{2}$$

Substituting x = 0 in (1) we get

$$\frac{-\pi}{2} = \frac{-\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{\pi n^2}$$

$$\frac{-\pi}{4} = \frac{-1}{\pi} + \sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}$$

Put n = 1, 2, 3 .....

$$\frac{\pi^2}{4} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2}$$

$$\therefore \qquad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \ldots\ldots$$

$$\text{i.e.} \qquad \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Example 3.22 : *What is the Fourier expansion of the periodic function whose definition in one period is* $f(x) = \begin{cases} -\pi & 0 < x < \pi \\ x - \pi & \pi < x < 2\pi \end{cases}$ **[Dec.-2003]**

State the value of the series at $x = \pi$ *and hence show that* $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$

Solution : As the interval is $(0, 2\pi)$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} -\pi\,dx + \int_{\pi}^{2\pi} (x-\pi)\,dx$$

$$= \frac{1}{\pi}\left\{-\pi(x)_0^{\pi} + \left[\frac{(x-\pi)^2}{2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{-\pi(\pi) + \left[\frac{\pi^2}{2} - 0\right]\right\}$$

$$= \frac{-\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} (-\pi)\cos nx\,dx + \int_{\pi}^{2\pi} (x-\pi)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{(-\pi)\left(\frac{\sin nx}{n}\right)_0^{\pi} + \left[(x-\pi)\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{(0-0) + \left[\left(0 + \frac{1}{n^2}\right) - \left(0 + \frac{(-1)^n}{n^2}\right)\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{1-(-1)^n}{n^2}\right\}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi}(-\pi)\sin nx\,dx + \int_{\pi}^{2\pi}(x-\pi)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{(-\pi)\left(\frac{-\cos nx}{n}\right)_0^{\pi} + \left[(x-\pi)\left(\frac{-\cos nx}{n}\right)-(1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\pi\left(\frac{(-1)^n - 1}{n}\right) + \left[\left(\frac{-\pi(-1)^{2n}}{n} + 0\right) - (0-0)\right]\right\}$$

$$= \frac{1}{n}[(-1)^n - 2]$$

Hence the Fourier series formula

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \textbf{becomes}$$

$$f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1}{\pi}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx + \left[\frac{(-1)^n - 2}{n}\right]\sin nx\right\}$$

As f(x) is discontinuous at $x = \pi$, therefore

$$f(\pi) = \frac{f(\pi-) + f(\pi+)}{2}$$

$$= \frac{\lim_{x\to\pi-} f(x) + \lim_{x\to\pi-} f(x)}{2}$$

$$= \frac{(-\pi) + (0)}{2} = \frac{-\pi}{2}$$

Substituting $x = \pi$ in Fourier series we get

$$\frac{-\pi}{2} = \frac{-\pi}{4} + \sum_{n=1}^{\infty}\frac{1-(-1)^n}{\pi n^2}\cdot(-1)^n$$

$$\frac{-\pi}{4} = \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{\pi n^2}$$

$$\frac{-\pi^2}{4} = \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}$$

Put n = 0, 1, 2, 3 ....

$$\frac{-\pi^2}{4} = \frac{-2}{1^2} - 0 - \frac{2}{3^2} - 0 - \frac{2}{5^2} \ldots\ldots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \ldots\ldots$$

i.e. $$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$$

Example 3.23 : ***Obtain half-range sine series to represent***

$$f(x) = \begin{cases} \dfrac{2x}{3} & 0 \le x \le \dfrac{\pi}{3} \\[2mm] \dfrac{\pi - x}{3} & \dfrac{\pi}{3} \le x \le \pi \end{cases}$$

Solution : To find sine series for f(x) in $(0, \pi)$

$$\therefore a_0 = 0,\ a_n = 0,\ b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/3} \left(\frac{2x}{3}\right) \sin nx\, dx + \int_{\pi/3}^{\pi} \left(\frac{\pi - x}{3}\right) \sin nx\, dx\right]$$

$$= \frac{2}{\pi}\left\{\left[\frac{2x}{3}\left(\frac{-\cos nx}{n}\right) - \frac{2}{3}\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi/3}\right.$$

$$\left. + \left[\left(\frac{\pi - x}{3}\right)\left(\frac{-\cos nx}{n}\right) - \left(\frac{-1}{3}\right)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi/3}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\left[\left(\frac{-2\pi}{9n}\cos\frac{n\pi}{3} + \frac{2}{3n^2}\sin\frac{n\pi}{3}\right) - (0+0)\right]\right.$$

$$\left. + \left[(0-0) - \left(\frac{-2\pi}{9n}\cos\frac{n\pi}{3} + \frac{1}{3n^2}\sin\frac{n\pi}{3}\right)\right]\right\}$$

$$= \frac{2}{\pi n^2}\sin\frac{n\pi}{3}$$

Thus the Fourier sine series formula

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{becomes}$$

$$= \sum_{n=1}^{\infty} \frac{2}{\pi n^2} \sin\frac{n\pi}{3} \cdot \sin nx$$

Example 3.24 : *Find half-range cosine series for the function*

$$f(x) = \begin{cases} x & 0 < x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x < \pi \end{cases}$$

and also graph the function for cosine series.

Solution :

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right\}$$

$$= \frac{2}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi/2} + \left(\frac{(\pi - x)^2}{2}\right)_{\pi/2}^{\pi}\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi/2}\right.$$

$$\left. + \left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - 0 - \frac{1}{n^2}\right]\right.$$

$$\left. + \left[0 - \frac{(-1)^n}{n^2} - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right]\right\}$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - (-1)^n - 1\right]$$

$\therefore$ The Fourier cosine series formula

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{becomes}$$

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2\cos\frac{n\pi}{2} - (-1)^n - 1\right]\cos nx$$

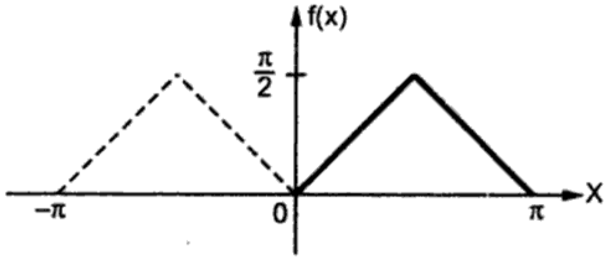


Fig. 3.24

For cosine series the graph will be symmetric about y-axis.

Type 3 (b) : Example involving polynomial in x in arbitrary interval.

Example 3.25 : *Find Fourier expansion of $f(x) = 4 - x^2$ in $0 < x < 2$. Graph the function and state the value of the series for $x = 0, 1, 2, 10, 11$.* [May-97]

Solution : Here $(0, 2L) = (0, 2) \Rightarrow L = 1$

$$\therefore \quad a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$$

$$= \frac{1}{1}\int_0^{2}(4 - x^2)\,dx$$

$$= \left(4x - \frac{x^3}{3}\right)_0^2 = \frac{16}{3}$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{1}\int_0^{2}(4 - x^2)\cos(n\pi x)\,dx$$

$$= \left[(4 - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$= \left\{\left(0 - \frac{4(1)}{n^2\pi^2} + 0\right) - (0 - 0 - 0)\right\}$$

$$= \frac{-4}{n^2\pi^2}$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{1}\int_0^{2} (4 - x^2)\sin n\pi x\,dx$$

$$= \left\{(4-x^2)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-2x)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right\}_0^2$$

$$= \left\{\left[0 - 0 - \frac{2}{n^3\pi^3}\right] - \left[\frac{-4}{n\pi} - 0 - \frac{2}{n^3\pi^3}\right]\right\} = \frac{4}{n\pi}$$

The Fourier series formula

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{becomes}$$

$$4 - x^2 = \frac{8}{3} + \sum_{n=1}^{\infty}\left(\frac{-4}{n^2\pi^2}\right)\cos(n\pi x) + \left(\frac{4}{n\pi}\right)\sin(n\pi x)$$

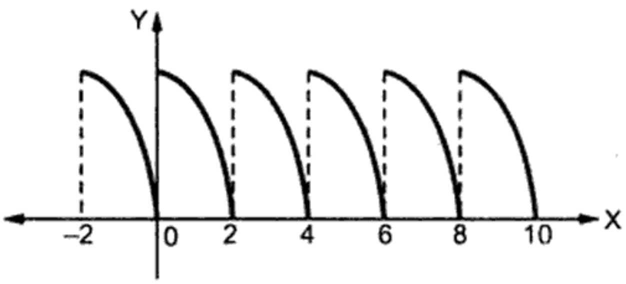


Fig. 3.25

Clearly the function is discontinuous at the points 0, 2, 4, 6, 8, 10 ... etc

$$\therefore \quad f(0) = \frac{f(0-) + f(0+)}{2} = \frac{0+4}{2} = 2$$

As the function is periodic with period 2.

$\therefore f(0) = f(2) = f(4) = f(6) = f(8) = f(10) = 2$

Now $\quad f(x) = 4 - x^2 \quad 0 < x < 2$

$\therefore \quad f(1) = 4 - 1 = 3$

Thus $f(1) = f(3) = f(5) = f(7) = f(9) = f(11) = 3$

➡ **Example 3.26 :** *Find the Fourier series for $f(x) = 2x - x^2$ in $(0, 3)$. Also graph the function.*

Solution : Here $(0, 2L) = (0, 3) \quad \Rightarrow 2L = 3 \Rightarrow L = \frac{3}{2}$

$$\therefore \quad a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$$

$$= \frac{1}{\frac{3}{2}}\int_0^{3} (2x - x^2)\,dx$$

$$= \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3 = 0$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{3}\int_0^{3} (2x - x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[(2x - x^2)\left(\frac{3}{2n\pi}\sin\frac{2n\pi x}{3}\right) - (2 - 2x)\left(\frac{-9}{4n^2\pi^2}\cos\frac{2n\pi x}{3}\right)\right.$$

$$\left. + (-2)\left(\frac{-27}{8n^3\pi^3}\sin\frac{2n\pi x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{-9}{n^2\pi^2}(-1)^{2n} - \frac{9}{2n^2\pi^2}\right] \qquad \text{As } \cos 2n\pi = (-1)^{2n} = 1$$

and $\sin 2n\pi = 0$

$$= \frac{-9}{n^2\pi^2}$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{3}\int_0^{3} (2x - x^2)\sin\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[\left(2x - x^2\right)\left(\frac{-3}{2n\pi}\cos\frac{2n\pi x}{3}\right) - (2 - 2x)\left(\frac{-9}{4n^2\pi^2}\sin\frac{2n\pi x}{3}\right)\right.$$

$$\left. + (-2)\left(\frac{27}{8n^3\pi^3}\cos\frac{2n\pi x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left\{\left[\frac{9}{2n\pi}(-1)^{2n} - 0 - \frac{27}{4n^3\pi^3}(-1)^{2n}\right] - \left[0 - 0 - \frac{27}{4n^3\pi^3}\right]\right\}$$

$$= \frac{3}{n\pi}$$

Substituting a_0, a_n, b_n in

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

We get

$$2x - x^2 = 0 + \sum_{n=1}^{\infty} \frac{-9}{n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)$$

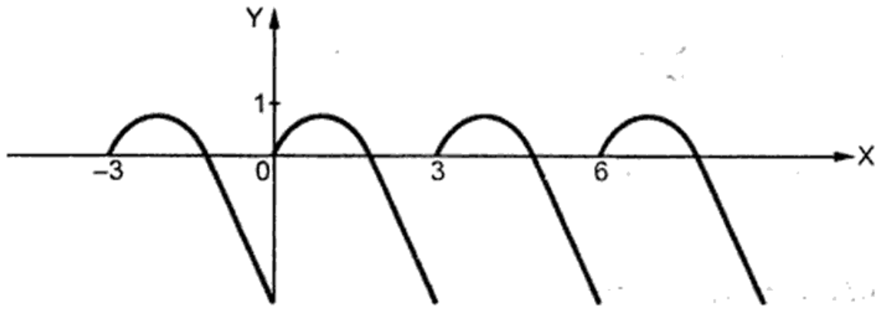


Fig. 3.26

Example 3.27 : *Find half range sine series for the function*

$$f(x) = \begin{cases} \frac{2k}{l}x & 0 \le x \le l/2 \\ \frac{2k}{l}(l - x) & l/2 \le x \le l \end{cases}$$

[May-2003]

Solution :

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \qquad L = l$$

$$= \frac{2}{l}\left[\int_0^{l/2} \frac{2k}{l}x \cdot \sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} \frac{2k}{l}(l - x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{4k}{l^2}\left[\int_0^{l/2} x \cdot \sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} (l - x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{4k}{l^2}\left\{\left[x\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right) - \left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^{l/2}\right.$$

$$\left. + \left[(l - x)\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right) - (-1)\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{l/2}^{l}\right\}$$

$$= \frac{4k}{l^2}\left\{\left[\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\right.$$

$$\left. + \left[\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\right\}$$

$$= \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$\therefore$ The Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\,dx \quad \text{becomes}$$

$$f(x) = \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cdot\sin\frac{n\pi x}{l}$$

Example 3.28 : *Obtain half-range sine series for*

$$f(x) = \begin{cases} \frac{1}{4} - x & 0 < x < \frac{1}{2} \\ x - \frac{3}{4} & \frac{1}{2} < x < 1 \end{cases}$$ **[May-99]**

Solution : $$b_n = \frac{2}{L}\int_0^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad L = 1$$

$$= \frac{2}{1}\left\{\int_0^{1/2}\left(\frac{1}{4} - x\right)\sin n\pi x\,dx + \int_{1/2}^{1}\left(x - \frac{3}{4}\right)\sin n\pi x\,dx\right\}$$

$$= \frac{2}{1}\left\{\left[\left(\frac{1}{4} - x\right)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^{1/2}\right.$$

$$\left. + \left[\left(x - \frac{3}{4}\right)\left(\frac{-\cos n\pi x}{n\pi}\right) - (1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_{1/2}^{1}\right\}$$

$$= 2\left\{\left[\left(\frac{1}{4n\pi}\cos\frac{n\pi}{2} - \frac{1}{n^2\pi^2}\sin\frac{n\pi}{2}\right) - \left(\frac{-1}{4n\pi} - 0\right)\right]\right.$$

$$\left. + \left[\left(\frac{-1}{4n\pi}(-1)^n - 0\right) - \left(\frac{1}{4n\pi}\cos\frac{n\pi}{2} + \frac{1}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]\right\}$$

$$= 2\left\{\frac{1-(-1)^n}{4n\pi} - \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$= \frac{2\,[1-(-1)^n]}{4n\pi} - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Thus the Fourier series formula

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{becomes}$$

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{2\,[1-(-1)^n]}{4n\pi} - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\sin n\pi x$$

$$= \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin \pi x + \left(\frac{1}{3\pi} - \frac{4}{3^2\pi^2}\right)\sin 3\pi x$$

$$+ \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 4\pi x \ \ldots\ldots$$

Exercise 3.3 : Problems on Type 3

1) *Find Fourier series for $f(x) = \pi^2 - x^2$ in $-\pi < x < \pi$ and $f(x) = f(x + 2\pi)$. Graph the function hence deduce that*

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots\ldots\ldots\ldots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \ldots\ldots\ldots\ldots$$

[May-2005]

Hint : Refer Example 3.16.

[Ans. : $a_0 = \frac{4\pi^2}{3}$, $a_n = \frac{4(-1)^{n+1}}{n^2}$]

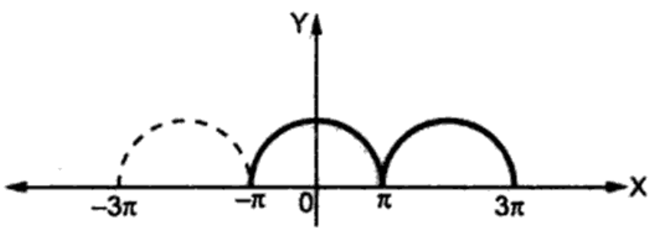


Fig. 3.27

2) *Find the Fourier series for* $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ *in* $(-\pi, \pi)$

[Ans. : $a_0 = 0,\ a_n = \frac{(-1)^{n+1}}{n^2}$**]**

3) *Find Fourier expansion of* $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

Hence deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}$ .... *and* $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ ....

Hint : Refer Example 3.18.

4) *Find the Fourier series for* $g(x) = x - x^2$ *and* $f(x) = x + x^2$ *in* $(-\pi, \pi)$

Hence deduce $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}$ ....

Hint : Refer Example 3.19.

5) *Obtain the Fourier series for* **[Dec.-2000]**

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi} & -\pi < x < 0 \\ 1 - \frac{2x}{\pi} & 0 < x < \pi \end{cases}$$

Graph the function and hence deduce that

$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}$ ....

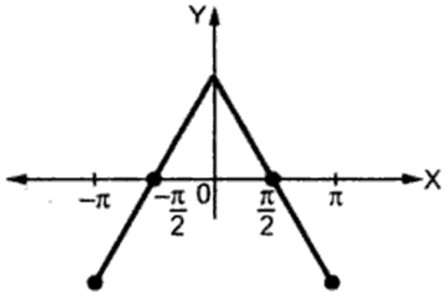


Fig. 3.28

Hint : Refer Example 3.20.

[Ans. $a_0 = 0,\ a_n = \frac{4}{n^2\pi^2}[1 - (-1)^n]$**]**

6) *Find the Fourier series for*

$f(x) = \begin{cases} -x & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$ *and* $f(x + 2\pi) = f(x)$

Hint : Refer Example 3.20.

[Ans. : $a_0 = \pi,\ a_n = \frac{2}{\pi} \cdot \frac{(-1)^n - 1}{n^2}$**]**

7) *Find half range Fourier sine series for* **[Dec.-2001]**

$$f(x) = \begin{cases} x & 0 < x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x < \pi \end{cases}$$

[Ans. : $b_n = \frac{4}{\pi n^2} \sin \frac{n\pi}{2}$**]**

8) *Expand $f(x) = lx - x^2$, $0 < x < l$ in half range (i) cosine series (ii) sine series.*

Deduce from sine series $\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3}$ ....

[Ans. : $a_0 = \frac{l^2}{3}, a_n = \frac{-2l^2}{n^2\pi^2}[1 + (-1)^n], b_n = \frac{4l^2}{n^3\pi^3}[1 - (-1)^n]$**]**

9) *If $f(x) = x^2$, $0 < x < 2$ Find (i) Half range cosine series (ii) Sine series (iii) Fourier series.*

[Ans. : (i) $a_0 = \frac{\delta}{3} a_n = \frac{16(-1)^n}{n^2\pi^2}$ **(ii)** $8\left[\frac{(-1)^n}{-n\pi} + \frac{(-1)^n - 2}{n^3\pi^3}\right]$

(iii) $a_0 = \frac{8}{3}, a_n = \frac{4}{n^2\pi^2}, b_n = \frac{-4}{n\pi}$**]**

10) *Find half range cosine series for f(x)*

$$f(x) = \begin{cases} kx & 0 \le x \le \frac{l}{2} \\ k(l-x) & \frac{l}{2} \le x \le l \end{cases}$$

[Dec.-2000]

Hint : Refer Example 3.26.

[Ans. : $a_0 = \frac{kl}{2}, a_n = \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - (-1)^n - 1\right]$

$$f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \frac{1}{10^2}\cos\frac{10\pi x}{l} \ldots\right]$$

11) *Expand* $f(t) = \begin{cases} t & 0 < t < \frac{\pi}{8} \\ \frac{\pi}{4} - t & \frac{\pi}{8} < t < \frac{\pi}{4} \end{cases}$ *in* $\left(0, \frac{\pi}{4}\right)$ *as a range sine series.*

[Ans. : $f(t) = \frac{1}{\pi}\left[\sin 4t - \frac{1}{9}\sin 12t + \frac{1}{25}\sin 20t - \frac{1}{44}\sin 28t \ldots\right]$**]**

12) $$f(x) = \begin{cases} \frac{\pi}{3} & 0 < x < \frac{\pi}{3} \\ 0 & \frac{\pi}{3} < x < \frac{2\pi}{3} \\ \frac{-\pi}{3} & \frac{2\pi}{3} < x < \pi \end{cases}$$

[Dec.-96]

Prove that (i) $f(x) = \frac{2}{\sqrt{3}}\left[\cos x - \frac{1}{5}\cos 5x + \frac{1}{7}\cos 7x \ldots\right]$

(ii) $f(x) = \sin 2x + \frac{1}{2}\sin 4x + \frac{1}{10}\sin 10x \ldots$

13) *Find Fourier series for $f(x) = x - x^2$ in $-1 < x < 1$* **[Dec.-94, May-93]**

[Ans. : $f(x) = \frac{-1}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x$**]**

14) *Determine the Fourier expansion for*

$$f(x)=\begin{cases}0 & -2<x<-1\\ 1+x & -1<x<0\\ 1-x & 0<x<1\\ 0 & 1<x<2\end{cases}\quad \textit{period 4}$$

[Ans. : $f(x)=\frac{1}{4}+\frac{4}{\pi}\sum_{1}^{\infty}\frac{1}{n^2}\left[1-\cos\frac{n\pi}{2}\right]\cos\frac{n\pi x}{2}$**]**

15) *Graph the following function and find its Fourier series* **[Dec.- 92]**

$$f(x)=\begin{cases}0 & -5<x<0\\ 3 & 0<x<5\end{cases}\quad \textit{period 10}$$

[Ans. : $a_0=3,\ a_n=0,\ b_n=\frac{3\,[1-(-1)^n]}{n\pi}$**]**

16) *If* $f(x)=\pi x \quad 0<x<1$

$\quad = \pi(2-x) \quad 1<x<2$

Show that $f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{0}^{\infty}\frac{\cos(2n+1)x\pi}{(2n+1)^2}$ **[May-94, May-96]**

17) *Find Fourier series for f(x) in* $-\pi<x<\pi$

$$f(x)=\begin{cases}\pi+x & -\pi\le x\le\frac{-\pi}{2}\\ \frac{\pi}{2} & \frac{-\pi}{2}\le x\le\frac{\pi}{2}\\ \pi-x & \frac{\pi}{2}\le x\le\pi\end{cases}$$

[Ans. : f(x) is even $f(x)=\frac{3\pi}{8}+\frac{4}{\pi}\sum_{1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{4}\cdot\cos nx$**]**

18) *Find the Fourier expansion of*

$$f(t)=\begin{cases}t & 0<t<1\\ 1-t & 1<t<2\end{cases}\quad \textit{period 2}$$

[Ans. : $a_0=0,\ a_n=\frac{2\,[(-1)^n-1]}{n^2\pi^2},\ b_n=\frac{1-(-1)^n}{n\pi}$**]**

19) *Obtain Fourier series expansion for* $f(x)=2-\frac{x^2}{2},\ 0\le x\le 2$

[Ans. : $f(x)=\frac{4}{3}+\sum_{n=1}^{\infty}\left(\frac{-2}{n^2\pi^2}\cos n\pi x+\frac{2}{n\pi}\sin n\pi x\right)$**]**

20) *Obtain the Fourier series for* $f(x)=|x|,\ -2\le x\le 2$

[Ans. : $|x|=1-\frac{8}{\pi^2}\left[\cos\frac{\pi x}{2}+\frac{1}{9}\cos\frac{3\pi x}{2}+\frac{1}{25}\cos\frac{5\pi x}{2}\ldots\right]$**]**

Type 4 : Function reducible to even or odd with change of origin.

Note : When the given interval is $0\le x\le 2\pi$ and the graph of the function is symmetric or antisymmetric about the line $x=\pi$ then the substitution $x-\pi=z$ shift the origin to $x=\pi$.

$$\therefore \quad x=0 \Rightarrow z=-\pi$$

$$x=\pi \Rightarrow z=0$$

$$x=2\pi \Rightarrow z=\pi$$

$\therefore$ The interval $(0, 2\pi)$ for x gives the interval $(-\pi, \pi)$ for z and if the function of the new independent variable z is even or odd in $(-\pi, \pi)$ then we can get the Fourier series in less calculations.

Example 3.29 : *Obtain the Fourier expansion of*

$$f(x) = \left(\frac{\pi - x}{2}\right)^2 \quad 0 \le x \le 2\pi$$

Hence deduce that (i) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots\ldots$

(ii) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \ldots\ldots$

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \ldots\ldots$

Solution :

$$f(x) = \frac{(\pi - x)^2}{4} \qquad 0 \le x \le 2\pi$$

Put $x - \pi = z$

$$\therefore \qquad x = \pi + z$$

$\therefore$ The function becomes

$$F(z) = \frac{z^2}{4} \qquad -\pi \le z \le \pi$$

which is an even function in $-\pi \le z \le \pi$

$\therefore b_n = 0$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} F(z)\, dz$$

$$= \frac{2}{\pi} \cdot \int_0^{\pi} \frac{z^2}{4} dz$$

$$= \frac{\pi^2}{6}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} F(z) \cos nz\, dz$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{z^2}{4} \cos nz\, dz$$

$$= \frac{1}{2\pi}\left[z^2\left(\frac{\sin nz}{n}\right) - 2z\left(\frac{-\cos nz}{n^2}\right) + 2\left(\frac{-\sin nz}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\left(0 + \frac{2\pi(-1)^n}{n^2} + 0\right) - (0 - 0 - 0)\right]$$

$$= \frac{(-1)^n}{n^2}$$

$\therefore$ The Fourier series for F(z) is given by

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz$$

$$\frac{z^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nz$$

Now As $z = x - \pi$

$$\therefore \quad \cos nz = \cos n(x - \pi)$$

$$= \cos nx \cdot \cos n\pi + \sin nx \sin n\pi$$

$$= \cos nx(-1)^n + 0$$

$$\therefore \quad \frac{(x-\pi)^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cdot (-1)^n \cos nx$$

$$\frac{(x-\pi)^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$$

For deductions put $x = 0$ and π

$$x = 0 \Rightarrow \quad \frac{\pi^2}{4} - \frac{\pi^2}{12} = \sum_{1}^{\infty} \frac{1}{n^2}$$

$$\therefore \quad \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} \ldots\ldots$$

$$x = \pi \Rightarrow \quad 0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\therefore \quad \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{-(-1)^n}{n^2}$$

$$\therefore \quad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} \ldots\ldots\ldots$$

Adding above two series we get

$$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2} \cdots\cdots\cdots$$

Example 3.30 : ***Prove that if*** $0 \le x \le 2\pi$

$$\frac{1}{12}x(\pi - x)(2\pi - x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$$

Solution :

$$f(x) = \frac{x}{12}(\pi - x)(2\pi - x) \qquad 0 \le x \le 2\pi$$

Put $x = z + \pi$

$$F(z) = \frac{(z+\pi)}{12}(-z)(\pi - z) \qquad -\pi \le z \le \pi$$

$$= \frac{-z}{12}(\pi^2 - z^2)$$

$$= \frac{z}{12}(z^2 - \pi^2) \qquad -\pi \le z \le \pi$$

which is an odd function of z.

$$\therefore a_0 = 0,\ a_n = 0,\ b_n = \frac{2}{\pi}\int_0^{\pi} F(z)\sin nz\,dz$$

$$b_n = \frac{2}{\pi}\cdot\frac{1}{12}\int_0^{\pi}(z^3 - \pi^2 z)\sin nz\,dz$$

$$= \frac{1}{6\pi}\Bigg[(z^3 - \pi^2 z)\left(\frac{-\cos nz}{n}\right) - (3z^2 - \pi^2)\left(\frac{-\sin nz}{n^2}\right)$$

$$+ (6z)\left(\frac{\cos nz}{n^3}\right) - 6\left(\frac{\sin nz}{n^4}\right)\Bigg]_0^{\pi}$$

$$= \frac{1}{6\pi}\left[\frac{6\pi(-1)^n}{n^3}\right] = \frac{(-1)^n}{n^3}$$

$\therefore$ The Fourier series for F(z) becomes

$$F(z) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin nz$$

Put $z = x - \pi$

$$f(x) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n(x - \pi)$$

$$\sin n(x-\pi) = \sin nx \cos n\pi - \cos nx \sin n\pi$$

$$= \sin nx(-1)^n - 0$$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} \frac{1}{n^3} \sin nx$$

Example 3.31 : *Determine the Fourier series for the function $f(x) = \sqrt{1-\cos x}$ in the interval $0 \le x \le 2\pi$ and hence deduce that* $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$. **[May-93, May-96]**

Solution :

$$f(x) = \sqrt{1-\cos x}$$

$$= \sqrt{2\sin^2\frac{x}{2}}$$

$$= \sqrt{2}\sin\frac{x}{2}$$

$$f(x) = \sqrt{2}\sin\frac{x}{2} \qquad 0 \le x \le 2\pi$$

Put $x = z + \pi$

$$F(z) = \sqrt{2}\sin\left(\frac{z+\pi}{2}\right) \qquad -\pi \le z \le \pi$$

$$= \sqrt{2}\cos\frac{z}{2} \qquad -\pi \le z \le \pi$$

which is an even function of z in $(-\pi, \pi)$

$\therefore$ $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} F(z)\,dz$$

$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{2}\cos\frac{z}{2}\,dz$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{\sin\frac{z}{2}}{\frac{1}{2}}\right]_0^{\pi}$$

$$= \frac{4\sqrt{2}}{\pi}\left[\sin\frac{\pi}{2} - 0\right]$$

$$= \frac{4\sqrt{2}}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} F(z)\cos nz\,dz$$

$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{2}\cos\left(\frac{z}{2}\right)\cdot\cos nz\ dz$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi} \cos\left(\frac{1}{2}+n\right)z + \cos\left(\frac{1}{2}-n\right)z\ dz$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{\sin\left(\frac{1}{2}+n\right)z}{\frac{1}{2}+n} + \frac{\sin\left(\frac{1}{2}-n\right)z}{\frac{1}{2}-n}\right]_0^{\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{\sin\left(\frac{1}{2}+n\right)\pi}{\frac{1}{2}+n} + \frac{\sin\left(\frac{1}{2}-n\right)\pi}{\frac{1}{2}-n}\right]$$

$$= \frac{\sqrt{2}}{\pi}\cdot 2\left[\frac{\cos n\pi}{1+2n} + \frac{\cos n\pi}{1-2n}\right]$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{2}{1-4n^2}\right](-1)^n$$

$$= \frac{4\sqrt{2}(-1)^n}{\pi(1-4n^2)}$$

∴ The Fourier series formula for F(z) is

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz$$

$$F(z) = \frac{2\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1-4n^2}\cos nz$$

Put $z = x - \pi$

$$f(x) = \frac{2\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1-4n^2}\cdot\cos n(x-\pi)$$

Now $\cos n(x-\pi) = \cos nx \cos n\pi + \sin nx \sin n\pi$

$$= \cos nx(-1)^n + 0$$

$$\therefore \quad \sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{1-4n^2} \cdot (-1)^n \cos nx$$

$$\sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{1}{1-4n^2} \cos nx$$

Put $x = 0$

$$0 = \frac{2\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{1}{1-4n^2}$$

$$\therefore \quad \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$$

Example 3.32 : *Find the Fourier series for* $f(x) = \begin{cases} a, & 0 < x < \pi \\ -a, & \pi < x < 2\pi \end{cases}$

Hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$

Solution : First we graph f(x) in $0 < x < 2\pi$

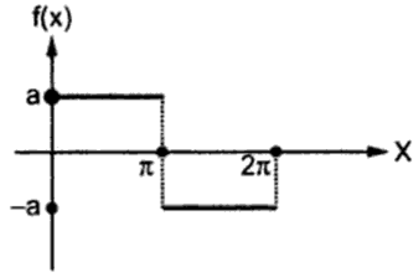


Fig 3.29

Since graph is antisymmetric about $x = \pi$

i.e. putting x

$$x - \pi = z$$

i.e. $$x = z + \pi$$

then f(x) reduces to f(z).

where $$f(z) = \begin{cases} a, & -\pi < z < 0 \\ -a, & 0 < z < \pi \end{cases}$$

which is an odd function.

$$\therefore \quad a_0 = a_n = 0$$

$\therefore$ Fourier series for F(z) is

$$F(z) = \sum b_n \sin z$$

where, $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nz\, dz$$

$$= \frac{2}{\pi}\int_0^{\pi} (-a) \sin nz\, dz$$

$$= -\frac{2a}{\pi}\left[\left(-\frac{\cos nz}{n}\right)\right]_0^{\pi}$$

$$= \frac{2a}{n\pi}[\cos n\pi - 1]$$

$$\therefore \quad F(z) = \frac{2a}{\pi}\left[\frac{(\cos n\pi - 1)}{n} \sin nz\right]$$

But $$z = x - \pi$$

$$\therefore \quad f(x) = \frac{2a}{\pi}\sum_{n=1}^{\infty}\left(\frac{\cos n\pi - 1}{n}\right)(-1)^n \sin nx$$

$\therefore$ f(x) is continuous at $x = \frac{\pi}{2}$

$\therefore$ Fourier series converges to $x = \frac{\pi}{2}$

$\therefore$ Fourier series converges to $f\left(\frac{\pi}{2}\right) = a$

$$\therefore \quad a = \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n}(-1)^n \sin\frac{n\pi}{2}$$

$$\frac{\pi}{2} = \frac{-2}{1}(-1) + 0 + \frac{(-2)}{3}(-1)(-1) + 0 + \frac{(-2)}{5}(-1)(1) + \ldots\ldots$$

i.e. $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$$

Example 3.33 : ***Express in terms of Fourier series.***

$$f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}, \; \textit{period} = 2\pi$$

Solution : Graph of f(x) in $0 \le x \le 2\pi$ is f(x).

Since the graph is symmetric about $x = \pi$ shifting the origin to $x = \pi$

$\therefore$ Put $$x - \pi = z$$

i.e. $$x = \pi + z$$

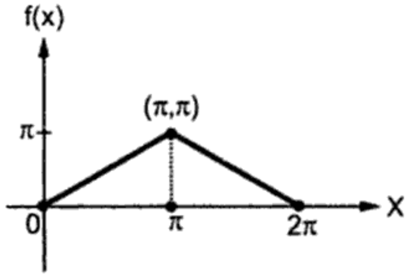


Fig. 3.30

$\therefore$ f(x) reduces to F(z)

where $$F(z) = \begin{cases} \pi + z, & -\pi \le z \le 0 \\ \pi - z, & 0 \le z \le \pi \end{cases}$$

i.e. F(z) is an even function.

$\therefore b_n = 0$ and Fourier series for F(z) is

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz$$

where $$a_0 = \frac{2}{\pi}\int_0^{\pi} F(z)\, dz = \frac{2}{\pi}\int_0^{\pi} (\pi - z)\, dz$$

$$= \frac{2}{\pi}\left[\frac{(\pi - z)^2}{-2}\right]_0^{\pi} = -\frac{2}{\pi}\left[0 - \pi^2\right] = \frac{\pi^2}{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} F(z)\cos nz\, dz = \frac{2}{\pi}\int_0^{\pi} (\pi - z)\cos nz\, dz$$

$$= \frac{2}{\pi}\left[(\pi - z)\frac{\sin nz}{n} - (-1)\left(\frac{-\cos nz}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{1}{n^2}(\cos n\pi - 1)\right] = \frac{2}{\pi n^2}(1 - \cos n\pi)$$

$$\therefore \quad F(z) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(1 - \cos n\pi)}{n^2}\cos nz$$

But $z = x - \pi$

$$\therefore \quad \cos nz = \cos n(x - \pi) = (-1)^n \cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{n^2}(-1)^n \cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2}\cos nx$$

Example 3.34 : $f(x) = \pi x,\ 0 \le x \le 1$
$= \pi(2 - x),\ 1 \le x \le 2,$ *period* 2

Show that in the interval

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi x$$

Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$ [May-2001]

Solution : Here L = 1,

Graph of the function in $0 \le x \le 2$ is

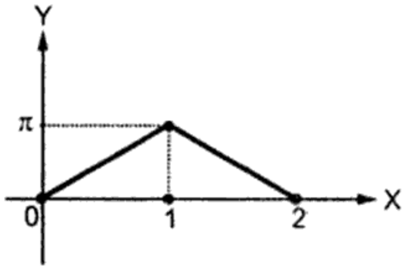


Fig 3.31

Since graph is symmetric about x = 1

∴ Shifting the origin at x = 1

i.e. using the transformation

$$x - 1 = z \Rightarrow x = 1 + z$$

∴ f(x) reduces to

$$F(z) = \begin{cases} \pi(1 + z), & -1 \le z \le 0 \\ \pi(1 - z), & 0 \le z \le 1 \end{cases}$$

which is a even function in $-1 \le z \le 0$

$$\therefore \qquad b_n = 0$$

∴ Fourier series for F(z) is

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi z$$

where
$$a_0 = \frac{2}{1}\int_0^1 F(z)\,dz$$

$$= \frac{2}{1}\int_0^1 \pi(1 - z)\,dz$$

$$= 2\pi\left[\frac{(1-z)^2}{-2}\right]_0^1$$

$$= -\pi[0-1] = \pi$$

$$a_n = \frac{2}{1}\int_0^1 \pi(1-z)\cos n\pi z\, dz$$

$$= 2\pi\left[(1-z)\frac{\sin n\pi z}{n\pi} - (-1)\left(\frac{-\cos n\pi z}{n^2\pi^2}\right)\right]_0^1$$

$$= \frac{-2\pi}{n^2\pi^2}[\cos n\pi - 1]$$

$$= \frac{-2\pi}{n^2\pi^2}[1 - \cos n\pi]$$

$$= \begin{cases} \dfrac{4}{n^2\pi^2}, & \text{if } n \text{ odd} \\ 0\ , & \text{if } n \text{ even} \end{cases}$$

$$= \frac{4}{\pi(2n+1)^2} \qquad n = 0, 1, 2 \,.........$$

$$\therefore \qquad F(z) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi z$$

But $z = x - 1$

$$\cos[(2n+1)\pi z] = \cos[(2n+1)\pi\,(x-1)] \quad \{\text{As } \cos[n\pi - \theta] = -\cos\theta \text{ if n is odd}\}$$

$$= -\cos(2n+1)\pi x$$

$$\therefore \qquad F(z) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}(2n+1)^2\,(2n+1)\pi x$$

At $x = 0$, if F(x) is continuous

$\therefore$ F.S. converges to $f(0) = \pi$

$$\therefore \qquad \pi = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi x$$

$$\therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \,.......$$

Exercise 3.4 : Problems on Type 4

1) $f(x) = mx \qquad 0 < x < \pi$

$= -mx + 2m\pi \qquad \pi < x < 2\pi,\ f(x) = f(x + 2\pi)$

Prove that

$$f(x) = \frac{m\pi}{2} - \frac{4m}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x \,........\right]$$

2) *Find the Fourier series for the function f(x).*

$$f(x) = \begin{cases} x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases},$$

$f(x) = f(x + 2\pi)$

Hint : Put $x - \frac{\pi}{2} = z,\ x = z + \frac{\pi}{2}$

$$\therefore \qquad x = -\frac{\pi}{2} \Rightarrow z = -\pi$$

$$x = \frac{\pi}{2} \Rightarrow z = 0$$

$$x = \frac{3\pi}{2} \Rightarrow z = \pi$$

$$F(z) = \begin{cases} z + \frac{\pi}{2} & -\pi < z < 0 \\ \frac{\pi}{2} - z & 0 < z < \pi \end{cases}$$

which is an even function of z.

[Ans. : $f(x) = \frac{4}{\pi}\left[\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x \,........\right]$**]**

3) *Find the Fourier series for* $f(x) = \begin{cases} 8 & 0 < x < 2 \\ -8 & 2 < x < 4 \end{cases}$

[Ans. : $f(x) = \frac{32}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} \,........\right]$**]**

Type 5 : Harmonic Analysis

In most of the practical examples what we get is not the function f(x) but we have the numerical data i.e. the value of the function and independent variables. The process of finding the Fourier series for available numerical data is called Harmonic Analysis.

The Fourier coefficients are evaluated by using the following formulae.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= 2\,[\text{Mean value of } f(x) \text{ in } (0, 2\pi)]$$

$$= \frac{2\sum f(x)}{N} = \frac{2\sum y}{N}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= 2\,[\text{Mean value of } f(x)\cos nx \text{ in } (0, 2\pi)]$$

$$= \frac{2\sum f(x)\cos nx}{N}$$

and $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= 2\,[\text{Mean value of } f(x)\sin nx \text{ in } (0, 2\pi)]$$

$$= \frac{2\sum f(x)\sin nx}{N}$$

3.10 Fundamental Harmonic

The term $a_1 \cos x + b_1 \sin x$ is called as fundamental or first harmonic.

The term $a_2 \cos 2x + b_2 \sin 2x$ is called second harmonic.

Example 3.35 : ***The following table gives variation of periodic current over a period.***

t sec.	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
A (amp.)	1.98	1.30	1.05	1.30	– 0.88	– 0.25	1.98

Show that there is a direct current part of 0.75 amp in variable current and obtain the amplitude of first harmonic. **[Dec.- 2004]**

Solution : Here n = 6, Period = T

$$\left[\therefore\ (0, 2L) = (0, T) \Rightarrow \left(\frac{n\pi x}{L}\right) = \left(\frac{n\pi t}{T/2}\right)\right]$$

$\therefore$ $$a_0 = 2 \times \text{Mean value of } A$$

$$a_1 = 2 \times \text{Mean value of } A \times \cos\frac{\pi t}{T/2}$$

$$b_1 = 2 \times \text{Mean value of } A \sin\frac{\pi t}{T/2}$$

t	$2\pi t/T$	A	$A \cos \frac{\pi t}{T/2}$	$A \sin \frac{\pi t}{T/2}$
0	0	1.98	1.98	0
T/6	$\pi/3$	1.30	0.65	1.1258
T/3	$2\pi/3$	1.05	– 0.525	0.909
T/2	π	– 1.30	– 1.30	–
2T/3	$4\pi/3$	+ 0.88	0.44	+ 0.762
5T/6	$5\pi/3$	+ 0.25	– 0.125	0.2165
Total		4.5	1.12	3.0133

$$\therefore \quad a_0 = 2 \times \frac{\sum A}{6} = \frac{2 \times 4.5}{6} = \frac{4.5}{3} = 1.5$$

$$a_1 = \frac{2 \times \sum A \cos \frac{\pi t}{T/2}}{6} = \frac{2 \times 1.12}{6} = 0.373$$

$$b_1 = \frac{2 \times \sum A \sin \frac{\pi t}{T/2}}{6} = \frac{3.0133}{2} = 1.004$$

$\therefore$ Required Fourier series representation is

$$A = \frac{a_0}{2} + a_1 \cos \frac{2\pi t}{T} + b_1 \sin \frac{2\pi t}{T} + \ldots$$

i.e.
$$A = 0.75 + 0.373 \cos \frac{2\pi t}{T} + 1.004 \sin \frac{2\pi t}{T} + \ldots$$

Where direct current = 0.75 amp

Amplitude of first harmonic

$$= \sqrt{a_1^2 + b_1^2}$$

$$= \sqrt{(0.373)^2 + (1.004)^2}$$

$$= 1.07$$

Example 3.36 : *A turning moment T units of the crank shaft of a steam engine is given for the series of values of the crank angle θ in degrees.* **[Dec.- 2004]**

θ	0	30	60	90	120	150	180
T	0	5224	8097	7850	5499	2626	0

Find the first four term in the series of sines to represent T, also calculate T when θ = 75°.

Solution : This is half range sine series problem

$$\therefore \quad T = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x$$

where
$$b_1 = \frac{2 \times \sum T \sin x}{6}$$
$$b_2 = \frac{2 \times \sum T \sin 2x}{6}$$
$$b_3 = \frac{2 \times \sum T \sin 3x}{6}$$
$$b_4 = \frac{2 \times \sum T \sin 4x}{6}$$

θ	T	T sin θ	T sin 2θ	T sin 3θ	T sin 4θ
0	0	0	0	0	0
30	5224	2612	4524.1	5224	4524.1
60	8097	7012.2	7012.2	0	– 7012.2
90	7850	7850	0	– 7850	0
120	5499	4762.2	– 4762.2	0	4762.2
150	2626	1313	– 2274.1	2626	– 2274.18
Total		23549.4	5400	0	26.92

$$\therefore \quad b_1 = \frac{2 \times 23549.4}{6} = 7849.8$$
$$b_2 = \frac{2 \times 5400}{6} = 1800$$
$$b_3 = \frac{2 \times 0}{6} = 0$$
$$b_4 = \frac{2 \times 26.92}{6} = 8.9733$$

$$\therefore \quad T = 7849.8 \sin \theta + 1800 \sin 2\theta + 8.9733 \sin 4\theta + \ldots$$

Example 3.37 : *The turning moment T on the crank shaft of a steam engine for the crank angle for the crank angle θ is given as*

θ°	*T*	θ°	*T*
0	0	180	7.9
30	2.7	210	6.8
60	5.2	240	5.5

90	7.0	270	4.1
120	8.1	300	2.6
150	8.3	330	1.2

Example T as a Fourier series up to first two harmonics.

Solution : We will form the table to calculate a′ s and b′ s of the Fourier coefficients.

θ°	T = y	y cos x	y cos 2x	y sin x	y sin 2x
0	0	0	0	0	0
30	2.7	2.338	1.35	1.35	2.338
60	5.2	2.6	− 2.6	4.503	4.503
90	7.0	0	− 7	7	0
120	8.1	− 4.05	− 4.05	7.015	− 7.015
150	8.3	− 7.188	4.15	4.15	− 7.188
180	7.9	− 7.9	7.9	0	0
210	6.8	− 5.889	3.4	− 3.4	5.889
240	5.5	− 2.75	− 2.75	− 4.763	4.763
270	4.1	0	− 4.1	− 4.1	0
300	2.6	1.3	− 1.3	− 2.252	− 2.252
330	1.2	1.039	− 0.6	− 0.6	− 1.039
	$\sum = 59.4$	$\sum = -20.5$	$\sum = -4.4$	$\sum = 8.903$	$\sum = -0.001$

Let the Fourier series upto the second harmonic is

$$Y = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin x) \quad \dots (1)$$

$$a_0 = \frac{2}{12}\sum y = \frac{59.4}{6} = 9.9 \text{ i.e. } a_0 = 9.9$$

$$a_1 = \frac{2}{12}\sum y \cos x = \frac{-20.5}{6} = -3.42$$

$$b_1 = \frac{2}{12}\sum y \sin x = \frac{8.903}{6} = 1.48$$

$$a_2 = \frac{2}{12}\sum y \cdot \cos 2x = \frac{-4.4}{6} = -0.73$$

$$b_2 = \frac{2}{12}\sum y \cdot \sin 2x = \frac{-0.001}{6} = -0.00017$$

Substituting the values of a's and b's in (1) we get

$$Y = 4.95 - 3.42 \cos x + 1.48 \sin x - 0.73 \cos 2x - 0.00017 \sin 2x$$

Example 3.38 : *Find the first two harmonics of the Fourier series for y from the data.*

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
y	2.34	3.01	3.69	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

Solution : The Fourier series expansion upto second harmonic is given by

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x)$$

We will form the table to calculate a's and b's of the Fourier coefficients.

x	y = f(x)	y cos x	y . cos 2x	y sin x	y . sin 2x
0°	2.34	2.34	0	0	0
30°	3.01	2.6067	1.505	1.505	2.6067
60°	3.69	1.845	– 1.845	3.1955	3.1955
90°	4.15	0	– 4.15	4.15	0
120°	3.69	– 1.845	– 1.845	3.1955	– 3.1955
150°	2.20	– 1.9052	1.1	1.1	– 1.9052
180°	*0.083	– 0.083	0.083	0	0
210°	0.51	– 0.4417	0.225	– 0.255	0.4417
240°	0.88	– 0.44	0.44	– 0.7621	0.7621
270°	1.09	0	– 1.09	– 1.09	0
300°	1.19	0.595	– 0.595	– 1.0305	– 1.0305
330°	1.64	1.4202	0.82	– 0.82	– 1.4202
	$\sum = 24.473$	$\sum = -4.092$	$\sum = -3.862$	$\sum = 9.1884$	$\sum = -0.5454$

Here

$$n = 12$$

$$a_0 = 2 \text{ Mean value of } f(n) \text{ in } (0, \pi)$$

$$= 2 \cdot \frac{\sum f(n)}{n} = 2 \times \frac{24.473}{12} = 4.078$$

$$a_1 = \frac{2}{12} \sum y \cos x = \frac{-4.092}{6} = 0.682$$

$$a_2 = 2 \times \frac{\sum y \cdot \cos 2x}{12} = -0.6437$$

$$b_1 = 2 \times \frac{\sum y \sin 2x}{12} = 1.5314$$

$$b_2 = 2 \times \frac{\sum y \ \sin 2x}{12} = -\ 0.0909$$

Hence Fourier series upto second harmonic is

f(x) = 2.0394 + (0.682) cos x – 0.6437 cos 2x + 1.5314 sin x – 0.0909 sin 2x

Example 3.39 : ***Obtain the constant term and the coefficients of the first cosine and sine terms in the exapnsion of y from the table.*** **[Dec.-2005, Dec.-2007]**

x	:	0	1	2	3	4	5
y	:	9	18	24	28	26	20

Solution : Here Period = 6, we have

$$\therefore \quad (0, 2L) = (0, 6)$$

$$\therefore \quad \left(\frac{n\pi x}{L}\right) = \left(\frac{n\pi x}{3}\right)$$

$\therefore$ The Fourier series to represent y is (0, 5) is

$$y = \frac{1}{2} a_0 + a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3} \qquad \text{... (1)}$$

where $a_0 = 2$ [mean value of y is (0, 5)]

$$a_1 = 2\left[\text{mean value of } y \cdot \cos\frac{\pi x}{3} \text{ in}(0,\ 5)\right]$$

$$b_1 = 2\left[\text{mean value of } y \cdot \sin\frac{\pi x}{3} \text{ in}(0,\ 5)\right]$$

The values are tabulated as follows

x	π x/3	y	$y \cdot \sin\frac{\pi x}{3}$	$y \cdot \cos\frac{\pi x}{3}$
0	0	9	0	9
1	π/3	18	15.589	9
2	2π/3	24	20.785	– 12
3	π	28	0	– 28
4	4π/3	26	– 22.517	– 13
5	5π/3	20	– 17.321	
		$\sum = 125$	$\sum = -\ 3.404$	$\sum = -\ 25$

$$n = 6$$

$$\therefore \quad a_0 = 2\frac{\sum y}{n} = 2 \times \frac{125}{6} = 41.66$$

$$a_1 = \frac{2\sum y \cos\frac{\pi x}{3}}{n} = -\frac{25}{3} = -8.33$$

$$b_1 = \frac{2\sum y \sin\frac{\pi x}{3}}{n} = -\frac{3.404}{3} = -1.15$$

Substituting a's and b's in (1) we obtain

$$y = 20.83 - 8.33 \cos\frac{\pi x}{3} - 1.15 \sin\frac{\pi x}{3}$$

Example 3.40 : *A machine completes its cycle of operations every time as certain pulley completes a revolution. The displacement f being the angle in degree turned through by the pulley. Find a Fourier series to represent f (x) for all values of x.*

x	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$f(x)$	7.976	8.226	7.204	5.676	3.674	1.764	0.552	0.262	0.204	2.492	4.636	6.824

Solution : See Table on next page.

$$a_0 = 2 \times \text{Mean value of } f(x) = 2 \times \frac{50.09}{12} = 8.34$$

$$a_1 = 2 \times \text{Mean value of } f(x) \cos x = 2 \times \frac{14.733}{12} = 2.45$$

$$a_2 = 2 \times \text{Mean value of } f(x) \cos 2x = 2 \times \frac{0.721}{12} = 0.12$$

$$a_3 = 2 \times \text{Mean value of } f(x) \cos 3x = 2 \times \frac{0.460}{12} = 0.08$$

$$b_1 = 2 \times \text{Mean value of } f(x) \sin x = 2 \times \frac{19.206}{12} = 3.16$$

$$b_2 = 2 \times \text{Mean value of } f(x) \sin 2x = 2 \times \frac{0.207}{12} = 0.003$$

$$b_3 = 2 \times \text{Mean value of } f(x) \sin 3x = 2 \times \frac{0.062}{12} = 0.01$$

Fourier series is

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots$$

$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots$$

$$= 4.17 + 2.45 \cos x + 0.12 \cos 2x + 0.08 \cos 3x + 3.16 \sin x$$

$$+ 0.03 \sin 2x + 0.01 \sin 3x + \ldots$$

x	f(x)	f(x) sin x	f(x) sin 2x	f(x) sin 3x	f(x) cos x	f(x) cos 2x	f(x) cos 3x
30°	7.976	3.988	6.939	7.976	6.939	3.988	0
60°	8.026	6.983	6.983	0	– 4.013	4.013	– 8.026
90°	7.204	7.204	0	– 7.204	0	– 7.204	0
120°	5.676	4.938	– 4.939	0	– 2.838	– 2.838	5.676
150°	3.674	1.837	– 3.196	– 3.196	– 3.196	1.837	0
180°	1.764	0	0	– 1.764	– 1.764	1.764	– 1.764
210°	0.552	– 0.276	0.480	0.480	– 0.480	0.276	0
240°	0.262	– 0.228	0.228	– 0.131	– 0.131	0.131	0.262
270°	0.904	– 0.904	0	0	0	– 0.904	0
300°	2.492	– 2.168	– 2.168	1.146	1.246	– 1.296	– 2.492
330°	4.736	– 2.368	– 4.120	4.120	4.120	2.368	0
360°	6.824	0	0	0	6.824	6.824	6.824
	50.09	19.206	0.207	0.062	14.733	0.721	0.460

Exercise 3.5 : Problems on Type 5

1) *In a machine the displacement $f(x)$ of a given point is given for a certain angle $x°$ as follows :*

$x°$	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
$f(x)$	7.9	8.0	7.2	5.6	3.6	1.7	0.5	0.2	0.9	2.5	4.7	6.8

Find the coefficient of sin 2x in the Fourier series representing the above variations. **[Ans. : – 0.072]**

2) *The displacement $f(x)$ of a part of a machine is tabulated with corresponding angular moment 'x' of the crank. Express $f(x)$ as a Fourier series upto third harmonic.*

$x°$	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
$f(x)$	1.80	1.10	0.30	0.16	0.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

[Ans. : f(x) = 1.26 + 0.04 cos x + 0.53 cos 2x – 0.1 cos 3x + ...

– 0.63 sin x – 0.23 sin 2x + 0.085 sin 3x + ...]

3) *The following values of y give the displacement in cm of a certain machine part of the rotation x of the flywheel. Expand $f(x)$ in the form of a Fourier series.*

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$
$f(x)$	0	9.2	14.4	17.8	17.3	11.7

[Ans. : f (x) = 11.733 – 7.733 cos 2x – 2.833 cos 4x + ...

– 1.566 sin 2x – 0.116 sin 4x + ...]

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y	1.0	1.4	1.9	1.7	1.5	1.2	1.0

4) *The following twelve values of y correspond to equidistant values of the angle $x°$ in $(0°, 360°)$.*

$x°$	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
y	10.5	20.2	26.4	29.3	27	21.5	12.8	1.6	– 19.2	– 18.0	– 15.8	– 0.4

Determine approximately the Fourier expansion for y in terms of x upto third harmonic.

[Ans. : y = 88.4 (– 1.7 cos x – 23.1 sin x) + (3.1 cos 2x + 1.1 sin 2x)

+ 4.45 cos 3x + 0.6 sin 3x]

5) *The following values of y give the displacement in inches of a certain machine part for the rotation x of the flywheel. Expand y in terms of a Fourier series :*

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$
y	0	9.2	14.4	17.8	17.3	11.7

[Ans. : y = 11.733 – (7.733 cos 2x + 1.566 sin 2x)
– 2.833 cos 4x + 0.116 sin 4 x) ....]

6) *The following table gives the variation of periodic current over a period :*

t, sec	0	$\frac{T}{6}$	$\frac{T}{3}$	$\frac{T}{2}$	$\frac{2T}{3}$	$\frac{5\pi}{3}$	T
A, amp	1.98	1.30	1.05	1.30	– 0.88	– 0.25	1.98

Show by practical harmonic analysis, that there is a direct current part of 0.75 ampere in the variable current and obtain the amplitude of the first harmonic. **[Ans. : 1.072]**

Hint *: The amplitude of the first harmonic* = $\sqrt{a_1^2 + b_1^2}$

7) *Obtain the first three coefficients in the Fourier cosine series for y, where y is given in the following table*

x	0	1	2	3	4	5
y	4	8	15	7	6	2

[Ans. : y = 7 – 2.8 cos x + 1.5 cos 2x + 2.7 cos 3x]

8) *The turning moment T is given for a series of values of the crank angle θ degrees/ Expand T in half range cosine series.*

θ°	0	30	60	90	120	150	180
T	0	5224	8097	7850	5499	2626	0

[Ans. : t = 4882.6 + 1186.4 cos θ – 3656.7 cos 7 cos 2θ + ...]

9) *In a machine the displacement y of a given point is given for a certain angle θ as follows :*

θ°	0	30	60	90	120	150	180	210	240	270	300	330
y	7.9	8	7.2	5.6	3.6	1.7	0.5	0.2	0.9	2.5	4.7	6.8

Find the coefficients of sin 2θ in the Fourier seiries representing the above variation.

[Ans. : – 0.0731]

10) *Determine the first two harmonic of the Fourier series for the following values :*

$\theta°$	0	30	60	90	120	150	180	210	240	270	300	330
y	2.34	3.01	3.68	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

[Ans. : $y = 2.102 + (0.558 \cos x + 1.531 \sin x) + (0.354 \cos 2x + 0.145 \sin 2x)$]

11) *The turning moment T on the crankshaft of a steam engine for the crank angle $\theta°$ is given as follows :*

$\theta°$	0	15	30	45	60	75	90	105	120	135	150	165	180
T	0	2.7	5.2	7.0	8.1	8.3	7.9	6.8	5.5	4.1	2.6	1.2	0

Expand T in a series of sines upto the second harmonics. **[Ans. : $7.8 \sin \theta + 1.5 \sin 2\theta - 9.2 \sin 3\theta + 11.6 \sin 4\theta$]**

12) *Analyze the current i into its constituent harmonics as far as the fifth harmonic, the values of i and $\theta°$ being given as follows. :*

$\theta°$	0	30	60	90	120	150	180	210	240	270	300	330
i	0	24	33.5	27.5	18.2	13.0	0	– 24	– 33.5	– 27.5	– 18.2	– 30

[Ans. : $i = (5.7 \cos \theta + 30.3 \sin \theta) - (5.1 \cos 3\theta - 3.2 \sin 3\theta) - (0.6 \cos 5\theta - 0.4 \sin 5\theta)$]

13) *Using tabulated values of x and y given in the table, obtain Fourier series upto third harminic to represent the ralation between x and y.*

$\theta°$	0	30	60	90	120	150	180	210	240	270	300	330
y	0	1.0	1.732	2.0	1.732	1.0	0	0	0	0	0	0

[Ans. : $y = \frac{2}{\pi}\left[1 + \frac{\pi}{2} \sin x - \frac{2}{3} \cos 2x\right]$]

University Questions

May - 2003

1. *An alternating current I after passing through the rectifier has the form*

$$I = I_0 \sin x \qquad 0 \le x \le \pi$$
$$= 0 \qquad \pi < x < 2\pi$$

where I_0 is the maximum current and period is 2π. Obtain coefficients of Fourier series expansion for I. **[7 Marks]**

2. *Find the half-range sine series for the function*

$$f(x) = \begin{cases} \frac{2k}{l} x & 0 \le x \le \frac{l}{2} \\ \frac{2k}{l}(l - x) & \frac{l}{2} \le x \le l \end{cases}$$

[6 Marks]

Dec. - 2003

1. *Find the Fourier series for the periodic function defined as*

$$f(x) = -\pi \qquad 0 < x < \pi$$
$$= x - \pi \quad \pi < x < 2\pi$$

State the value of f(x) at $x = \pi$ and hence show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

[9 Marks]

2. *Show that, if* $0 < x < \pi$

$$\cos x = \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{m}{4m^2 - 1} \sin 2mx$$

[7 Marks]

May - 2004

1. *Expand f(x) = x sin x in the interval* $0 \le x \le 2\pi$. **[8 Marks]**

2. *Obtain the constant term and the coefficient of the first sine and cosine terms in the Fourier expansion of y as given in the following table.* **[8 Marks]**

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Dec. - 2004

1. *Find a Fourier series for*

$$f(x) = x + \frac{x^2}{4} \text{ in } (-\pi, \pi) \text{ and } f(x + 2\pi) = f(x)$$

Hence show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots = \frac{\pi^2}{12}$ **[8 Marks]**

2. *The following table gives variation of periodic current over a period.*

t (secs)	0	T/6	T/3	T/2	2T/3	5T/6	T
A(amp)	1.98	1.30	1.05	1.30	– 0.88	– 0.25	1.98

Show that there is a direct current part of 0.75 amperes in variable current and obtain amplitude of 1st harmonic. **[8 Marks]**

May - 2005

1. *Find Fourier series to represent the function* $f(x) = \pi^2 - x^2$ *in the interval* $-\pi \le x \le \pi$ *and* $f(x + 2\pi) = f(x)$. *Deduce that,*

i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$

ii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8^2}$ **[8 Marks]**

2. *The turning moment T units of the clerk shaft of a steam engine is given for a series of values of the crank angle 0 in degrees*

θ°	0	30	60	90	120	150	180
T	0	5224	8097	7850	5499	2626	0

Find the first four terms in a series of sines to represents T. Also calculate T, when $\theta = 75°$.

[8 Marks]

Dec. - 2005

1. *Expand* $f(x) = x \sin x$ *as a Fourier series in the interval* $0 \le x \le 2\pi$. **[8 Marks]**
2. *Obtain the constant term and the coefficients of first sine and cosine terms in the Fourier expansion of y as given in the following table :* **[8 Marks]**

x	0	1	2	3	4	5
y	9	18	24	28	26	20

May - 2006

1. *Obtain the Fourier series for the periodic function defined in the interval* $0 \le x \le 2\pi$ *as :*

$$f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & -\pi \le x \le 0 \end{cases}$$

and hence deduce that $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ... = \frac{1}{2}$ **[8 Marks]**

2. *The turning moment T on the crankshaft of a steam engine for crank angle* θ° *is given as follows :*

θ°	0	30	60	90	120	150	180
T	0	5.2	8.1	7.9	5.5	2.6	0

Expand T in a series of sines upto the third harmonic. Find also the value of T when $0 = 105°$.

[8 Marks]

Dec. - 2006

1. *Determine the Fourier coefficients of the function*

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ \cos \pi x, & 0 < x < 1 \end{cases} \quad \text{Period is 2.}$$

Find also the value of the series at x = 1. **[8 Marks]**

2. *Find the Fourier series upto the second harmonic to represent the function given below :* **[8 Marks]**

x	:	0	30	60	90	120	150	180
f(x)	:	2.34	3.01	3.69	4.15	3.69	2.20	0.83
x	:	210	240	270	300	330		
f(x)	:	0.51	0.88	1.09	1.19	1.64		

May - 2007

1. *Find a Fourier series for :*

   $f(x) = x^2$ *in* $(-\pi, \pi)$ *and hence deduce that*

   1) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\ldots = \frac{\pi^2}{12}$ 2) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots = \frac{\pi^2}{6}$ **[8 Marks]**

2. *Find the first two harmonic of the Fourier series for y from the data :*

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
y	2.34	3.01	3.69	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

[8 Marks]

Dec. - 2007

1. *Expand* $f(x) = x \sin x$ *in the interval* $0 \le x \le 2\pi$. **[8 Marks]**

2. *Obtain the constant term and the coefficients of the first cosine and sine terms in the expansion of y from the table :* **[8 Marks]**

x	0	1	2	3	4	5
y	9	18	24	28	26	20

May - 2008

1. *Find Fourier Series for the periodic function :*

$$f(x) = \begin{cases} 0 & -2 < x < -1 \\ 1 + x & -1 < x < 0 \quad \text{Period } 4 \\ 1 - x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

[8 Marks]

2. *The displacement y of a part of mechanism is tabulated with corresponding angular movement $x°$ of the crank. Find the Fourier series for y upto second harmonic.*

x°	0	30	60	90	120	150	180	210	240	270	300	330
y	1.80	1.10	0.30	0.16	1.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

[8 Marks]

Dec. - 2008

1. a) *Obtain the Fourier series for the periodic function defined in the interval* $-\pi < x < \pi$ *as* $f(x) = x + \frac{x^2}{4}$.

   Hence show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots = \frac{\pi^2}{12}$. **[8 Marks]**

2. *Determine the first two harmonics of the Fourier series for the following data :* **[8 Marks]**

x	0	1	2	3	4	5
y	9	18	24	28	26	20

❑❑❑

4

Reduction Formulae

4.1 Introduction

The term "reduction formula" means a formula which gives the relation between the integral and its simpler form. It reduces a given integral to a known integration form by repeated application of integration by parts.

Reduction formulae :

1) $$\int_0^{\pi/2} \cos^n x\, dx = \int_0^{\pi/2} \sin^n x dx \quad \text{(Wallis formula)}$$

$$= \frac{[(n-1) \text{ subtract } 2 \ldots .2 \text{ or } 1]}{[(n) \text{ subtract } 2 \ldots .2 \text{ or } 1]} \times \left(\frac{\pi}{2} \text{ if n is even}\right)$$

2) $$\int_0^{\pi/2} \sin^m \cos^n x\, dx = \left\{\frac{[(m-1) \text{ subtract } 2 \ldots 2 \text{ or } 1][(n-1) \text{ subtract } 2 \ldots 2 \text{ or } 1]}{[(m+n) \text{ subtract } 2 \ldots 2 \text{ or } 1]}\right\}$$

$$\times \left(\frac{\pi}{2} \begin{array}{c} \text{if m, n} \\ \text{both even} \end{array}\right)$$

$$= \left\{\frac{[(m-1) \text{ subtract } 2 \ldots 2 \text{ or } 1][(n-1) \text{ subtract } 2 \ldots 2 \text{ or } 1]}{[(m+n) \text{ subtract } 2 \ldots 2 \text{ or } 1]}\right\}$$

$\times$ (1) otherwise

3) $$\int_0^{\pi/2} \sin^m x \cos x\, dx = \frac{1}{m+1}$$

4) $$\int_0^{\pi/2} \cos^m x . \sin x . dx = \frac{1}{m+1}$$

Conversion formulae :

1) $$\int_0^{2\pi} \sin^m x . \cos^n x\, dx = \begin{cases} = 4\int_0^{\pi/2} \sin^m x . \cos^n x\, dx & \text{if m, n even} \\ = 0 & \text{otherwise} \end{cases}$$

2) $\int_0^{\pi} \sin^m x \cos^n x \, dx = \begin{cases} = 2\int_0^{\pi/2} \sin^m x . \cos^n x \, dx & \text{if n even for any m} \\ = 0 & \text{if n is odd} \end{cases}$

3) $\int_0^{2\pi} \sin^n x \, dx = \begin{cases} = 4\int_0^{\pi/2} \sin^n x \, dx & \text{if n even} \\ = 0 & \text{if n odd} \end{cases}$

4) $\int_0^{2\pi} \cos^n x \, dx = \begin{cases} = 4\int_0^{\pi/2} \cos^n x \, dx & \text{if n even} \\ = 0 & \text{if n odd} \end{cases}$

5) $\int_0^{\pi} \sin^n x \, dx = 2\int_0^{\pi/2} \sin^n x \, dx$ for any n.

i.e for all values of n.

6) $\int_0^{\pi} \cos^n x \, dx = \begin{cases} = 2\int_0^{\pi/2} \cos^n x \, dx & \text{if n even.} \\ = 0 & \text{if n odd} \end{cases}$

4.2 Type 1

Reduction formulae for trigonometric functions :

Example 4.1 : *Find reduction formula for* $\int_0^{\pi/2} \sin^n x . dx$ *and* $\int_0^{\pi/2} \cos^n x . dx$.

Solution : Step 1 :

Let $I_n = \int \sin^n x dx$

$$= \int \underset{u}{\sin^{n-1} x} \quad \underset{v}{\sin x dx}$$

Step 2 :

Using the rule of integration by parts,

$$\int u . v dx = u \int v dx - \int \left[\frac{du}{dx} \int v \, dx \right] dx$$

Step 3 :

$$I_n = \sin^{n-1} x(-\cos x) - \int (n-1) \sin^{n-2} x \cos x(-\cos x) dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^{n}x\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\,I_{n-2} - (n-1)\,I_n$$

Step 4 : Simplify

$$\therefore \quad I_n + (n-1)\,I_n = -\sin^{n-1}x\cos x + (n-1)\,I_{n-2}$$

$$\therefore \quad [1+(n-1)]\,I_n = -\sin^{n-1}x\cos x + (n-1)\,I_{n-2}$$

$$\therefore \quad n\,I_n = -\sin^{n-1}x\cos x + (n-1)\,I_{n-2}$$

$$I_n = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\,I_{n-2}$$

This is the required reduction formula for $\int \sin^n x$.

Step 5 :

Now let $\quad U_n = \int_0^{\pi/2} \sin^n x\,dx$

Hence substituting the limits

$$U_n = \left[-\frac{\sin^{n-1}x\cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\,U_{n-2}$$

$$U_n = \frac{n-1}{n}\,U_{n-2}$$

Step 6 : Applying the formula successively we get

$$U_{n-2} = \frac{n-3}{n-2}\,U_{n-4}$$

$$U_{n-4} = \frac{n-5}{n-4}\,U_{n-6}$$

$$U_{n-6} = \frac{n-7}{n-6}\,U_{n-8} \qquad \text{.... And so on.}$$

If n is even then the last term will be $U_2 = \frac{1}{2}U_0$

Where $\quad U_0 = \int_0^{\pi/2} \sin^0 x\,dx = \frac{\pi}{2}$

If n is odd then the last term will be $U_3 = \frac{2}{3}U_1$

Where $$U_1 = \int_0^{\pi/2} \sin^1 x dx = 1$$

Combining all we get,

If n is even,

$$\int_0^{\pi/2} \sin^n x.dx = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)....\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$$

If n is odd,

$$\int_0^{\pi/2} \sin^n x.dx = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)....\left(\frac{6}{7}\right)\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)\cdot 1$$

We know that,

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\int_0^{\pi/2} \sin^n x.dx = \int_0^{\pi/2} \sin^n\left(\frac{\pi}{2} - x\right)dx = \int_0^{\pi/2} \cos^n x\cdot dx$$

Thus if n is even,

$$\int_0^{\pi/2} \cos^n x.dx = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)....\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$$

If n is odd,

$$\int_0^{\pi/2} \cos^n x.dx = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)....\left(\frac{6}{7}\right)\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)\cdot 1$$

Combining all these cases we can write,

$$\int_0^{\pi/2} \cos^n x dx = \int_0^{\pi/2} \sin^n x dx$$

$$= \frac{[(n-1) \text{ subtract } 2....2 \text{ or } 1]}{[(n) \text{ subtract } 2....2 \text{ or } 1]} \times \left(\frac{\pi}{2} \text{ if n is even}\right)$$

Example 4.2 : *Find R. F. for* $\int_0^{\pi/4} \sin^n x\cdot dx$ *and hence S.T.* $\int_0^{\pi/4} \sin^6 x\cdot dx = \frac{5\pi}{64} - \frac{11}{48}$

Solution : Step 1 :

Let $$I_n = \int \sin^n x dx$$

Following the same steps of above problem we get.

$$\therefore \qquad I_n = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n} I_{n-2}$$

This is the required reduction formula for $\int \sin^n x$.

Step 2 :

Now let $\quad U_n = \int_0^{\pi/4} \sin^n x dx$

Hence putting the limits

$$U_n = \left[-\frac{\sin^{n-1}x\cos x}{n}\right]_0^{\pi/4} + \frac{n-1}{n} U_{n-2}$$

$$= -\frac{1}{n}\left[\left(\frac{1}{\sqrt{2}}\right)^{n-1}\left(\frac{1}{\sqrt{2}}\right) - 0\right] + \frac{n-1}{n} U_{n-2}$$

$$= -\frac{1}{n}\left[\left(\frac{1}{\sqrt{2}}\right)^{n}\right] + \frac{n-1}{n} U_{n-2}$$

$$\therefore \qquad U_n = -\frac{1}{2^{n/2}\cdot n} + \frac{n-1}{n} U_{n-2}$$

Step 3 :

To find $\int_0^{\pi/4} \sin^6 x \cdot dx$ put n = 6

i.e. $\quad U_6 = \frac{1}{2^3\cdot 6} + \frac{5}{6} U_{6-2}$

$$= -\frac{1}{48} + \frac{5}{6} U_4$$

Where $\quad U_4 = -\frac{1}{2^2\cdot 4} + \frac{3}{4} U_2$

$$= -\frac{1}{16} + \frac{3}{4} U_2$$

$$U_2 = -\frac{1}{4} + \frac{1}{2} U_0$$

Where $\quad U_0 = \int_0^{\pi/4} \sin^0 x dx = \int_0^{\pi/4} 1 dx = \frac{\pi}{4}$

$$\therefore \quad \int_0^{\pi/4} \sin^6 x dx = -\frac{1}{48} + \frac{5}{6}\left[-\frac{1}{16} + \frac{3}{4}\left(-\frac{1}{4} + \frac{1}{2}\frac{\pi}{4}\right)\right]$$

$$= -\frac{1}{48} + \frac{5}{6}\left[\frac{-1}{16} - \frac{3}{16} + \frac{3\pi}{32}\right]$$

$$= -\frac{1}{48} - \frac{5}{96} - \frac{15}{96} + \frac{5\pi}{64}$$

$$= -\frac{22}{96} + \frac{5\pi}{64}$$

$$= \frac{5\pi}{64} - \frac{11}{48}$$

Example 4.3 : *Find R.F. for* $\int_0^{\pi/4} \cos^{2n} x \cdot dx$ *and hence show that*

$$\int_0^{\pi/4} \cos^6 x \cdot dx = \frac{11}{48} + \frac{5\pi}{64}$$

Solution : Step 1 :

Let $\quad I_n = \int \cos^{2n} x dx$

$$= \int \underset{u}{\cos^{2n-1} x}\ \underset{v}{\cos x} dx \qquad \text{... (Note this step)}$$

Step 2 :

Using the rule of by parts,

$$\int u \cdot v dx = u\int v\, dx - \int\left[\frac{du}{dx}\int v\, dx\right] dx$$

Step 3 :

$$= \cos^{2n-1}x(\sin x) - \int (2n-1)\cos^{2n-2}x(-\sin x)(\sin x)\, dx$$

$$= \cos^{2n-1}x(\sin x) + (2n-1)\int \cos^{2n-2}x \sin^2 x\, dx$$

$$= \cos^{2n-1}x(\sin x) + (2n-1)\int \cos^{2n-2}x(1-\cos^2 x)\, dx$$

$$= \cos^{2n-1}x(\sin x) + (2n-1)\left[\int \cos^{2n-2}x\, dx - \int \cos^{2n} x dx\right]$$

$$= \cos^{2n-1}x(\sin x) + (2n-1)\int \cos^{2n-2}x\, dx - (2n-1) I_n$$

$$[1 + 2n - 1]\, I_n = \cos^{2n-1}x \sin x + (2n-1) I_{n-1}$$

$$I_n = \frac{\cos^{2n-1}x \sin x}{2n} + \frac{2n-1}{2n} I_{n-1} \qquad \text{... (1)}$$

Step 4 :

Let, $\quad U_n = \int_0^{\pi/4} \cos^{2n} x\, dx$ Substituting limits in equation (1) we get

Put the limits,

$$U_n = \left[\frac{\cos^{2n-1}x \sin x}{2n}\right]_0^{\pi/4} + \frac{2n-1}{2n} U_{n-1}$$

Step 5 : Simplify

$$= \frac{1}{2n}\left[\left(\frac{1}{\sqrt{2}}\right)^{2n-1}\frac{1}{\sqrt{2}} - 0\right] + \frac{2n-1}{2n} U_{n-1}$$

$$= \frac{1}{2n}\left[\left(\frac{1}{\sqrt{2}}\right)^{2n}\right] + \frac{2n-1}{2n} U_{n-1}$$

$$= \frac{1}{2n}\,\frac{1}{2^n} + \frac{2n-1}{2n} U_{n-1}$$

$$U_n = \frac{1}{n}\cdot\frac{1}{2^{n+1}} + \frac{2n-1}{2n} U_{n-1} \qquad \text{... (1)}$$

Step 6 : Put n = 3 in (1), we get

Now $$\int_0^{\pi/4} \cos^6 x\,dx = \frac{1}{3.2^4} + \frac{6-1}{6} U_2$$

$$= \frac{1}{48} + \frac{5}{6}\left[\frac{1}{2.2^3} + \frac{3}{4} U_1\right]$$

$$= \frac{1}{48} + \frac{5}{6}\left[\frac{1}{16} + \frac{3}{4} U_1\right]$$

$$= \frac{1}{48} + \frac{5}{96} + \frac{15}{24}\left[\frac{1}{1.2^2} + \frac{1}{2} U_0\right]$$

$$= \frac{1}{48} + \frac{5}{96} + \frac{15}{24.4} + \frac{15}{48} U_0$$

$$= \frac{1}{48} + \frac{20}{96} + \frac{15}{48}\,\frac{\pi}{4}$$

$$= \frac{11}{48} + \frac{5\pi}{16 \times 4} = \frac{11}{48} + \frac{5\pi}{64}$$

Example 4.4 : *Find R.F.* $\int_0^{\pi/3} \cos^n x\,dx$ *and evaluate* $\int_0^{\pi/3} \cos^6 x\,dx$.

Solution : Step 1 :

Let $$I_n = \int_0^{\pi/3} \cos^n x\,dx$$

$$= \int_0^{\pi/3} \cos^{n-1}x \cdot \cos x \cdot dx \qquad \text{... (Note this step)}$$

Step 2 : Using the rule of by parts

$$\int u.v dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx$$

Step 3 :

$$= \left[\cos^{n-1}x \cdot \sin x\right]_0^{\pi/3} - \int_0^{\pi/3}(n-1)\cos^{n-2}x(-\sin x)\sin x \cdot dx$$

$$= \left(\frac{1}{2}\right)^{n-1}\frac{\sqrt{3}}{2} + (n-1)\int_0^{\pi/3}\cos^{n-2}x\left(1-\cos^2 x\right)dx$$

$$= \frac{\sqrt{3}}{2^n} + (n-1)\int_0^{\pi/3}\cos^{n-2}x dx - (n-1)\int_0^{\pi/3}\cos^n x dx$$

Step 4 :

Simplifying we get,

$$I_n = \frac{\sqrt{3}}{2^n} + (n-1)I_{n-2} - (n-1)I_n$$

$$I_n = \frac{\sqrt{3}}{n.2^n} + \frac{n-1}{n}I_{n-2}$$

Step 5 :

Now put $n = 6, I_6 = \dfrac{\sqrt{3}}{6.2^6} + \dfrac{5}{6}I_4$

Put $n = 4, I_4 = \dfrac{\sqrt{3}}{4.2^4} + \dfrac{3}{4}\cdot I_2$

Put $n = 2,$

$$I_2 = \frac{\sqrt{3}}{2.2^2} + \frac{1}{2}I_0 = \frac{\sqrt{3}}{8} + \frac{1}{2}\int_0^{\pi/3}\cos^0 x dx = \frac{\sqrt{3}}{8} + \frac{\pi}{6}$$

Resubstituting we get

$$\therefore \quad I_6 = \frac{\sqrt{3}}{384} + \frac{5}{6}\left[\frac{\sqrt{3}}{64} + \frac{3}{4}\left(\frac{\sqrt{3}}{8} + \frac{\pi}{6}\right)\right] = \frac{3\sqrt{3}}{8} + \frac{5\pi}{48}$$

Example 4.5 : *Obtain R.F. for* $I_{m,n} = \int_0^{\pi/2} \sin^m x \cdot \cos^n x \cdot dx$

Solution : Step 1 :

Let
$$U_{m,n} = \int \sin^m x \cos^n x dx$$

$$= \int \sin^m x \cos^{n-1} x \cos x dx \qquad \text{... (Note this step)}$$

$$= \int \underbrace{\cos^{n-1} x}_{u} \underbrace{(\sin^m x \cos x)\, dx}_{v}$$

Step 2 : Using the rule of by parts,

$$\int u.v dx = u\int v\, dx - \int\left[\frac{du}{dx}\int v\, dx\right] dx \quad \text{and}$$

Since,
$$\int \sin^m x \cos x dx = \frac{\sin^{m+1} x}{m+1}$$

Step 3 :

$$U_{m,n} = \cos^{n-1} x\left[\frac{\sin^{m+1} x}{m+1}\right] + (n-1)\int \cos^{n-2} x \sin x\left[\frac{\sin^{m+1}}{m+1}\right] dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x \sin^{m+2} x dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x \sin^m x \sin^2 x dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x \sin^m x (1 - \cos^2 x)\, dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x \sin^m x dx$$

$$- \frac{n-1}{m+1}\int \cos^n x \sin^m x dx$$

$$\therefore \quad U_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} U_{m,n-2} - \frac{(n-1)}{m+1} U_{m,n}$$

Step 4 : Simplifying we get

$$\therefore \quad \left[1 + \frac{(n-1)}{m+1}\right] U_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} U_{m,n-2}$$

$$\therefore \quad \left[\frac{(m+1)+(n-1)}{m+1}\right] U_{m,n} = \frac{\cos^{n-1}x \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} U_{m,n-2}$$

$$\therefore \quad \left[\frac{m+n}{m+1}\right] U_{m,n} = \frac{\cos^{n-1}x \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} U_{m,n-2}$$

$$\therefore \quad U_{m,n} = \frac{\cos^{n-1}x \sin^{m+1}x}{m+n} + \frac{n-1}{m+n} U_{m,n-2}$$

This is the reduction formula for $U_{m,n}$.

Step 5 :

Let $$I_{m,n} = \int_0^{\pi/2} \sin^m x \cdot \cos^n x \cdot dx$$

$$\therefore \quad I_{m,n} = \left(\frac{\cos^{n-1}x \sin^{m+1}x}{m+1}\right)_0^{\pi/2} + \frac{n-1}{m+1} U_{m,n-2}$$

$$\therefore \quad I_{m,n} = (0-0) + \frac{n-1}{m+n} I_{m,n-2}$$

$$\therefore \quad I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$$

Applying the same formula repeatedly we get,

$$\therefore \quad I_{m,n-2} = \frac{n-3}{m+n-2} I_{m,n-4}$$

$$\therefore \quad I_{m,n-4} = \frac{n-5}{m+n-4} I_{m,n-6}$$

$$\therefore \quad I_{m,n-6} = \frac{n-7}{m+n-6} I_{m,n-8} \text{ and so on.}$$

Case one : If n is even then,

$$\therefore \quad I_{m,n} = \left(\frac{n-1}{m+n}\right)\left(\frac{n-3}{m+n-2}\right)\left(\frac{n-5}{m+n-4}\right)\cdots\left(\frac{1}{m+2}\right) I_{m,0}$$

Now $$I_{m,0} = \int_0^{\pi/2} \sin^m x \cdot \cos^0 x \cdot dx = \int_0^{\pi/2} \sin^m x \cdot dx$$

Now there are two possibilities.

If m is even,

$$\int_0^{\pi/2} \sin^m x \cdot dx = \left(\frac{m-1}{m}\right)\left(\frac{m-3}{m-2}\right)\left(\frac{m-5}{m-4}\right)\cdots\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$$

If m is odd,

$$\int_0^{\pi/2} \sin^m x \cdot dx = \left(\frac{m-1}{m}\right)\left(\frac{m-3}{m-2}\right)\left(\frac{m-5}{m-4}\right)\cdots\left(\frac{6}{7}\right)\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)\cdot 1$$

Case two : If n is odd then,

$$\therefore \quad I_{m,n} = \left(\frac{n-1}{m+n}\right)\left(\frac{n-3}{m+n-2}\right)\left(\frac{n-5}{m+n-4}\right)\cdots\left(\frac{2}{m+3}\right) I_{m,1}$$

Now $\quad I_{m,1} = \int_0^{\pi/2} \sin^m x \cdot \cos^1 x \cdot dx = \dfrac{1}{m+1}$ for any m.

Combining all the cases we get

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x dx = \frac{[(m-1)\text{ subtract } 2\ldots.2 \text{ or } 1][(n-1)\text{ subtract } 2\ldots.2 \text{ or } 1]}{[(m+n)(m+n-2)\text{ subtract } 2\ldots.2 \text{ or } 1]} \times \frac{\pi}{2}$$

(if m, n both even)

$$= \frac{[(m-1)\text{ subtract } 2\ldots.2 \text{ or } 1][(n-1)\text{ subtract } 2\ldots.2 \text{ or } 1]}{[(m+n)(m+n-2)\text{ subtract } 2\ldots.2 \text{ or } 1]} \times (1)$$

(otherwise)

Example 4.6 : *Find R.F. for* $\int \tan^n x.dx$

Solution : Step 1 :

Let $\quad I_n = \int \tan^n x dx$

$$= \int \tan^{n-2} x \cdot \tan^2 x \cdot dx \quad \ldots \text{(Note this step)}$$

Step 2 :

Using the result of $\tan^2 x$ we have,

$$I_n = \int \tan^{n-2} x \cdot (\sec^2 x - 1)\, dx$$

$$I_n = \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx$$

Step 3 :

As $\quad \int f(x)^n f'(x) = \dfrac{[f(x)]^{n+1}}{n+1}$

$$\therefore \quad I_n = \frac{\tan^{n-1}}{n-1} - I_{n-2}$$

$$\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$$

This is the required reduction formula.

Example 4.7 : *If* $U_n = \int_0^{\pi/4} \tan^n\theta \cdot d\theta$ *show that* $n(U_{n+1} + U_{n-1}) = 1$ **(Dec.-2000)**

Solution : Step 1 :

$$U_n = \int_0^{\pi/4} \tan^n\theta \cdot d\theta$$

and

$$U_{n+1} = \int_0^{\pi/4} \tan^{n+1}\theta \cdot d\theta$$

Step 2 :

$$U_{n+1} = \int_0^{\pi/4} \tan^{n-1}\theta \tan^2\theta \, d\theta$$

$$= \int_0^{\pi/4} \tan^{n-1}\theta(\sec^2\theta - 1)d\theta$$

$$= \int_0^{\pi/4} \tan^{n-1}\theta\sec^2\theta d\theta - \int_0^{\pi/4} \tan^{n-1}\theta d\theta$$

Step 3 :

$$\therefore \quad U_{n+1} = \left[\frac{\tan^n\theta}{n}\right]_0^{\pi/4} - U_{n-1} = \frac{1}{n} - U_{n-1}$$

$$U_{n+1} + U_{n-1} = \frac{1}{n}$$

$$n(U_{n+1} + U_{n-1}) = 1$$

Example 4.8 : *If* $U_n = \int \tanh^n x \cdot dx$ *then show that* $U_n = \frac{\tanh^{n-1} x}{n-1} + U_{n-2}$

Solution : Step 1 :

we have,

$$U_n = \int \tanh^n x \, dx$$

$$= \int \underbrace{\tanh^{n-2} x}_{u} \; \underbrace{\tanh^2 x \cdot dx}_{v} \qquad \text{... (Note this step)}$$

Step 2 : Using the trigonometric formula

$$U_n = \int \tanh^{n-2} x \cdot [1 - \sec h^2 x] \cdot dx$$

$$= \int \tanh^{n-2} x \cdot dx - \int (\tanh x)^{n-2} \cdot (\sec h^2 x) \cdot dx$$

$$= \int \tanh^{n-2}x \cdot dx - \frac{(\tan x)^{n-1}}{n-1}$$

$$\therefore \qquad U_n = U_{n-2} - \frac{\tanh^{n-1}x}{n-1}$$

Example 4.9 : *Find R.F. for $I_n = \int \sec^n x\, dx$ and hence find $\int_0^{\pi/4} \sec^6 x \cdot dx$.* **(Dec.-2004)**

Solution : Step 1 :

We have, $\qquad I_n = \int \sec^n x dx$

$$= \int \underset{u}{\underline{\sec^{n-2}x}} \; \underset{v}{\underline{\sec^2 x dx}} \qquad \text{... (Note this step)}$$

Step 2 :

Using the rule of by parts

$$\int u \cdot v dx = u \int v dx - \int \left[\frac{du}{dx} \int v\, dx\right] dx$$

$$I_n = \sec^{n-2}x \tan x - \int (n-2) \sec^{n-3}x(\sec x \tan x) \tan x dx$$

$$= \sec^{n-2}x \tan x - (n-2) \int \sec^{n-2}x(\sec^2 x - 1)\, dx$$

$$= \sec^{n-2}x \tan x - (n-2) \int \sec^n x dx + (n-2) \int \sec^{n-2} x dx$$

$$I_n = \sec^{n-2}x \tan x - (n-2)\, I_n + (n-2)\, I_{n-2}$$

$$(n-1)\, I_n = \sec^{n-2}x \tan x + (n-2)\, I_{n-2}$$

Step 3 :

Simplifying we get,

$$I_n = \frac{\sec^{n-2}x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$\int \sec^n x dx = \frac{\sec^{n-2}x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx$$

This is the required reduction formula.

Step 4 :

To find $\int_0^{\pi/4} \sec^n x\, dx$ put the limits. Let $U_n = \int_0^{\pi/4} \sec^n x\, dx$

$$\int_0^{\pi/4} \sec^n x\, dx = \left(\frac{\sec^{n-2}x \tan x}{n-1}\right)_0^{\pi/4} + \frac{n-2}{n-1} \int_0^{\pi/4} \sec^{n-2} x.\, dx$$

$$U_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} U_{n-2}$$

To find $\int_0^{\pi/4} \sec^6 x . dx$. put n = 6, 4, 2

$$U_6 = \frac{(\sqrt{2})^4}{5} + \frac{4}{5} U_4$$

$$U_4 = \frac{(\sqrt{2})^2}{3} + \frac{2}{3} U_2$$

$$U_2 = \int_0^{\pi/4} \sec^2 x . dx. = [\tan x]_0^{\pi/4} = 1 \text{ on simplification}$$

$$\therefore \quad U_6 = \frac{4}{5} + \frac{4}{5}\left(\frac{2}{3} + \frac{2}{3}(1)\right)$$

$$U_6 = \frac{28}{15}$$

Example 4.10 : *If* $I_{m,n} = \int \cos^m x \cdot \sin nx \cdot dx$ *then show that* $(m+n)\, I_{m,n} = -\cos^m x \cdot \cos nx + m\, I_{m-1,\, n-1}$ *and hence evaluate* $\int_0^{\pi/2} \cos^5 x \cdot \sin 3x \cdot dx$

Solution : Step 1 :

We have, $\quad I_{m,n} = \int \underset{u}{\cos^m x} \cdot \underset{v}{\sin nx} \cdot dx$

Step 2 :

Using the rule of by parts

$$\int u \cdot v dx = u \int v\, dx - \int \left[\frac{du}{dx} \int v\, dx\right] dx$$

Step 3 :

$$= \left[-\frac{\cos nx}{n} \cdot \cos^m x\right] - \int \left(-\frac{\cos nx}{n}\right) m \cos^{m-1} x \cdot (-\sin x)\, dx$$

$$= -\frac{\cos nx \cdot \cos^m x}{n} - \frac{m}{n} \int \cos^{m-1} x \cdot (\cos nx \cdot \sin x)\, dx \qquad \dots (1)$$

Step 4 :

Consider

$$\sin(n-1)x = \sin(nx - x) = \sin nx \cos x - \cos nx \cdot \sin x$$

$$\cos nx \cdot \sin x = \sin nx \cdot \cos x - \sin(n-1)x$$

$\therefore$ Equation (1) becomes,

$$= \frac{-\cos nx \cdot \cos^m x}{n} - \frac{m}{n}\int \cos^{m-1} x\left[\sin nx \cos x - \sin(n-1)x\right]$$

$$= \frac{-\cos nx \cdot \cos^m x}{n} - \frac{m}{n}\left[\int \cos^m x \sin nx - \int \cos^{m-1} x \sin(n-1)\, x dx\right]$$

$$\therefore \quad I_{m,n} = \frac{-\cos nx \cdot \cos^m x}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

$$\therefore \quad \left(1+\frac{m}{n}\right) I_{m,n} = \frac{-\cos nx \cdot \cos^m x}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$(m+n) I_{m,n} = -\cos^m x \cdot \cos nx + m I_{m-1,n-1}$$

$$I_{m,n} = \frac{-\cos^m x \cdot \cos nx}{(m+n)} + \frac{m}{(m+n)} I_{m-1,n-1} \qquad \dots (2)$$

To evaluate $\int_0^{\pi/2} \cos^5 x \cdot \sin 3x \cdot dx$, Let $U_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx \, dx$

Substituting the limits we get,

$$U_{m,n} = \left(\frac{-\cos^m x \cdot \cos nx}{(m+n)}\right)_0^{\pi/2} + \frac{m}{(m+n)} U_{m-1,n-1}$$

$$U_{m,n} = \left(\frac{1}{(m+n)}\right) + \frac{m}{(m+n)} U_{m-1,n-1}$$

Put m = 5 and n = 3

$$U_{5,3} = \frac{1}{8} + \frac{5}{8} U_{4,2}$$

$$U_{4,2} = \frac{1}{6} + \frac{4}{6} U_{3,1}$$

$$U_{3,1} = \int_0^{\pi/2} \cos^3 x \cdot \sin x \cdot dx = \frac{1}{4}, \quad \textbf{Resubstituting we get}$$

$$U_{5,3} = \frac{1}{8} + \frac{5}{8}\left(\frac{1}{6} + \frac{4}{6}\cdot\frac{1}{4}\right) = \frac{1}{3}$$

Example 4.11 : *If* $I_n = \int_0^{\pi/2} \cos^n x \cos nx\,dx$ *prove that* $I_n = \frac{1}{2} I_{n-1} = \frac{\pi}{2^{n+1}}$

[Dec.-2002, May-2006]

Solution :

$$I_n = \int_0^{\pi/2} \underset{u}{\cos^n x}\ \underset{v}{\cos nx}\,dx$$

Applying integration by parts,

$$= \left[\cos^n x \cdot \frac{\sin nx}{n}\right]_0^{\pi/2} - \int_0^{\pi/2} n \cos^{n-1} x(-\sin x)\cdot\frac{\sin nx}{n}\,dx$$

$$= 0 + \int_0^{\pi/2} \cos^{n-1} x \cdot \sin x \cdot \sin nx\,dx$$

We know that,

$$\cos(n-1)x = \cos nx \cos x + \sin nx \sin x$$

$$\therefore \quad \boxed{\sin nx \sin x = \cos(n-1)x - \cos nx \cos x}$$

$\therefore$ The integral becomes,

$$I_n = \int_0^{\pi/2} \cos^{n-1} x\left[\cos(n-1)x - \cos nx \cos x\right]dx.$$

$$= \int_0^{\pi/2} \cos^{n-1} x \cos(n-1)x\,dx - \int_0^{\pi/2} \cos^n x \cos nx\,dx$$

$$I_n = I_{n-1} - I_n$$

$$\therefore \quad 2I_n = I_{n-1}$$

$$\therefore \quad I_n = \frac{1}{2} I_{n-1}$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot I_{n-2}$$

$$= \frac{1}{2^2} I_{n-2}$$

$$= \frac{1}{2^2}\cdot\frac{1}{2}\cdot I_{n-3}$$

$$= \frac{1}{2^3} I_{n-3}$$

$$\vdots$$

$$= \frac{1}{2^n} I_{n-(n)}$$

$$= \frac{1}{2^n} I_0$$

Now $\qquad I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2}$

$\therefore \qquad I_n = \frac{1}{2^n} \cdot \frac{\pi}{2}$

$$= \frac{\pi}{2^{n+1}}$$

Example 4.12 : *Find R.F.for* $I_n = \int \frac{\sin nx}{\sin x} \cdot dx$ *and show that* $\int_0^{\pi} \frac{\sin nx}{\sin x} \cdot dx = \pi$ *or* 0 *according as n is odd or even respectively.*

Solution : Step 1 :

We are given that,

$$I_n = \int \frac{\sin nx}{\sin x} \cdot dx \qquad \text{... (1)}$$

$$\boxed{\sin C - \sin D = 2 \sin \left[\frac{C-D}{2}\right] \cdot \cos \left[\frac{C+D}{2}\right]}$$

Step 2 :

We know that

$$\sin nx - \sin (n-2) x = 2 \sin x \cdot \cos (n-1) x$$

$$\therefore \qquad \frac{\sin nx - \sin(n-2)x}{\sin x} = 2 \cos (n-1) x$$

Step 3 :

Integrating we get,

$$\int \frac{\sin nx}{\sin x} \cdot dx - \int \frac{\sin(n-2) x}{\sin x} \cdot dx = 2\int \cos(n-1)x \cdot dx$$

$$\therefore \qquad I_n - I_{n-2} = 2\left[\frac{\sin(n-1) x}{n-1}\right] \qquad \text{from (1)}$$

$$\therefore \qquad I_n = I_{n-2} + \frac{2}{n-1} [\sin (n-1) x] \qquad \text{... (2)}$$

which is required R.F. for I_n ?

Step 4 : Again, we have from equation (2), Let $U_n = \int_0^{\pi} \frac{\sin nx}{\sin x} dx$

$$U_n = \int_0^{\pi} \frac{\sin nx}{\sin x} \cdot dx = U_{n-1} + \frac{2}{n-1}[\sin(x-1) x]_0^{\pi}$$

$$\therefore \qquad U_n = U_{n-2}$$

$$\therefore \qquad U_{n-2} = U_{n-4}$$

$$\therefore \qquad U_{n-4} = U_{n-6} = \text{.... and so on}$$

It follows that if n is even,

$$U_n = U_0 = \int_0^{\pi} \frac{\sin 0x}{\sin x} \cdot dx = 0$$

And if n is odd, then

$$U_n = U_1 = \int_0^{\pi} \frac{\sin x}{\sin x} \cdot dx = \pi$$

Example 4.13 : *If* $I_n = \int_0^{\pi/4} \frac{\sin(2n-1)x}{\sin x} dx$ *then prove that* $n(I_{n+1} - I_n) = \sin \frac{n\pi}{2}$. *Hence find* I_3 **(May-2001)**

Solution : From I_n we can write $I_{n+1} = \int_0^{\pi/4} \frac{\sin(2n+1)x}{\sin x} dx$

$$\therefore \qquad I_{n+1} - I_n = \int_0^{\pi/4} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x} dx$$

$$= \int_0^{\pi/4} \frac{2\cos 2nx \cdot \sin x}{\sin x} dx$$

$$= 2\left[\frac{\sin 2nx}{2n}\right]_0^{\pi/4}$$

$$I_{n+1} - I_n = \frac{1}{n} \sin \frac{n\pi}{2}$$

To find I_3,

Put $n = 2$, $I_3 - I_2 = 0$ i.e. $I_3 = I_2$

Put $n = 1$, $I_2 - I_1 = 1$

$$\therefore \qquad I_2 = 1 + I_1$$

But $$I_1 = \int_0^{\pi/4} 1 \cdot dx = \frac{\pi}{4}$$

$$\therefore \qquad I_3 = I_2 = 1 + \frac{\pi}{4}$$

Exercise 4.1

1) *Show that* $\int_0^{\pi/2} \cos^n x \cdot dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot 1$ *if n is odd*

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} \text{ if } n \text{ is even}$$

2) *Show that* $\int \cos^{2n} \theta \cdot d\theta = \frac{1}{2n} \tan\theta \cdot \cos^{2n}\theta + \left(1 - \frac{1}{2n}\right) \int \cos^{2n-2}\theta \cdot d\theta$

3) $U_n = \int_0^{\pi/4} \sin^{2n} x\, dx$ *prove that* $U_n = \left(1 - \frac{1}{2n}\right) U_{n-1} - \frac{1}{n 2^{n+1}}$ **(Dec.-2004)**

4) *If* $U_n = \int_0^{\pi/4} \tan^n \theta \cdot d\theta$ *show that* $U_n = \frac{1}{n-1} - U_{n-2}$ **(May-1998)**

5) *Find R.F. for* $\int_0^{\pi/4} \tan^6 x\, dx$ *and evaluate* $\int_0^a x^5 \left(2a^2 - x^2\right)^{-3} dx$ **[Ans. : $\frac{1}{2}\log 2 - \frac{1}{4}$]**

6) *Find R.F. for* $\int \cot^n x \cdot dx$

7) *If* $I_n = \int_{\pi/4}^{\pi/2} \cot^n \theta d\theta$ *prove that* $I_n = \frac{+1}{n-1} - I_{n-2}$. *Hence evaluate* $\int_{\pi/4}^{\pi/2} \cot^6 \theta d\theta$ **(May-2005)**

8) *Find R.F. for* $\int \operatorname{cosec}^n x \cdot dx$ *and show that* $\int_{\pi/4}^{\pi/2} \operatorname{cosec}^6 \theta d\theta = \frac{28}{15}$

9) *Prove that* $\int \sin n\theta \cdot \sec\theta \cdot d\theta = -2\frac{\cos(n-1)\theta}{n-1} - \int \sin(n-1)\theta \cdot \sec\theta \cdot d\theta$

Hint : *Use* $\frac{\sin n\theta + \sin(n-2)\theta}{\cos\theta} = 2\sin(n-1)\theta$... *and hence evaluate* $\int_0^{\pi/2} \frac{\cos 5\theta \sin 3\theta}{\cos\theta} d\theta$

Hint : *First find* $\int_0^{\pi/2} \sin n\theta \cdot \sec\theta \cdot d\theta$ *then use* $\cos 5\theta \cdot \sin 3\theta = \frac{1}{2}(\sin 8\theta - \sin 2\theta)$ **[Ans. : $\frac{-181}{105}$]**

10) *If* $U_n = \int \cos n\theta \cdot \operatorname{cosec}\theta$ *then show that* $U_n - U_{n-2} = \frac{2\cos(n-1)\theta}{n-1}$ *and show that* $\int_0^{\pi/2} \frac{\sin 3\theta \cdot \sin 5\theta}{\sin\theta} d\theta = \frac{71}{105}$

Hint : *Use* $\sin 3\theta \cdot \sin 5\theta = \frac{1}{2}(\cos 2\theta - \cos 8\theta)$

11) *If* $I_n = \int_0^{\pi/4} \frac{\cos(2n-1)x}{\sin x} dx$ *prove that* $n\left(I_{n+1} - I_n\right) = 1$

4.3 Type 2 : Combination of Algebraic and Trigonometric Functions

Example 4.14 : *If* $I_n = \int_0^{\pi/2} x \cdot \sin^n x \cdot dx$ *show that*

$I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n^2}$ *and hence evaluate* I_5 **[May-2000, May-2002]**

Solution :

Step 1 :

We have,
$$I_n = \int_0^{\pi/2} \underset{v}{(x \sin x)} \cdot \underset{u}{\sin^{n-1} x} \cdot dx \quad \text{... (Note this step)}$$

Step 2 :

Using the rule of by parts
$$\int u \cdot v dx = u \int v\, dx - \int \left[\frac{du}{dx} \int v\, dx\right] dx$$

Step 3 :

$$\int x \sin x \cdot dx = x(-\cos x) - (1)(-\sin x)$$
$$= -x \cos x + \sin x$$
$$I_n = \left[(-x \cos x + \sin x) \sin^{n-1} x\right]_0^{\pi/2}$$
$$- \int_0^{\pi/2} [-x \cos x + \sin x](n-1) \sin^{n-2} x \cdot \cos x \cdot dx$$
$$= 1 - (n-1) \int_0^{\pi/2} \left[-x \sin^{n-2} x \cdot \cos^2 x + \sin^{n-1} x \cdot \cos x\right] \cdot dx$$
$$= 1 + (n-1)\left[\int_0^{\pi/2} x \sin^{n-2} x\left(1 - \sin^2 x\right) dx - \int_0^{\pi/2} \sin^{n-1} x \cos x dx\right]$$

Since $\int_0^{\pi/2} \sin^{n-1} x \cdot \cos x dx = \frac{1}{n}$ $\quad$ [Use $\int_0^{\pi/2} \sin^m x \cdot \cos x\, dx = \frac{1}{m+1}$]

$$= 1 + (n-1) \int_0^{\pi/2} x \sin^{n-2} x \cdot dx - (n-1) \int_0^{\pi/2} x \cdot \sin^n x \cdot dx - (n-1)\frac{1}{n}$$
$$= 1 + (n-1) I_{n-2} - (n-1) I_n - \frac{(n-1)}{n}$$
$$n I_n = \frac{1}{n} + (n-1) I_{n-2}$$

Step 4 : Simplify,

$$\therefore \quad I_n = \frac{1}{n^2} + \frac{n-1}{n} \cdot I_{(n-2)} \qquad \ldots (1)$$

is the required reduction formula.

From equation (1), substituting n = 5, 3, 1

$$I_5 = \frac{1}{25} + \frac{4}{5} I_3$$

$$I_3 = \frac{1}{9} + \frac{2}{3} I_1$$

And
$$I_1 = \int_0^{\pi/2} x \sin x \cdot dx$$

$$= \left[(x)(-\cos x) - (1)(-\sin x)\right]_0^{\pi/2} = 1$$

$$\therefore \quad I_5 = \frac{1}{25} + \frac{4}{5}\left(\frac{1}{9} + \frac{2}{3} \cdot 1\right) = \frac{149}{225}$$

Example 4.15 : *If* $I_n = \int_0^{\pi/2} x \cdot \cos^n x \cdot dx$ *prove that,*

$I_n = -\frac{1}{n^2} + \frac{n-1}{n} I_{n-2}$ *and find* I_3 **[May-2004, May-2005, May-2006]**

Hint : $I_n = \int_0^{\pi/2} \underset{v}{(x \cos x)} \cdot \underset{u}{\cos^{n-1}} x \cdot dx$ *and note* $\int x \cos x \, dx = (x \sin x + \cos x)$

Solution : Given : $I_n = \int_0^{\pi/2} x \cos^n x dx = \int_0^{\pi/2} \underset{u}{x \cos^{n-1} x} \cdot \underset{v}{\cos x} \cdot dx$ ... (Note this step)

$$I_n = \left[x \cos^{n-1} x \cdot (\sin x)\right]_0^{\pi/2}$$

$$- \int_0^{\pi/2} \left[\cos^{(n-1)} x (1) + x \cdot (n-1) \cos^{n-2} x(-\sin x)\right] \sin x dx$$

$$= [0] - \int_0^{\pi/2} \cos^{n-1} x \cdot \sin x dx + (n-1) \int_0^{\pi/2} x \cos^{n-2} x \cdot \sin^2 x \, dx$$

$$= \left[\frac{\cos^n x}{n}\right]_0^{\pi/2} + (n-1) \int_0^{\pi/2} x \cos^{n-2} x (1 - \cos^2 x) \, dx$$

$$= \frac{-1}{n} + (n-1)\int_0^{\pi/2} x\cos^{n-2}x dx - (n-1)\int_0^{\pi/2} x \cdot \cos^n x \cdot dx$$

$$= \frac{-1}{n} + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n = \frac{-1}{n^2} + \frac{n-1}{n} I_{n-2}$$

For finding I_3 put n = 3

$$\therefore \quad I_3 = \frac{-1}{9} + \frac{2}{3} I_1$$

Also
$$I_1 = \int_0^{\pi/2} \underset{u}{x} \underset{v}{\cos x}\, dx$$

$$= [x \sin x + \cos x]_0^{\pi/2} = \frac{\pi}{2} - 1$$

Thus
$$I_3 = \frac{-1}{9} + \frac{2}{3}\left(\frac{\pi}{2} - 1\right)$$

$$\therefore \quad I_3 = \frac{\pi}{3} - \frac{7}{9}$$

Example 4.16 : *If* $I_n = \int_0^{\pi/2} x^n \cdot \sin x \cdot dx$ *then show that*

$$I_n + n(n-1) I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

(Dec.-2003)

Solution : We are given that,

$$I_n = \int_0^{\pi/2} \underset{u}{x^n} \underset{v}{\sin x} dx$$

Using the rule of by parts

$$\int u \cdot v dx = u\int v\, dx - \int \left[\frac{du}{dx}\int v dx\right] dx$$

$$\therefore \quad I_n = \left[x^n(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} nx^{n-1}(-\cos x)\, dx$$

$$= 0 + n\int_0^{\pi/2} x^{n-1} \cos x\, dx$$

Again integrating by parts,

$$= \left[n \cdot x^{n-1} \cdot \sin x\right]_0^{\pi/2} - n\int_0^{\pi/2} (n-1)\, x^{n-2} \sin x \, dx$$

Simplifying we get,

$$I_n = n \cdot \left(\frac{\pi}{2}\right)^{n-1} (1) - 0 - n(n-1)\int_0^{\pi/2} x^{n-2} \sin x dx$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\, I_{n-2}$$

$$\therefore \qquad I_n + n(n-1)\, I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

Example 4.17 : *If* $I_n = \int_0^{\infty} e^{-x} \cdot \sin^n x \cdot dx$ *then show that* $(n^2+1)\, I_n = n(n-1)\, I_{n-2}$ *and evaluate* I_4 **(May-1999, Dec.-1999)**

Solution : We are given that,

$$I_n = \int_0^{\infty} \underset{v}{e^{-x}} \cdot \underset{u}{\sin^n x} \cdot dx$$

Using the rule of by parts, $\int u \cdot v dx = u\int v dx - \int \left[\frac{du}{dx}\int v\, dx\right] dx$

$$\therefore \qquad I_n = \left[\sin^n x \cdot (-e^{-x})\right]_0^{\infty} - \int_0^{\infty} n \sin^{n-1} x \cos x \,(-e^{-x})\, dx$$

$$= 0 + n\int_0^{\infty} \underset{v}{e^{-x}} \,(\underset{u}{\sin^{n-1} x \cos x})\, dx$$

$$= \left[n \cdot \sin^{n-1} x \cos x (-e^{-x})\right]_0^{\infty} - n$$

$$\int_0^{\infty} \left[\sin^{n-1} x(-\sin x) + (n-1)\sin^{n-2} x \cdot \cos^2 x\right](-e^{-x} dx)$$

$$= 0 + n\int_0^{\infty} \left[-\sin^n x e^{-x} + (n-1)\sin^{n-2} x (1-\sin^2 x) \cdot e^{-x}\right] dx$$

$$= -n\int_0^{\infty} e^{-x} \sin^n x dx + n(n-1)\left[\int_0^{\infty} e^{-x}(\sin^{n-2} x - \sin^n x)\, dx\right]$$

Simplify,

$$= -n I_n + n(n-1)[I_{n-2} - I_n]$$

$$I_n = -n^2 I_n + n(n-1)[I_{n-2}]$$

$$I_n = \frac{n(n-1)}{n^2+1} I_{n-2} \quad \dots (1)$$

Now put n = 4, in equation (1) we get,

$$I_4 = \frac{4(3)}{17} I_2$$

$$I_2 = \frac{2(1)}{5} I_0$$

$$I_0 = \int_0^{\infty} e^{-x} dx = x$$

Resubstituting we get $I_4 = \frac{24}{85}$

Example 4.18 : *If* $I_n = \int_0^{\pi/2} x^n [\sin x + \cos x] dx$. *Put the limits for integration as 0 to* $\pi/2$.

and show that $I_n = \left[\left(\frac{\pi}{2}\right)^n + n\left(\frac{\pi}{2}\right)^{n-1}\right] - n(n-1) I_{n-2}$

(Dec.-1998)

Solution : Step 1 :

We are given that,

$$I_n = \int_0^{\pi/2} x^n [\sin x + \cos x] dx = \int_0^{\pi/2} \underset{u}{x^n} \underset{v}{\sin x} dx + \int_0^{\pi/2} x^n \cos x dx$$

Step 2 :

Using the rule of by parts

$$\int u \cdot v dx = u \int v dx - \int \frac{du}{dx}\left(\int v\, dx\right) dx$$

Step 3 :

$$= \left[x^n(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} n x^{n-1}(-\cos x) dx + \left[x^n \cdot \sin x\right]_0^{\pi/2}$$

$$- \int_0^{\pi/2} n x^{n-1} \sin x dx$$

$$= 0 + n\int_0^{\pi/2} x^{n-1}\cos x dx + \left(\frac{\pi}{2}\right)^n (1) - 0 - n\int_0^{\pi/2} x^{n-1}\sin x dx$$

Simplifying and again using the rule of by parts,

$$= \left(\frac{\pi}{2}\right)^n + \left[nx^{n-1}\sin x\right]_0^{\pi/2} - n\int_0^{\pi/2} (n-1)\, x^{n-2}\sin x dx$$

$$-\left[n\cdot x^{n-1}(-\cos x)\right]_0^{\pi/2} + n\int_0^{\pi/2} (n-1)\, x^{n-2}(-\cos x)\, dx$$

$$= \left(\frac{\pi}{2}\right)^n + n\left(\frac{\pi}{2}\right)^{n-1}(1) - 0 - n(n-1)\int_0^{\pi/2} x^{n-2}(\sin x + \cos x)\, dx$$

$$I_n = \left(\frac{\pi}{2}\right)^n + n\left(\frac{\pi}{2}\right)^{n-1} - n\,(n-1)\, I_{n-2}$$

Exercise 4.2

1) *Show that* $\int x^m\cdot\cos nx\cdot dx = \frac{x^m}{n}\sin nx + \frac{m}{n^2}x^{m-1}\cdot\cos nx - \frac{m(m-1)}{n^2}\int x^{m-2}\cdot\cos nx\cdot dx$

Hint : *Integrate by parts twice.*

2) *If* $I_n = \int_0^{\infty} e^{-px}\sin^n x\, dx\ (n \ge 2, p > 0)$ *then produce that* $(n^2 + p^2)I_n = n(n-1)I_{n-2}$

3) *If* $I_n = \int_0^{\infty} e^{-2x}\sin^n x\, dx$ *prove that* $I_n = \frac{n(n-1)}{n^2+4}I_{n-2}$

4) *Find a reduction formula for* $\int e^{mx}\cos^n x\, dx$

5) *If* $I_n = \int_0^{\pi/2} x^n\sin(2p+1)x\, dx$ *prove that* $(2p+1)^2 I_n + n(n-1)\, I_{n-2} = (-1)^p\cdot n\cdot\left(\frac{\pi}{2}\right)^{n-1}$

4.4 Type 3 : Algebraic Functions

➡ **Example 4.19 :** *If* $I_n = \int \frac{x^n}{(a^2+x^2)^{3/2}}\cdot dx$ *then show that*

$$(n-2)\, I_n = \frac{x^{n-1}}{(a^2+x^2)^{1/2}} - (n-1)\, a^2\, I_{n-2}$$

Solution : $I_n = \int x^n(a^2+x^2)^{-3/2}\, dx$

$$= \int \underbrace{x^{n-1}}_{u}\underbrace{(a^2+x^2)^{-3/2}x}_{v}\, dx$$

Using the rule of by parts,

$$\int u \cdot v dx = u\int v\, dx - \int \frac{du}{dx}\int v\, dx\, dx$$

Here we take $u = x^{n-1}$ and $v = \left(a^2 + x^2\right)^{-3/2}$

$$\therefore \quad I_n = x^{n-1}\frac{1}{2}\left(\frac{\left(a^2+x^2\right)^{-1/2}}{\frac{-1}{2}}\right) - \int (n-1)x^{n-2}\,\frac{1}{2}\left(\frac{\left(a^2+x^2\right)^{-1/2}}{\frac{-1}{2}}\right)dx$$

$$= -x^{n-1}\left(a^2+x^2\right)^{-1/2} + (n-1)\int x^{n-2}\left(a^2+x^2\right)^{-3/2}\left(a^2+x^2\right)dx$$

$$= -x^{n-1}\left(a^2 + x^2\right)^{-1/2} + (n-1)\int x^{n-2}\cdot a^2\left(a^2 + x^2\right)^{-3/2}dx$$

$$+(n-1)\int x^{n-2}\cdot x^2\left(a^2 + x^2\right)^{-3/2}dx$$

$$I_n = -x^{n-1}\left(a^2 + x^2\right)^{\frac{-1}{2}} + (n-1)\,a^2\cdot I_{n-2} + (n-1)\,I_n$$

Simplify,

$$(2-n)\,I_n = -x^{n-1}\left(a^2 + x^2\right)^{\frac{-1}{2}} + (n-1)\,a^2\, I_{n-2}$$

$$(n-2)\,I_n = \frac{x^{n-1}}{\left(a^2 + x^2\right)^{\frac{1}{2}}} - (n-1)\,a^2\, I_{n-2}$$

Example 4.20 : *If* $I_n = \int x^n(a-x)^{1/2}\cdot dx$ *then show that* $(2n+3)\,I_n = 2an\,I_{n-1} - 2x^n\,(a-x)^{3/2}$

Solution : We are given that,

$$I_n = \int x^n\,(a-x)^{1/2}\,dx$$

Using the rule of by parts,

$$\int u \cdot v dx = u\int v\, dx - \int \frac{du}{dx}\int v\, dx\, dx$$

$$= -\frac{2}{3}x^n(a-x)^{3/2} + \frac{2n}{3}\int x^{n-1}\cdot(a-x)^{3/2}\cdot dx$$

$$= -\frac{2}{3}x^n\,(a-x)^{3/2} + \frac{2n}{3}\int x^{n-1}\,(a-x)(a-x)^{1/2}\cdot dx$$

$$= -\frac{2}{3}x^n(a-x)^{3/2} + \frac{2an}{3}\int x^{n-1}\cdot(a-x)^{1/2}\cdot dx$$

$$-\frac{2n}{3}\int x^n\cdot(a-x)^{1/2}\cdot dx$$

$$= -\frac{2}{3}x^n(a-x)^{3/2} + \frac{2an}{3}\int x^{n-1}\cdot(a-x)^{1/2}\cdot dx - \frac{2n}{3}I_n$$

Transferring the last term to L.H.S. and simplifying,

$$(2n+3)\,I_n = 2an\,I_{n-1} - 2x^n(a-x)^{3/2}$$

Example 4.21 : *If* $I_{m,n} = \int x^m\cdot(\log x)^n\cdot dx$ *then show that*

$$I_{m,n} = \frac{x^{m+1}}{m+1}\cdot(\log x)^n - \frac{n}{m+1}I_{m,n-1}$$

Solution : We are given that

$$I_{m,n} = \int x^m\cdot(\log x)^n\cdot dx$$

Using the rule of by parts,

$$\int u\cdot v\,dx = u\int v\,dx - \int \frac{du}{dx}\int v\,dx\,dx$$

$$I_{m,n} = \left[\frac{x^{m+1}}{m+1}\cdot(\log x)^n\right] - \int \frac{x^{m+1}}{m+1}\cdot n(\log x)^{n-1}\cdot\frac{1}{x}\cdot dx$$

$$= \frac{x^{m+1}}{m+1}\cdot(\log x)^n - \frac{n}{m+1}\int x^m\cdot(\log x)^{n-1}\cdot dx$$

$$I_{m,n} = \frac{x^{m+1}}{m+1}\cdot(\log x)^n - \frac{n}{m+1}I_{m,n-1}$$

Example 4.22 : *If* $f(m,n) = \int x^m(1-x)^n\,dx$ *then prove that*

$$f(m,n) = \frac{x^{m+1}(1-x)^n}{m+n+1} + \frac{n}{m+n+1}f(m,n-1)$$

(May-2004)

Solution : Given : $f(m,n) = \int \underset{u}{(1-x)^n}\cdot\underset{v}{x^m}\,dx$

Use integration by parts,

$$= (1-x)^n\cdot\frac{x^{m+1}}{m+1} - \int n(1-x)^{n-1}(-1)\cdot\frac{x^{m+1}}{m+1}dx$$

$$= \frac{(1-x)^n x^{m+1}}{m+1} + \frac{n}{m+1}\int \underline{(1-x)^{n-1}\cdot x^m\cdot[x]\,dx}$$

$$= \frac{(1-x)^n x^{m+1}}{m+1} - \frac{n}{m+1}\int \underline{(1-x)^{n-1} x^m [(1-x)-1]}\,dx$$

... (Note this step)

$$= \frac{(1-x)^n x^{m+1}}{m+1} - \frac{n}{m+1}\int \left[(1-x)^n x^m - (1-x)^{n-1}\cdot x^m\right]dx$$

$$f(m,n) = \frac{(1-x)^n x^{m+1}}{m+1} - \frac{n}{m+1}[f(m,n) - f(m,n-1)]$$

$$\therefore \quad \left[1+\frac{n}{m+1}\right]f(m,n) = \frac{(1-x)^n x^{m+1}}{m+1} + \frac{n}{m+1} f(m,n-1)$$

$$\therefore \quad \left[\frac{m+n+1}{m+1}\right]f(m,n) = \frac{x^{m+1}(1-x)^n}{m+1} + \frac{n}{m+1} f(m,n-1)$$

$$\therefore \quad f(m,n) = \frac{x^{m+1}(1-x)^n}{m+n+1} + \frac{n}{m+n+1} f(m,n-1)$$

Exercise 4.3

1) $I_n = \int (\log x)^n dx$ *prove that* $I_n + n I_{n-1} = x(\log x)^n$

2) *If* $I_n = \int (x^2 + a^2)^{n/2}$ *then* $I_n = \frac{x(x^2+a^2)^{n/2}}{n+1} + \frac{na^2}{n+1}\int (a^2 + x^2)^{\frac{n}{2}-1}\cdot dx$

Hint : $\int 1\cdot(x^2 + a^2)^{n/2}\cdot dx$ *(Use by parts)*

3) *If* $I_n = \int x^n (a^2 - x^2)^{1/2} dx$ *show that* $(n+2) I_n = -x^{n-1}(a^2 - x^2)^{3/2} + a^2(n-1) I_{n-2}$

4.5 Type 4 : Find the Direct Value

Example 4.23 : ***Evaluate*** $\int_4^6 \sin^4 \pi x \cos^2 2\pi x\, dx$ **(May-2003)**

Solution : Let $I = \int_4^6 \sin^4 \pi x \cdot \cos^2 2\pi x\, dx$

Put $\pi x = 4\pi + t \Rightarrow \pi dx = dt$

Now, as $\sin(4\pi + t) = \sin t$ and $\cos(8\pi + 2t) = \cos 2t$

x	4	6
t	0	2π

∴ The integral becomes

$$I = \frac{1}{\pi}\int_0^{2\pi} \sin^4 t\cdot\cos^2 2t\,dt$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \sin^4 t\cos^2 2t\,dt$$

Using formula : $\cos 2t = (1-2\sin^2 t)$

$$= \frac{4}{\pi}\int_0^{\pi/2} \sin^4 t\left(1-2\sin^2 t\right)^2\,dt$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \sin^4 t\left(1-4\sin^2 t+4\sin^4 t\right)\,dt$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \left(\sin^4 t-4\sin^6 t+4\sin^8 t\right)\,dt \qquad \text{... (Walli's formula)}$$

$$= \frac{4}{\pi}\left\{\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-4\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}+4\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right\}$$

$$= \frac{4}{\pi}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\left\{1-\frac{4\cdot5}{6}+\frac{4\cdot7\cdot5}{8\cdot6}\right\}$$

$$= \frac{7}{16}$$

Example 4.24 : $\int_0^{\pi} x\sin^7 x\cos^4 x\,dx = \frac{16\pi}{1155}$ **(Dec.-1999)**

Solution : We are given that,

$$I = \int_0^{\pi} x\sin^7 x\cos^4 x\,dx \qquad \text{... (1)}$$

$$= \int_0^{\pi} (\pi-x)\sin^7(\pi-x)\cos^4(\pi-x)\cdot dx$$

$$\left(\text{... Using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

Using $\sin(\pi-x) = \sin x \quad \cos(\pi-x) = -\cos x$

$$I = \int_0^{\pi} (\pi-x)\sin^7 x\cos^4 x\cdot dx \qquad \text{... (2)}$$

Adding equation (1) and equation (2),

$$2I = \int_0^{\pi} \pi \sin^7 x \cos^4 x dx = \pi \cdot 2 \int_0^{\pi/2} \sin^7 x \cos^4 x dx$$

$$I = \pi \cdot \frac{6 \cdot 4 \cdot 2 \cdot 3 \cdot 1}{11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1} = \frac{16\pi}{1155} \qquad \text{... Ans}$$

Example 4.25 : *Considering* $\int_0^1 x^{2n+1}\left(1 - x^2\right)^{-1/2} dx$ *show that*

$$\frac{1}{2n+2} + \frac{1}{2} \cdot \frac{1}{2n+4} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{2n+6} + \ldots + = \frac{2 \cdot 4 \cdot 6 \ldots 2n}{3 \cdot 5 \cdot 7 \ldots (2n+1)}$$

Solution : Let $I = \int_0^1 x^{2n+1}\left(1 - x^2\right)^{-1/2} dx$

We have by binomial series,

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \ldots$$

Take $z = -x^2$, $n = -\frac{1}{2}$

$$\left(1 - x^2\right)^{-1/2} = 1 + \frac{1}{2} \cdot x^2 + \frac{3}{8} x^4 + \ldots$$

$$\therefore \quad I = \int_0^1 x^{2n+1}\left(1 + \frac{1}{2} x^2 + \frac{3}{8} x^4 + \ldots\right) dx$$

$$= \int_0^1 \left(x^{2n+1} + \frac{1}{2} x^{2n+3} + \frac{3}{8} x^{2n+5} + \ldots\right) dx$$

$$= \left[\frac{x^{2n+2}}{2n+2} + \frac{1}{2} \cdot \frac{x^{2n+4}}{2n+4} + \frac{3}{8} \frac{x^{2n+6}}{2n+6} + \ldots\right]_0^1$$

$$= \frac{1}{2n+2} + \frac{1}{2} \frac{1}{2n+4} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{2n+6} + \ldots \qquad \text{... (1)}$$

Also, $\int_0^1 x^{2n+1}\left(1 - x^2\right)^{-1/2} dx = \int_0^1 \frac{x^{2n+1}}{\sqrt{1 - x^2}} dx$

$x = \sin \theta$, $\quad dx = \cos \theta \, d\theta$

Limits :

x	0	1
θ	0	π/2

$$= \int_0^{\pi/2} \frac{\sin^{2n+1}\theta \cos\theta \, d\theta}{\cos\theta} = \int_0^{\pi/2} \sin^{2n+1}\theta d\theta$$

$$= \frac{2n}{2n+1} \frac{2n-2}{2n-1} \ldots\ldots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1 \qquad \ldots (2)$$

Since (1) and (2) are the values of the same integral, it follows that,

$$\frac{1}{2n+2} + \frac{1}{2} \cdot \frac{1}{2n+4} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{2n+6} + \ldots = \frac{2 \cdot 4 \cdot 6 \ldots 2n}{3 \cdot 5 \cdot 7 \ldots (2n+1)}$$

Example 4.26 : $\int_0^{2a} x^3 \sqrt{2ax - x^2} \cdot dx = \frac{7\pi a^5}{8}$

Solution : Step 1 :

Let $$I = \int_0^{2a} x^3 \sqrt{2ax - x^2} \cdot dx$$

Step 2 :

Put $x = 2a \sin^2\theta$, $dx = 4a \sin\theta \cos\theta \, d\theta$

x	0	2a
θ	0	π/2

$$\therefore \quad I = \int_0^{\pi/2} (2a \sin^2\theta)^3 \sqrt{4a^2 \sin^2\theta - 4a^2 \sin^4\theta} \, (4a \sin\theta \cos\theta d\theta)$$

$$= \int_0^{\pi/2} (8a^3) \sin^6\theta \, 2a \sin\theta \cos\theta \, (4a \sin\theta \cos\theta) \, d\theta$$

$$= 64a^5 \int_0^{\pi/2} \sin^8\theta \cos^2\theta \, d\theta$$

$$I = 64a^5 \frac{7 \cdot 5 \cdot 3 \cdot 1 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} \qquad \text{(As m and n are both even)}$$

$$\int_0^{2a} x^3 \sqrt{2ax - x^2} \, dx = \frac{7\pi a^5}{8} \qquad \ldots \textbf{Ans.}$$

Exercise 4.4

1) $\int_0^{\pi/2} x \cdot \sin^6 x \cdot \cos^4 x \cdot dx = \frac{3\pi^2}{512}$

2) *Evaluate* $\int_0^{\infty} \frac{x^8}{(1+x^6)^{7/2}} \cdot dx$

Hint : Put $x^3 = \tan\theta$ [Ans. : $\frac{2}{45}$]

3) *By considering the value of* $\int_0^1 (1-x^2)^n \cdot dx$ *show that,*

$$1 - \frac{n}{1 \cdot 3} + \frac{n(n-1)}{1 \cdot 2 \cdot 5} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 7} + \ldots = \frac{2 \cdot 4 \cdot 6 \ldots (2n)}{3 \cdot 5 \cdot 7 \ldots (2n+1)}$$

4) *Evaluate* $\int_0^{2a} x^{7/2}(2a-x)^{-1/2} dx$ **[Ans. : $\frac{35\pi a^4}{8}$]**

5) *Evaluate* $\int_0^{\pi} x \sin^5 x \cos^4 x \, dx$ **[Ans. : $\frac{8\pi}{315}$]**

6) *Evaluate* $\int_0^{\pi/2} \cos^3 2\theta \sin^4 \theta d\theta$ **[Ans. : $\frac{16}{105}$]**

7) *Evaluate* $\int_{-\pi/2}^{\pi/2} \cos^3 \theta (1+\sin\theta)^2 d\theta$ **[Ans. : $\frac{8}{5}$]**

8) *Evaluate* $\int_0^{2\pi} \sin^2 \theta (1+\cos\theta)^4 d\theta$ **[Ans. : $\frac{21\pi}{8}$]**

9) *Evaluate* $\int_0^1 x^5 \sin^{-1} x \, dx$ **[Ans. : $\frac{11\pi}{192}$]**

10) *Prove that* $\int_0^{\infty} \frac{dx}{(1+x^2)^n} = \frac{(2n-2)!}{2^{2n-2}[(n-1)!]^2} \cdot \frac{\pi}{2}$

11) *Prove that* $\int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}} dx = \frac{(2n)!}{2^{2n}(n!)^2} \cdot \frac{\pi}{2}$

University Questions

May - 2003

1. *Evaluate* $\int_4^6 \sin^4 \pi x \cos^2 2\pi x \, dx$. **[5 Marks]**

Dec. - 2003

1. *If* $I_n = \int_0^{\pi/2} x^n \sin x \, dx$, *then show that* $I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1) I_{n-2}$ **[5 Marks]**

May - 2004

1. *If* $U_n = \int_0^{\pi/2} \theta \cos^n \theta d\theta$ *then show that* $U_n = -\frac{1}{n^2} + \frac{n-1}{n} U_{n-2}$ *hence evaluate* U_4. **[5 Marks]**

2. *If* $f(m,n) = \int x^m (1-x)^n dx$ *then show that* $f(m,n) = \frac{x^{m+1}(1-x)^n}{m+n+1} + \frac{n}{m+n+1} f(m,n-1)$. **[5 Marks]**

Dec. - 2004

1. *If* $u_n = \int_0^{\pi/4} \sin^{2n} x dx$ *rove that* $u_n = \left(1 - \frac{1}{2n}\right) u_{n-1} - \frac{1}{n 2^{n+1}}$ **[5 Marks]**

2. *If* $I_n = \int_0^{\pi/4} \sec^n \theta d\theta$ *Prove that* $I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{(n-2)}{n-1} I_{n-2}$ *hence evaluate* $\int_0^{\pi/4} \sec^6 \theta d\theta$ **[5 Marks]**

May - 2005

1. *If* $I_n = \int_0^{\pi/2} x^n \cos^n x dx$, *then show that* $I_n = -\frac{1}{n^2} + \frac{(n-1)}{n} I_{n-2}$ **[5 Marks]**

2. *If* $I_n = \int_{\pi/4}^{\pi/2} \cot^n \theta d\theta$, *prove that* $I_n = \frac{1}{n-1} - I_{n-2}$. *Hence evaluate* $\int_{\pi/4}^{\pi/2} \cot^6 \theta d\theta$. **[5 Marks]**

Dec. - 2005

1. *If* $I_n = \int_0^{\pi/2} (x^n \sin(2p+1)) x \cdot dx$, *prove that* $(2p+1)^2 I_n + n(n-1) I_{n-2} = (-1)^p \cdot n \left(\frac{\pi}{2}\right)^{n-1}$ **[5 Marks]**

May - 2006

1. *If* $I_n = \int_0^{\pi/2} \cos^n x \cos nx \, dx$, *prove that* $I_n = \frac{1}{2} I_{n-1} = \frac{\pi}{2^{n+1}}$ **[5 Mark]**

2. *Establish the reduction formula connecting*

   $I_n = \int_0^{\pi/2} x \cos^n x \, dx$ *with* I_{n-2} **[5 Marks]**

Dec. - 2006

1. *Evaluate* $\int_4^6 \sin^4 \pi x \cos^2 2\pi x \, dx$ **[4 Marks]**

2. *If* $I_n = \int_0^{\pi/4} \sin^{2n} x dx$, *prove that* $I_n = \left(1 - \frac{1}{2n}\right) I_{n-1} - \frac{1}{n 2^{n+1}}$. **[5 Marks]**

May - 2007

1. *If* $I_n = \int_0^{\pi/4} \sin^{2n} x \, dx$, *prove that*

   $I_n = \left(1 - \frac{1}{2n}\right) I_{n-1} - \frac{1}{n 2^{n+1}}$ **[4 Marks]**

2. *Evaluate*

$$\int_0^{\pi} x\sin^7 x\cos^4 x\,dx$$

[4 Marks]

Dec. - 2007

1. *Find reduction formula for*

$I_n = \int_0^{\pi/4} \sec^n x\,dx$, *hence find* I_6. **[5 Marks]**

2. *If* $I_n = \int_0^{\pi/4} \frac{\sin(2n-1)x}{\sin x}\,dx$, *then prove that*

$n(I_{n+1} - I_n) = \sin\frac{n\pi}{2}$, *hence find* I_3. **[5 Marks]**

May - 2008

1. *If* $I_n = \int_0^{\infty} e^{-x}\sin^n x\,dx$,

show that $I_n = \frac{n(n-1)}{n^2+1} I_{n-2}$

hence find I_4. **[5 Marks]**

2. $I_{m,n} = \int \cos^m x\,\sin nx\,dx$,

find the reduction formula connecting $I_{m,n}$ *with* $I_{m-1,n-1}$. **[5 Marks]**

Dec. - 2008

1. *If* $I_n = \int_0^{\pi/2} x^n \sin(2p+1)x\,dx$, *prove that*

$(2p+1)^2 I_n + n(n-1) I_{n-2} = (-1)^p . n.\left(\frac{\pi}{2}\right)^{n-1}$ **[5 Marks]**

2. *If* $I_n = \int_{\pi/4}^{\pi/2} \cot^n\theta\,d\theta$, *prove that*

$I_n = \frac{1}{n-1} - I_{n-2}$. *Hence evaluate* $\int_{\pi/4}^{\pi/2} \cot^6\theta\,d\theta$ **[5 Marks]**

3. *Evaluate* $\int_4^6 \sin^4 \pi x\ \cos^2 2\pi x\,dx$. **[4 Marks]**

5

Beta and Gamma Function

Introduction

In this chapter, we are coming across two special types of improper integrals viz. Beta and Gamma function which plays important role to evaluate certain complicated integrals by handy techniques by expressing interms of these integrals. Also used in double integrals, Fourier transforms, Laplace transforms etc.

5.1 Gamma Function

Definition : The definite improper integral $\int_0^\infty e^{-t}\, t^{n-1}\, dt$ is denoted by the symbol $\overline{|n}$ and called as Gamma n or Euler's integral of the second kind or improper integral.

Thus

$$\boxed{\overline{|n} = \int_0^\infty e^{-t}\ t^{n-1}\ dt, \quad (n > 0)}$$

5.2 Properties of Gamma Functions

1) $\overline{|1} = 1$

Proof :

$$\overline{|n} = \int_0^\infty e^{-t}\ t^{n-1}\ dt$$

Put n = 1

$$\overline{|1} = \int_0^\infty e^{-t} \cdot t^0\ dt$$

$$= \left[\frac{e^{-t}}{-1}\right]_0^\infty = 1$$

2) Reduction formula for Gamma function

$$\overline{|n+1} = n\overline{|n}$$

Proof :

$$\overline{|n+1} = \int_0^\infty e^{-t}\ t^n\ dt$$

Integrating by parts.

$$= \left\{ \left(t^n \frac{e^{-t}}{-1} \right)_0^\infty - \int_0^\infty n\, t^{n-1} \frac{e^{-t}}{-1} dt \right\} \qquad \left(\because \lim_{t \to \infty} \frac{t^n}{e^t} = 0 \right)$$

$$= 0 + n \int_0^\infty e^{-t}\, t^{n-1}\, dt$$

$$= n\,\Gamma n$$

If n is positive integer

$$\Gamma(n+1) = n\,\Gamma n$$

$$= n(n-1)\,\Gamma(n-1)$$

$$= n(n-1)(n-2)\,\Gamma(n-2)$$

....

....

$$= n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1\, \Gamma 1 \qquad (\because \Gamma 1 = 1)$$

$$= n!$$

Hence $\Gamma(n+1) = n!$ if n is positive integer

3) $\Gamma 0 = \infty$

Proof : We know that $\Gamma(n+1) = n\,\Gamma n$

$$\therefore \qquad \Gamma n = \frac{\Gamma(n+1)}{n}$$

Put n = 0

$$\Gamma 0 = \frac{\Gamma 1}{0} = \frac{0!}{0} = \frac{1}{0} = \infty$$

$$= \infty$$

4) Alternative definition of Gamma function

$$\boxed{\Gamma n = 2 \int_0^\infty e^{-x^2} x^{2n-1}\, dx}$$

Proof : We know that

$$\Gamma n = \int_0^\infty e^{-t}\, t^{n-1}\, dt$$

Put $t = x^2$, $dt = 2x\, dx$

t	0	∞
x	0	∞

$$\therefore \quad \Gamma n = \int_0^\infty e^{-x^2} \cdot x^{2n-2} \cdot 2x dx$$

$$= 2\int_0^\infty e^{-x^2} \; x^{2n-1} \; dx$$

Additional Standard Results :

1) $\Gamma 1/2 = \sqrt{\pi}$

2) $\int_0^\infty e^{-kx} \; x^{n-1} dx = \frac{\Gamma n}{k^n}$

Proof 1) : $\Gamma 1/2 = 2\int_0^\infty e^{-x^2} x^{2(1/2)-1} \, dx$

$$= 2\int_0^\infty e^{-x^2} \, dx$$

$$= 2 \cdot \frac{\sqrt{\pi}}{2}$$

$$= \sqrt{\pi}$$

Proof 2) : Consider $\int_0^\infty e^{-kx} x^{n-1} \, dx$

Put $kx = t$

$x = \frac{t}{k}$

$dx = \frac{dt}{k}$

x	0	∞
t	0	∞

$$= \int_0^\infty e^{-t}\left(\frac{t}{k}\right)^{n-1} \frac{dt}{k}$$

$$= \frac{1}{k^n} \int_0^\infty e^{-t} \; t^{n-1} \; dt$$

$$= \frac{\Gamma n}{k^n} \text{ Hence the proof.}$$

We can use this result as a formula.

3) $$\Gamma P \, \Gamma 1 - P = \frac{\pi}{\sin P\pi} \quad (0 < p < 1)$$

3a) $$\Gamma 1/4 \, \Gamma 3/4 = \Gamma 1/4 \, \Gamma 1 - 1/4 \qquad \text{Put } P = \frac{1}{4}$$

$$= \frac{\pi}{\sin \pi/4}$$

$$= \frac{\pi}{1/\sqrt{2}}$$

$$= \pi\sqrt{2}$$

3b) $\Gamma 1/3 \, \Gamma 2/3$ put $P = \frac{1}{3}$ in (3),

$$\therefore \quad \Gamma 1/3 \, \Gamma 1 - 1/3 = \frac{\pi}{\sin\frac{\pi}{3}} \qquad \text{Put } P = \frac{1}{3}$$

$$= \frac{\pi}{\sqrt{3}/2}$$

$$\sqrt{1/3} \, \sqrt{2/3} = \frac{2\pi}{\sqrt{3}}$$

5.3 Illustrations on Gamma Function

Type 1 Reducible to the standard form

Example 5.1 : *Evaluate* $\int_0^\infty e^{-x^4} dx$ **(Dec.-2002)**

Solution : Put $x^4 = t$

$\therefore \quad x = t^{1/4}$

$dx = \frac{1}{4} t^{-3/4} \, dt$

x	0	∞
t	0	∞

Thus the integral becomes

$$\int_0^\infty e^{-x^4} dx = \int_0^\infty e^{-t} \cdot \frac{1}{4} t^{-3/4} dt$$

$$= \frac{1}{4} \int_0^\infty t^{1/4-1} e^{-t} dt = \frac{1}{4} \overline{|1/4}$$

Example 5.2 : *Evaluate* $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$

(Dec.-1992, Dec.-2001)

Solution : Put $\sqrt{x} = t$

$\therefore$ $x = t^2$

$dx = 2t\, dt$

x	0	∞
t	0	∞

Thus the integral becomes

$$\int_0^\infty x^{1/4} e^{-\sqrt{x}} dx = \int_0^\infty t^{2/4} e^{-t} \cdot 2t\, dt$$

$$= 2\int_0^\infty t^{3/2} e^{-t} dt$$

$$= 2\overline{|5/2} = 2 \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} = \frac{3\sqrt{\pi}}{2}$$

Example 5.3 : *Evaluate* $\int_0^\infty x^n e^{-x^m} dx$

(Dec.- 2004)

Solution : Put $x^m = t$

$x = t^{1/m}$

$$dx = \frac{1}{m} t^{1/m - 1} dt = \frac{1}{m} \cdot t^{\frac{1}{m} - 1} \cdot dt = \frac{1}{m} t^{\left(\frac{1-m}{m}\right)} dt$$

x	0	∞
t	0	∞

Thus given integral becomes

$$= \int_0^\infty t^{n/m} e^{-t} \cdot \frac{1}{m} t^{\left(\frac{1-m}{m}\right)} dt$$

$$= \frac{1}{m} \int_0^{\infty} t^{\left(\frac{n+1}{m}\right)-1} e^{-t}\, dt$$

$$= \frac{1}{m} \overline{|n + 1/m}$$

Exercise 5.1 (Problems on Type 1)

Prove that

1) $\int_0^{\infty} x^m e^{-ax^n} dx \quad (a > 0) = \frac{1}{n\, a^{(m+1)/n}} \overline{|m + 1/n}$

2) $\int_0^{\infty} e^{-h^2 x^2} dx = \frac{\sqrt{\pi}}{2h}$

3) $\int_0^{\infty} \sqrt{x}\, e^{-\sqrt[3]{x}} dx = \frac{315\sqrt{\pi}}{16}$

4) $\int_0^{\infty} x^2 e^{-x^4} dx \int_0^{\infty} e^{-x^4} dx = \frac{\pi}{8\sqrt{2}}$

5) $\int_0^{\infty} \sqrt{y}\, e^{-y^2} dy \int_0^{\infty} e^{-y^2} \frac{1}{\sqrt{y}} dy = \frac{\pi}{2\sqrt{2}}$

6) $\int_0^{\infty} x^2 e^{-h^2 x^2} = \frac{\sqrt{\pi}}{4h^3}$

7) $\int_0^{\infty} \sqrt{y}\, e^{-y^3} dy = \frac{\sqrt{\pi}}{3}$

8) $\int_0^{\infty} x^7 e^{-2x^2} dx = \frac{3}{16}$

9) $\int_0^{\infty} x^{n-1} e^{-h^2 x^2} dx = \frac{\overline{|n/2}}{2h^n}$

10) $\int_0^{\infty} x^{2/3} e^{-\sqrt[3]{x}} dx = 72$

11) $\int_0^{\infty} x^n e^{-\sqrt{ax}} dx = \frac{2(2n+1)!}{a^{n+1}}$

Type 2 Problems involving (Constant)$^{\text{(Variable)}}$

Example 5.4 : *Evaluate* $\int_0^{\infty} \frac{x^a}{a^x} dx \ (a > 0)$ **(Dec.-1993)**

Solution : Put $a^x = e^t$

$\therefore \quad x \log a = t$

$\therefore \quad x = \dfrac{t}{\log a}$

$\therefore \quad dx = \dfrac{dt}{\log a}$

Limits :

x	0	∞
t	0	∞

$\therefore$ The integral becomes

$$\int_0^\infty \frac{x^a}{a^x} dx = \int_0^\infty \left(\frac{t}{\log a}\right)^a \cdot \frac{1}{e^t} \cdot \frac{dt}{\log a}$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^\infty t^a \cdot e^{-t} \, dt$$

$$= \frac{\Gamma(a+1)}{(\log a)^{a+1}}$$

Example 5.5 : *Evaluate* $\int_0^\infty \frac{x^2}{3^{x^2}} dx$ (Dec.-2000)

Solution : Put $3^{x^2} = e^t$

$\therefore \quad x^2 \log 3 = t$

or $\quad x^2 = \dfrac{t}{\log 3}$

or $\quad x = \dfrac{t^{1/2}}{\sqrt{\log 3}}$

$\therefore \quad dx = \dfrac{\frac{1}{2} t^{-1/2} \, dt}{\sqrt{\log 3}}$

Limits :

x	0	∞
t	0	∞

$\therefore$ The integral becomes

$$\int_0^\infty \frac{x^2}{3^{x^2}} 2dx = \int_0^\infty \frac{t}{\log 3} \cdot \frac{1}{e^t} \cdot \frac{\frac{1}{2} t^{-1/2} \, dt}{\sqrt{\log 3}}$$

$$= \frac{1}{2(\log 3)^{3/2}} \int_0^\infty t^{1/2} e^{-t} \, dt$$

$$= \frac{1}{2(\log 3)^{3/2}} \Gamma(3/2) = \frac{1}{2(\log 3)^{3/2}} \frac{1}{2}\sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{4(\log 3)^{3/2}}$$

Exercise 5.2 Problems on Type 2

1) *Prove that* $\int_0^\infty \frac{dx}{3^{4x^2}} = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$

2) $\int_0^\infty \frac{x^2}{2^x}\, dx = \frac{2!}{(\log 2)^3}$

3) $\int_0^\infty 7^{-4x^2}\, dx = \frac{\sqrt{\pi}}{4\sqrt{\log 7}}$ **(May-2002)**

4) $\int_0^\infty a^{-4x^2}\, dx = \frac{\sqrt{\pi}}{4\sqrt{\log a}}$

5) $\int_0^\infty \frac{x^4}{4^x}\, dx = \frac{24}{(\log 4)^5}$

6) $\int_0^\infty \frac{x^5}{5^x}\, dx = \frac{120}{(\log 5)^6}$

7) $\int_0^\infty 5^{-4x^2}\, dx = \frac{\sqrt{\pi}}{4\sqrt{\log 5}}$

Type 3 Problems involving log x

Example 5.6 : *Evaluate* $\int_0^1 \frac{dx}{\sqrt{x \log \frac{1}{x}}}$

Solution : Put $\log 1/x = t$

$\therefore \quad -\log x = t$

or $\quad \log x = -t$

or $\quad x = e^{-t}$

$\therefore \quad dx = -e^{-t}\, dt$

x	0	1
t	∞	0

$\therefore$ The integral becomes

$$\int_0^1 \frac{dx}{\sqrt{x \log \frac{1}{x}}} = \int_\infty^0 \frac{-e^{-t}\, dt}{\sqrt{e^{-t} \cdot t}}$$

$$= \int_0^\infty t^{-1/2}\, e^{-t/2}\, dt$$

Now this problem is reduced to Type 1

$\therefore$ Put $\frac{t}{2} = u$, $t = 2u$, $dt = 2\ du$

x	0	∞
t	0	∞

$$\int_0^1 \frac{dx}{\sqrt{x \log \frac{1}{x}}} = \int_0^\infty (2u)^{-1/2} \cdot e^{-u} \cdot 2\,du$$

$$= \int_0^\infty \sqrt{2}\ u^{-1/2} \cdot e^{-u} \cdot du$$

$$= \sqrt{2} \cdot \Gamma(1/2)$$

$$= \sqrt{2} \cdot \sqrt{\pi}$$

$$= \sqrt{2\pi}$$

Example 5.7 : *Evaluate* $\int_0^1 x^m (\log x)^m\, dx,\ (m > 0)$

(May-1998)

Solution : Put $\log x = -t$, $x = e^{-t}$, $dx = -e^{-t}\ dt$

x	0	1
t	∞	0

$\therefore$ The integral becomes

$$\int_0^1 x^m (\log x)\,dx = \int_\infty^0 (e^{-t})^m \cdot (-t)^m \cdot (-e^{-t})\ dt$$

$$= \int_0^\infty e^{-(m+1)t} \cdot (-1)^m \cdot t^m\ dt$$

Put $(m + 1)\, t = u$

$\therefore \qquad t = \frac{u}{m+1}$

and $\qquad dt = \frac{du}{m+1}$

$$\therefore \int_0^1 x^m \log x\,dx = \int_0^\infty (-1)^m \cdot e^{-u} \cdot \left(\frac{u}{m+1}\right)^m \cdot \frac{du}{m+1}$$

$$= \frac{(-1)^m}{(m+1)^{m+1}} \int_0^\infty u^m \cdot e^{-u}\ du$$

$$= \frac{(-1)^m \overline{|m+1}}{(m+1)^{m+1}}$$

Example 5.8 : *Evaluate* $\int_0^1 (x \log x)^4 \, dx$

(May-2005)

Solution : Let $I = \int_0^1 (x \log x)^4 dx$ put $\log x = -t$, $x = e^{-t}$, $dx = -e^{-t}\, dt$

x	0	1
t	∞	0

From given integral

$$I = \int_\infty^0 e^{-4t} \cdot (-t)^4 (-e^{-t})\, dt$$

$$= \int_0^\infty t^4 \; e^{-5t} \; dt$$

Put $5t = u$, $t = \frac{u}{5}$, $dt = \frac{du}{5}$

$$= \int_0^\infty \left(\frac{u}{5}\right)^4 \cdot e^{-u} \cdot \frac{du}{5}$$

$$= \frac{1}{5^5} \cdot \int_0^\infty u^4 \; e^{-u} \; du$$

$$= \frac{\overline{|5}}{5^5} = \frac{4\,!}{5^5}$$

Exercise 5.3 Problems on Type 3

1) *Prove that* $\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \overline{|n}$

2) $\int_0^1 x^{a-1} \left(\log \frac{1}{x}\right)^{n-1} dx \; (a > 0) = \frac{\overline{|n}}{a^n}$

3) $\int_0^1 \frac{dx}{\sqrt{-\log x}} = \sqrt{\pi}$

4) $\int_0^1 x^m (\log x)^n \, dx = \frac{(-1)^n \overline{|n+1}}{(m+1)^{n+1}}$

5) $\int_0^1 \frac{x\, dx}{\sqrt{\log \frac{1}{x}}} = \sqrt{\frac{\pi}{2}}$

6) $\int_0^1 (\log x)^n \, dx = (-1)^n \overline{|n}$	
7) $\int_0^1 (x \log x)^3 \, dx = \frac{3!}{4^4}$	(Dec.-2003)
8) $\int_0^1 x^3 \left(\log \frac{1}{x}\right)^4 dx = \frac{3}{128}$	

Type 4 Problems involving sin ax or cos ax

Example 5.9 : *Show that* $\int_0^\infty x\, e^{-ax} \sin bx \, dx = \frac{2ab}{(a^2+b^2)^2}$

Solution : We know that Img $(e^{ibx}) = \sin bx$. $(\because e^{ibx} = \cos bx + i \sin bx)$

Thus the integral becomes

$$\int_0^\infty x\, e^{-ax} \sin bx\, dx = \text{Im} \int_0^\infty x\, e^{-ax}\, e^{ibx}\, dx$$

$$= \text{Im} \int_0^\infty x\, e^{-(a-ib)x}\, dx$$

Put $(a - ib)\, x = t$, $x = \frac{t}{a-ib}$, $dx = \frac{dt}{a-ib}$

x	0	∞
t	0	∞

$$= \text{Img} \int_0^\infty \left(\frac{t}{a-ib}\right) \cdot e^{-t} \cdot \frac{dt}{a-ib}$$

$$= \text{Img} \frac{1}{(a-ib)^2} \int_0^\infty t\, e^{-t}\, dt$$

$$= \text{Img} \frac{1}{(a-ib)^2} \overline{|2} = \text{Img}\left[\frac{1}{(a-ib)}\right]^2 \cdot 1$$

$$= \text{Img}\left[\frac{(a+ib)}{(a-ib)(a+ib)}\right]^2$$

$$= \text{Img} \frac{(a+ib)^2}{(a^2+b^2)^2}$$

$$= \text{Img}\,\frac{a^2 - b^2 + 2\,iab}{\left(a^2 + b^2\right)^2} = \text{Img}\left[\frac{a^2 - b^2}{\left(a^2 + b^2\right)^2} + i\frac{2ab}{\left(a^2 + b^2\right)^2}\right]$$

$$= \frac{2ab}{\left(a^2 + b^2\right)^2}$$

Example 5.10 : *Show that* $\int_0^\infty \cos\left(ax^{1/n}\right)dx = \frac{\Gamma(n+1)}{a^n}\cos\frac{n\pi}{2}$

Solution : We know that $\cos\left(ax^{1/n}\right) = \text{Real } e^{-i\left(ax^{1/n}\right)}$ $\left[\because e^{-iax^{1/n}} = \cos\left(ax^{1/n}\right) + i\sin\left(ax^{1/n}\right)\right]$

Thus the integral becomes

$$I = \text{Real}\int_0^\infty e^{-iax^{1/n}}\,dx$$

Put $\quad i\,a\,x^{1/n} = t$

$$\therefore \quad x^{1/n} = \frac{t}{i\,a},\; x = \frac{t^n}{(i\,a)^n}$$

$$dx = \frac{n\,t^{n-1}\,dt}{(i\,a)^n}$$

Limits :

x	0	∞
t	0	∞

$$\therefore \int_0^\infty \cos\left(ax^{1/n}\right)dx = \text{Real}\int_0^\infty e^{-t}\cdot\frac{n\,t^{n-1}\,dt}{(i\,a)^n}$$

$$= \text{Real}\,\frac{n}{a^n}\left(\frac{1}{i}\right)^n\int_0^\infty e^{-t}\,t^{n-1}\,dt$$

$$= \text{Real}\,\frac{n}{a^n}(-i)^n\,\Gamma n$$

$$= \text{Real}\,\frac{n\Gamma n}{a^n}\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)^n$$

$$= \frac{\Gamma(n+1)}{a^n}\cdot\cos\frac{n\pi}{2}$$

Exercise 5.4 Problem on Type 4

1) *Prove that* $\int_0^\infty x^{m-1} \cos ax\, dx = \frac{\Gamma m}{a^m} \cos \frac{m\pi}{2}$ **(May-2004)**

2) *Prove that* $\int_0^\infty x^{m-1} \sin ax\, dx = \frac{\Gamma m}{a^m} \sin \frac{m\pi}{2}$

3) *Prove that* $\int_0^\infty x^{n-1} e^{-ax} \cos bx\, dx = \frac{\Gamma n}{(a^2+b^2)^{n/2}} \cos\left(n \tan^{-1} \frac{b}{a}\right)$

4) *Prove that* $\int_0^\infty x^{n-1} e^{-ax} \sin bx\, dx = \frac{\Gamma n}{(a^2+b^2)^{n/2}} \sin\left(n \tan^{-1} \frac{b}{a}\right)$

Type 5 Problem on $\Gamma n+1 = n\Gamma n$

Example 5.11 : *If* $I_n = \dfrac{\frac{\sqrt{\pi}}{2}\Gamma (n+1)/2}{\Gamma n/2+1}$ *show that* $I_{n+2} = \dfrac{n+1}{n+2} I_n$ *hence find* I_5.

Solution : Given $$I_n = \frac{\frac{\sqrt{\pi}}{2}\Gamma (n+1)/2}{\Gamma n/2+1} \quad \text{... (1)}$$

Replacing n by n + 2 we get

$$I_{n+2} = \frac{\frac{\sqrt{\pi}}{2}\Gamma (n+3)/2}{\Gamma (n+2)/2+1}$$

Apply $\Gamma n+1 = n\Gamma n$

$$\therefore \quad I_{n+2} = \frac{\frac{\sqrt{\pi}}{2}\left(\frac{n+1}{2}\right)\Gamma (n+1)/2}{\left(\frac{n+2}{2}\right)\Gamma (n+2)/2}$$

$$= \frac{n+1}{n+2} \cdot \frac{\frac{\sqrt{\pi}}{2}\Gamma (n+1)/2}{\Gamma n/2+1}$$

$$I_{n+2} = \frac{n+1}{n+2} I_n$$

To find I_5 put n = 3

$$I_5 = \frac{4}{5} I_3$$

and $$I_3 = \frac{2}{3} I_1$$

To find I_1 put n = 1 in (1)

$$I_1 = \frac{\frac{\sqrt{\pi}}{2}\,\Gamma 1}{\Gamma(1/2+1)}$$

$$I_1 = \frac{\frac{\sqrt{\pi}}{2}}{\frac{1}{2}\Gamma\frac{1}{2}} = \frac{\frac{\sqrt{\pi}}{2}}{\frac{1}{2}\sqrt{\pi}} = 1$$

Thus $$I_5 = \frac{4}{5}\cdot\frac{2}{3}\cdot 1 = \frac{8}{15}$$

Example 5.12 : *Show that* $\frac{2^n\,\Gamma(n+1/2)}{\sqrt{\pi}} = 1\cdot3\cdot5\cdot\ldots\cdot(2n-1)$.

Solution : Consider $\Gamma(n+1/2)$

Apply $\Gamma(n+1) = n\,\Gamma n$ repeatedly, we get $\Gamma\left(n+\frac{1}{2}\right) = \Gamma\left(\left(n-\frac{1}{2}\right)+1\right)$

$$= \left(n-\frac{1}{2}\right)\Gamma(n-1/2)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma(n-3/2)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\Gamma(n-5/2)$$

....

....

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\ldots\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$$

$$= \frac{(2n-1)}{2}\cdot\frac{(2n-3)}{2}\cdot\frac{(2n-5)}{2}\ldots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

$$\Gamma(n+1/2) = \frac{(2n-1)(2n-3)(2n-5)\ldots 5\cdot 3\cdot 1}{2^n}\sqrt{\pi}$$

Thus $$\frac{2^n\,\Gamma(n+1/2)}{\sqrt{\pi}} = 1\cdot3\cdot5\ldots(2n-1).$$

5.4 Beta Function

Definition : A Beta function of m, n is denoted by $\beta(m,n)$ and defined as,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \quad (m>0, n>0)$$

The Beta function is also called as Euler's integral of the first kind.

Note : Beta function of negative numbers is not defined.

5.5 Properties of Beta Function

1) $\beta(m, n) = \beta(n, m)$

Proof : $$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

Put $1-x = t$, $-dx = dt$

x	0	1
t	1	0

$$= \int_1^0 (1-t)^{m-1}\, t^{n-1}\,(-dt)$$

$$= \int_0^1 t^{n-1}(1-t)^{m-1}\, dt$$

$$= \beta(n,m)$$

2) $$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$$

Proof : $$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

Put $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\, d\theta$

x	0	1
θ	0	π/2

$$= \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\cos^{2n-1}\theta\, d\theta$$

Thus
$$\boxed{\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta}$$

Substituting $m = \frac{p+1}{2}$, $n = \frac{q+1}{2}$

we get

$$\boxed{\int_0^{\pi/2} \sin^p\theta \, \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)}$$

3) $$\beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} \, dx$$

Proof : $$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} \, dx$$

Put $x = \frac{t}{1+t}$, $dx = \frac{dt}{(1+t)^2}$

x	0	1
t	0	∞

Substituting we get

$$\beta(m, n) = \int_0^{\infty} \left(\frac{t}{1+t}\right)^{m-1}\left(1 - \frac{t}{1+t}\right)^{n-1} \cdot \frac{dt}{(1+t)^2}$$

$$= \int_0^{\infty} \frac{t^{m-1}}{(1+t)^{m+n}} \, dt$$

As the variable is immaterial in definite Integral, hence we get the result.

$$\boxed{\beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} \, dx}$$

We can use this result as another definition of Beta function.

4) Relation between Beta and Gamma functions

$$\boxed{\beta(m, n) = \frac{\Gamma m \, \Gamma n}{\Gamma (m+n)}}$$

5) $$\boxed{\Gamma(1/2) = \sqrt{\pi}}$$

Proof : We know that

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Put $p = q = 0$

$$\int_0^{\pi/2} d\theta = \frac{1}{2}\frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1)}$$

$$\frac{\pi}{2} = \frac{1}{2}(\Gamma(1/2))^2$$

$$\therefore \quad \Gamma(1/2) = \sqrt{\pi}$$

6) Legendre's duplication formula : **(May-1997, Dec.-1998, Dec.-2001)**

$$\boxed{\Gamma(m)\,\Gamma(m + 1/2) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)}$$

Proof : We know that

$$\frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

Put $q = p$

$$\frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{p+1}{2}\right) = \int_0^{\pi/2} (\sin\theta \cos\theta)^p \, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^p d\theta$$

Put $2\theta = u$, $2\,d\theta = du$

Limits :

θ	0	π/2
u	0	π

$$\therefore \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{p+1}{2}\right) = \frac{1}{2^p}\int_0^{\pi} \sin^p u \cdot \frac{du}{2}$$

$$= \frac{1}{2 \cdot 2^p}\int_0^{\pi} \sin^p u \, du$$

$$\left\{\because \int_0^{2a} f(x)\,dx = \int_0^a [f(x) + f(2a-x)]dx\right\}$$

$$= \frac{1}{2\cdot 2^p}\int_0^{\pi/2}\left[\sin^p u + \sin^p(\pi - u)\right] du$$

$$= \frac{1}{2.2^p}\int_0^{\pi/2} 2\sin^p u\, du \qquad \{\because \sin(\pi - u) = \sin u\}$$

$$\frac{1}{2}\beta\left(\frac{p+1}{2}\ \frac{p+1}{2}\right) = \frac{2}{2\cdot 2^p}\int_0^{\pi/2} \sin^p u \cdot \cos^0 u\ du$$

Using the formula we get,

$$\frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{p+1}{2}\right) = \frac{1}{2^p}\cdot\frac{1}{2}\,\beta\left(\frac{p+1}{2}, \frac{0+1}{2}\right)$$

Put $\frac{p+1}{2} = m$

$\therefore\ p = 2m - 1$

$$\beta(m, m) = \frac{1}{2^{2m-1}}\,\beta\left(m, \frac{1}{2}\right)$$

$$\therefore \qquad \frac{\Gamma m\,\Gamma m}{\Gamma 2m} = \frac{1}{2^{2m-1}}\,\frac{\Gamma m\,\Gamma 1/2}{\Gamma m + 1/2}$$

$$\therefore \qquad \Gamma m\,\Gamma m + 1/2 = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma 2m$$

5.6 Illustrations on Beta Function

Type 1 Problems using basic definition

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx\ (m, n > 0)$$

Example 5.13 : *Prove that* $\int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}dx = \frac{(2n)\,!}{2^{2n}\,(n!)^2}\cdot\frac{\pi}{2}$, *where n is positive integer.*

Solution : Let $\quad I = \int_0^1 x^{2n}\left(1 - x^2\right)^{-1/2} dx$

Put $x^2 = t$, $x = t^{1/2}$, $dx = \frac{1}{2}t^{-1/2}\ dt$

Limits :

x	0	1
t	0	1

$$\therefore \qquad I = \int_0^1 t^n (1-t)^{-1/2} \cdot \frac{1}{2} t^{\;1/2} \, dt$$

$$= \frac{1}{2} \int_0^1 t^{n-1/2} (1-t)^{-1/2} \, dt$$

$$= \frac{1}{2} \beta\left(n + \frac{1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma(n+1/2)\,\Gamma(1/2)}{\Gamma(n+1)}$$

$$= \frac{\sqrt{\pi}}{2} \frac{\Gamma(n+1/2)}{n\,!} \qquad \left\{\begin{array}{l} \because \Gamma(1/2) = \sqrt{\pi} \\ \Gamma(n+1) = n\,! \end{array}\right\}$$

$$= \frac{\sqrt{\pi}}{2(n\,!)}\left(n - \frac{1}{2}\right) \Gamma(n-1/2) \qquad \left\{\because \Gamma(n+1) = n\,\Gamma(n)\right\}$$

Applying

$$= \frac{\sqrt{\pi}}{2\,(n)\,!}\left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right) \ldots\ldots \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2\,(n)\,!}\left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right) \ldots\ldots \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}$$

$$= \frac{\pi}{2} \cdot \frac{1}{(n)\,!} \frac{(2n-1)\,(2n-3) \ldots\ldots 5 \cdot 3 \cdot 1}{2^n}$$

Multiply and divide by (2n) (2n – 2) (2n – 4) .... 6·4·2.

$$= \frac{\pi}{2} \cdot \frac{1}{(n)\,!} \cdot \frac{1}{2^n} \cdot \frac{(2n)(2n-1)(2n-2)(2n-4) \ldots 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2n)(2n-2)(2n-4) \ldots 6 \cdot 4 \cdot 2}$$

$$= \frac{\pi}{2} \cdot \frac{1}{2^n\,(n)\,!} \cdot \frac{(2n)\,!}{2^n\,(n)\,(n-1)\,(n-2) \ldots\ldots 3 \cdot 2 \cdot 1}$$

$$\therefore \qquad I = \frac{\pi}{2} \cdot \frac{(2n)\,!}{2^{2n}\,(n\,!)^2}$$

Example 5.14 : *Prove that* $\int_0^1 \left(1 - x^{1/n}\right)^m dx = \frac{m!\; n!}{(m+n)!}$ **(May-2005)**

Solution : Let $\qquad I = \int_0^1 \left(1 - x^{1/n}\right)^m dx$

Put $x^{1/n} = t$, $x = t^n$, $dx = n\ t^{n-1}\ dt$

Limits :

x	0	1
t	0	1

$$\therefore \qquad I = \int_0^1 (1-t)^m \cdot n \cdot t^{n-1}\, dt$$

$$= n\int_0^1 t^{n-1}(1-t)^m\, dt$$

$$= n\,\beta(n, m+1)$$

$$= n\frac{\overline{|n}\ \overline{|m+1}}{\overline{|n+m+1}}$$

$$= \frac{\overline{|n+1}\ \overline{|m+1}}{\overline{|m+n+1}}$$

$$= \frac{n!\ m!}{(m+n)!}$$

Exercise 5.5

1) *Prove that* $\int_0^1 x^{m-1}\left(1-x^2\right)^{n-1} dx = \frac{1}{2}\beta\left(\frac{m}{2}, n\right)$ **(Dec.-1999)**

Hint : *Put* $x^2 = t$

2) *Prove that* $\int_0^{1/2} x^3\left(1-4x^2\right)^{1/2} dx = \frac{1}{120}$

Hint : *Put* $4x^2 = t$

3) *Prove that* $\int_0^2 x\sqrt[3]{8-x^3}\, dx = \frac{16\pi}{9\sqrt{3}}$

Hint : *Put* $x^3 = 8t$

4) *Evaluate* $\int_0^n x^n (n-x)^m\, dx$ **[Ans. :** $n^{n+m+1}\beta(n+1, m+1)$**]**

Hint : *Put* $x = nt$

5) *Evaluate* $\int_0^1 x^3\left(1-\sqrt{x}\right)^5 dx$ **[Ans. :** $\frac{1}{5148}$**]**

Hint : *Put* $\sqrt{x} = t$

6) *Prove that* $\int_0^1 \frac{dx}{\sqrt{1-x^m}} = \frac{\sqrt{\pi}}{m} \frac{\Gamma(1/m)}{\Gamma(1/m + 1/2)}$

Hint : *Put* $x^m = t$

7) *Prove that* $\int_0^1 \frac{dx}{\sqrt[3]{1-x^3}} = \frac{2\pi}{3\sqrt{3}}$

Hint : *Put* $x^3 = t$ *and use* $\Gamma p\, \Gamma(1-p) = \frac{\pi}{\sin p\pi}$

Type 2 Problems using the formula

$$\int_0^{\pi/2} \sin^p \theta \cos^q \theta \, d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Example 5.15 : *Show that* $\int_{-\pi/4}^{\pi/4} (\sin\theta + \cos\theta)^{1/3} d\theta = \frac{6\sqrt{\pi}}{2^{5/6}} \frac{\Gamma(2/3)}{\Gamma(1/6)}$

Solution : Consider

$$\sin\theta + \cos\theta = \sqrt{2}\left[(\sin\theta)\frac{1}{\sqrt{2}} + (\cos\theta)\frac{1}{\sqrt{2}}\right] = \sqrt{2}\left[\sin\theta \cos\frac{\pi}{4} + \cos\theta \sin\frac{\pi}{4}\right]$$

$$= \sqrt{2} \sin\left(\theta + \frac{\pi}{4}\right)$$

Thus the integral becomes

$$I = \int_{-\pi/4}^{\pi/4} \left[\sqrt{2} \sin\left(\theta + \frac{\pi}{4}\right)\right]^{1/3} d\theta$$

$$= 2^{1/6} \int_{-\pi/4}^{\pi/4} \sin^{1/3}\left(\theta + \frac{\pi}{4}\right) d\theta$$

Put $\theta + \frac{\pi}{4} = t$, $d\theta = dt$

Limits :			
	θ	$-\pi/4$	$\pi/4$
	t	0	$\pi/2$

$$\therefore \quad I = 2^{1/6} \int_0^{\pi/2} \sin^{1/3} t \, dt$$

$$= 2^{1/6} \int_0^{\pi/2} \sin^{1/3} t \cdot \cos^0 t \, dt$$

$$= 2^{1/6} \cdot \frac{1}{2}\beta\left(\frac{\frac{1}{3}+1}{2}, \frac{0+1}{2}\right)$$

$$= \frac{1}{2^{5/6}}\beta\left(\frac{2}{3},\frac{1}{2}\right)$$

$$= \frac{1}{2^{5/6}} \frac{\Gamma(2/3)\,\Gamma(1/2)}{\Gamma(2/3+1/2)}$$

$$= \frac{1}{2^{5/6}} \frac{\Gamma(2/3)\sqrt{\pi}}{\Gamma(5/6)}$$

$$= \frac{1}{2^{5/6}} \frac{\Gamma(2/3)\sqrt{\pi}}{\frac{1}{6}\Gamma(1/6)}$$

$$= \frac{6\sqrt{\pi}}{2^{5/6}} \frac{\Gamma(2/3)}{\Gamma(1/6)}$$

Example 5.16 : *Show that* $\int\limits_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

Solution : Let $I = \int\limits_0^\infty \frac{dx}{1+x^4}$

Put $x^2 = \tan\theta$, $x = \sqrt{\tan\theta}$, $dx = \frac{1}{2}(\tan\theta)^{-1/2} \cdot \sec^2\theta\, d\theta$

Limits :

x	0	∞
θ	0	π/2

Thus the integral becomes

$$I = \int\limits_0^{\pi/2} \frac{1}{1+\tan^2\theta}\,\frac{1}{2}\cdot\tan^{-1/2}\theta\cdot\sec^2\theta\cdot d\theta$$

$$= \frac{1}{2}\int\limits_0^{\pi/2} \tan^{-1/2}\theta\; d\theta$$

$$= \frac{1}{2}\int\limits_0^{\pi/2} \cot^{1/2}\theta\; d\theta$$

$$= \frac{1}{2}\int\limits_0^{\pi/2} \sin^{-1/2}\theta\;\cos^{1/2}\theta\; d\theta$$

Use, $$\int_0^{\pi/2} \sin^p\theta \, \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\therefore \quad I = \frac{1}{2}\cdot\frac{1}{2}\beta\left(\frac{-\frac{1}{2}+1}{2}, \frac{\frac{1}{2}+1}{2}\right)$$

$$= \frac{1}{4}\beta\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$= \frac{1}{4}\frac{\Gamma(1/4)\,\Gamma(3/4)}{\Gamma(1)}$$

$$= \frac{1}{4}\Gamma(1/4)\,\Gamma(1-1/4)$$

$$= \frac{1}{4}\frac{\pi}{\left(\sin\frac{\pi}{4}\right)} \qquad \left(\because \Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi}\right)$$

$$= \frac{1}{4}\frac{\pi}{\left(\frac{1}{\sqrt{2}}\right)}$$

$$\therefore \quad \int_0^{\infty}\frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

Example 5.17 : *Show that* $\int_0^{\pi/2} \tan^n x\,dx = \frac{\pi}{2}\sec\left(\frac{n\pi}{2}\right)$

Solution : Let $$I = \int_0^{\pi/2} \sin^n x \, \cos^{-n} x \, dx$$

$$= \frac{1}{2}\beta\left(\frac{1+n}{2}, \frac{1-n}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{1-n}{2}\right)}{\Gamma(1)}$$

$$= \frac{1}{2}\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(1-\left(\frac{1-n}{2}\right)\right)$$

$$= \frac{1}{2}\frac{\pi}{\sin\left(\frac{1+n}{2}\right)\pi} \qquad \because\left(p = \frac{1+n}{2}\right)$$

$$= \frac{\pi}{2\sin\left(\frac{\pi}{2} + \frac{n\pi}{2}\right)}$$

$$= \frac{\pi}{2\cos\frac{n\pi}{2}}$$

$$= \frac{\pi}{2}\sec\frac{n\pi}{2}$$

Exercise 5.6

1) *Evaluate* $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$ **(Dec.-2000) [Ans. : $\frac{\pi}{\sqrt{2}}$]**

2) *Evaluate* $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \cdot \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$ **(Dec.-2004) [Ans. : π]**

3) *Prove that* $\int_0^{\pi/2} \sin^m\theta\, d\theta \int_0^{\pi/2} \sin^{m+1}\theta\, d\theta = \frac{\pi}{2(m+1)}$

4) *Prove that* $\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \beta(m,n)$

***Hint :** Put* $x = \tan^2\theta$

5) *Prove that* $\int_0^{\infty} \left(\frac{x}{1+x^2}\right)^6 dx = \frac{3\pi}{512}$

***Hint :** Put* $x = \tan\theta$

6) *Prove that* $\int_0^{1} \sqrt{1-x^4}\, dx = \frac{1}{12}\sqrt{\frac{2}{\pi}}\left(\overline{|}1/4\right)^2$

Hint : $x^4 = \sin^2\theta$

7) ***Prove that*** $\int_0^{1} \frac{x^2 dx}{(1-x^4)^{1/2}} - \int_0^{1} \frac{dx}{(1+x^4)^{1/2}} = \frac{\pi}{4\sqrt{2}}$

Hint :** Put* $x^4 = \sin^2\theta$ ***in first and $x^4 = \tan^2\theta$ ***in second***

8) ***Show that*** $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta \cdot \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{\pi^2}{2}$

9) ***Prove that*** $\int_0^{\pi/2} \sin^4\theta \sec^{1/2}\theta\, d\theta = \frac{\sqrt{2}}{7\sqrt{\pi}}\left(\overline{|}1/4\right)^2$

Type 3 Problems having general form

$$\int_a^b (x-a)^l (b-x)^m \, dx$$

Hint : In this type use the standard substitution

$$x - a = (b-a)\, t$$

Example 5.18 : *Prove that* $\int_{-1}^{1} (1+x)^m (1-x)^n \, dx = 2^{m+n+1} \dfrac{m!\, n!}{(m+n+1)!}$ *where m, n are positive integers.* **(Dec.-2003)**

Solution : Let
$$I = \int_{-1}^{1} (1+x)^m (1-x)^n \, dx$$

$$a = -1, \quad b = 1$$

Put $1 + x = 2t$, $dx = 2\, dt$

x	− 1	1
t	0	1

$$\therefore \quad I = \int_0^1 (2t)^m (2-2t)^n \cdot 2\, dt$$

$$= 2^{m+n+1} \int_0^1 t^m (1-t)^n \, dt$$

$$= 2^{m+n+1} \beta(m+1, n+1)$$

$$= 2^{m+n+1} \frac{\Gamma(m+1)\, \Gamma(n+1)}{\Gamma(m+n+2)}$$

$$= 2^{m+n+1} \frac{m!\, n!}{(m+n+1)!}$$

Example 5.19 : *Evaluate* $\int_3^7 (x-3)^{1/4} (7-x)^{1/4} \, dx$

Solution : Put $x - 3 = 4t$, $dx = 4\, dt$

$$a = 3, \quad b = 7$$

x	3	7
t	0	1

$$I = \int_0^1 (4t)^{1/4} (4 - 4t)^{1/4} \cdot 4\,dt$$

$$= 4^{3/2} \int_0^1 t^{1/4} (1 - t)^{1/4}\, dt$$

$$= 8\,\beta\left(\frac{5}{4}, \frac{5}{4}\right) = 8\, \frac{\overline{|5/4}\, \overline{|5/4}}{\overline{|5/2}}$$

$$= 8\, \frac{\left(\frac{1}{4}\overline{|1/4}\right)^2}{\frac{3}{2}\,\frac{1}{2}\sqrt{\pi}} = \frac{2}{3\sqrt{\pi}} \left(\overline{|1/4}\right)^2$$

Exercise 5.7

1) *Evaluate* $\int_a^b (x-a)^m (b-x)^n\, dx$ **[Ans. : $(b - a)^{m+n+1}\, \beta(m + 1, n + 1)$]**

2) *Evaluate* $\int_2^5 (x-2)^{1/4} (5-x)^{1/4}\, dx$ **(May-2001) [Ans. : $\frac{1}{4}\sqrt{\frac{3}{\pi}}\left(\overline{|1/4}\right)^2$]**

3) *Evaluate* $\int_3^7 \sqrt{(x-3)(7-x)}\, dx$ **[Ans. : 2π]**

4) *Prove that* $\int_a^b \sqrt[n]{(x-a)(b-x)}\, dx = (b-a)^{\left(\frac{2}{n}-1\right)} \cdot \beta\left(\frac{1}{n}+1, \frac{1}{n}+1\right)$

5) *Prove that* $\int_5^9 \sqrt[4]{(9-x)(x-5)}\, dx = \frac{2\left(\overline{|1/4}\right)^2}{3\sqrt{\pi}}$

Type 4 Problems using the formula

$$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

Example 5.20 : *Show that* $\int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}}\, dy = \beta(m, n)$ **(Dec.-1994, Dec.-1996, May-1992, May-1993, May-2002, Dec.-2002, May-2004)**

Solution : We know that

$$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \qquad \text{... (Note this step)}$$

$$(0 < 1 < \infty)$$

$$= I_1 + I_2 \text{ (Say)} \quad \dots (1)$$

Consider $\quad I_2 = \int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

Put $x = \frac{1}{y}$, $dx = \frac{-1}{y^2} dy$

x	1	∞
y	1	0

$$I_2 = \int_1^0 \frac{\left(\frac{1}{y}\right)^{m-1}\left(-\frac{1}{y^2}dy\right)}{\left(1+\frac{1}{y}\right)^{m+n}}$$

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

Thus from equation (1),

$$I = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

As the variable is immaterial in definite integrals thus combining both integrals and using one variable we get.

$$I = \int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy$$

Example 5.21 : *Prove that* $\int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \frac{1}{a^n b^m} \beta(m, n)$ **(May-2000)**

Solution : Let $\quad I = \int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}} dx$

Put $bx = at$, $\therefore \; x = \frac{at}{b}$, $dx = \frac{a}{b} dt$

x	0	∞
t	0	∞

$$\therefore \qquad I = \int_0^\infty \frac{\left(\frac{at}{b}\right)^{m-1}}{(a+at)^{m+n}} \cdot \frac{a}{b}\, dt$$

$$= \int_0^\infty \frac{1}{a^{m+n}} \frac{a^{m-1+1}}{b^{m-1+1}} \frac{t^{m-1}}{(1+t)^{m+n}}\, dt$$

$$= \frac{1}{a^n b^m} \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, dt$$

$$= \frac{1}{a^n b^m} \beta(m, n)$$

Exercise 5.8

1) *Show that* $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}} dx = 0$ **(May-1999)**

2) *Show that* $\int_0^\infty \frac{x^6 - x^3}{(1+x^3)^5} \cdot x^2 dx = 0$

Hint : *Put* $x^3 = t$

3) *Show that* $\int_0^1 \frac{x^2 + x^3}{(1+x)^7} dx = \frac{1}{60}$

Hint : *Refer solved Example no. 5.20.*

4) *Prove that* $\int_1^\infty \frac{x^{n/2-1}}{(1+x)^n} dx = \frac{1}{2} \beta\left(\frac{n}{2}, \frac{n}{2}\right)$

Hint : *Put* $x = 1/t$

5) *Show that* $\int_0^\infty \frac{x^{n-1}}{(a+bx)^{m+n}} dx = \frac{1}{a^m b^n} \beta(m, n)$

Hint : *Put* $bx = at$ *or* $bx = a\tan^2\theta$

6) *Show that* $\int_0^\infty \frac{x^{p-1}\, dx}{(m+nx)^{p+q}} = \frac{1}{m^q n^p} \beta(p, q)$ **(Dec.-2000)**

7) *Show that* $\int_0^\infty \frac{x^8 - x^5}{(1+x^3)^5} dx = 0$ **(May-2003)**

8) $\int_0^\infty \frac{x - x^3}{(1+x)^5} dx = \frac{-1}{6}$

9) $\int_0^\infty \frac{x}{(1+x)^4} dx + \int_0^\infty \frac{x^{1/2}}{(1+x)^4} dx = \frac{1}{6} + \frac{\pi}{16}$

Type 5

Example 5.22 : *Show that* $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}} dx = \frac{\beta(m,n)}{a^n (a+b)^m}$

Solution : Let

$$I = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}\, dx}{(a+bx)^{m+n}}$$

$$= \int_0^1 \left(\frac{x}{a+bx}\right)^{m+n} \cdot x^{-1-n}(1-x)^{n-1}\, dx$$

Put $\frac{x}{a+bx} = \frac{t}{a+b}$

$\therefore \boxed{x = \frac{a\,t}{a+b-b\,t}}$

x	0	1
t	0	1

$$dx = \frac{a(a+b)\,dt}{(a+b-b\,t)^2}$$

Also $1 - x = 1 - \frac{a\,t}{a+b-b\,t} = \frac{(a+b)(1-t)}{(a+b-b\,t)}$

Substituting, we get

$$I = \int_0^1 \left(\frac{t}{a+b}\right)^{m+n} \cdot \left[\frac{a\,t}{a+b-b\,t}\right]^{-1-n} \left[\frac{(a+b)(1-t)}{(a+b-b\,t)}\right]^{n-1} \frac{a(a+b)\,dt}{(a+b-b\,t)^2}$$

$$= \frac{1}{a^n (a+b)^m} \int_0^1 t^{m-1}(1-t)^{n-1}\, dt \quad \text{(on symplification)}$$

$$= \frac{1}{a^n (a+b)^m} \beta(m,n)$$

Example 5.23 : *Prove that* $\int_0^\infty \frac{dx}{(e^x + e^{-x})^n} = \frac{1}{4}\beta\left(\frac{n}{2},\frac{n}{2}\right)$ *and hence evaluate* $\int_0^\infty \operatorname{sech}^8 x\, dx$

(May-1995)

Solution : Let

$$I = \int_0^\infty \frac{dx}{(e^x + e^{-x})^n}$$

We know that

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$$

if f (x) is an even function of x.

Thus

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(e^{x} + e^{-x})^{n}}$$

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{x}}{(e^{2x} + 1)^{n}}\, dx \qquad \ldots (1)$$

Put $\qquad e^{2x} = t$

$$2\, e^{2x} dx = dt$$

$$dx = \frac{dt}{2t}$$

x	$-\infty$	∞
0	0	∞

$\therefore$ From (1),

$$I = \frac{1}{2}\int_{0}^{\pi/2} \frac{t^{n/2}}{(1+t)^{n}} \frac{dt}{2t}$$

$$= \frac{1}{4}\int_{0}^{\pi/2} \frac{t^{n/2-1}}{(1+t)^{\frac{n}{2}+\frac{n}{2}}}\, dt$$

$$= \frac{1}{4}\beta\left(\frac{n}{2}, \frac{n}{2}\right)$$

Thus $\qquad \displaystyle\int_{0}^{\infty} \frac{dx}{(e^{x} + e^{-x})^{n}} = \frac{1}{4}\beta\left(\frac{n}{2}, \frac{n}{2}\right)$

i.e. $\qquad \displaystyle\int_{0}^{\infty} \frac{dx}{(2\cosh x)^{n}} = \frac{1}{4}\beta\left(\frac{n}{2}, \frac{n}{2}\right)$

Put n = 8, $\qquad \displaystyle\int_{0}^{\infty} \frac{dx}{(2\cosh x)^{8}} = \frac{1}{4}\beta(4, 4)$

$$\int_{0}^{\infty} \frac{1}{2^{8}} \operatorname{sech}^{8} x\, dx = \frac{1}{4} \cdot \frac{\lceil 4 \; \lceil 4}{\lceil 8}$$

$$\int_{0}^{\infty} \operatorname{sech}^{8} dx = \frac{2^{8}}{4} \cdot \frac{3!\, 3!}{7!} = \frac{16}{35}$$

Example 5.24 : *Show that* $\int_0^{\pi/2} \frac{\sin^{2m-1} x \cdot \cos^{2n-1} x \, dx}{(a \sin^2 x + b \cos^2 x)^{m+n}} = \frac{1}{2a^m b^n} \beta(m, n)$

(May-1994)

Solution :

$$I = \int_0^{\pi/2} \frac{\sin^{2m-1}x \cos^{2n-1}x \, dx}{(a \sin^2 x + b \cos^2 x)^{m+n}}$$

$$= \int_0^{\pi/2} \frac{\sin^{2m-1}x \cdot \cos^{2n-1}x \, dx}{\cos^{2m+2n}x \, (a \tan^2 x + b)^{m+n}}$$

$$= \int_0^{\pi/2} \frac{\tan^{2m-1}x \cdot \sec^2 x \, dx}{(a \tan^2 x + b)^{m+n}}$$

$$= \int_0^{\pi/2} \frac{\tan^{2m-2}x \cdot \tan x \sec^2 x \, dx}{(a \tan^2 x + b)^{m+n}}$$

Put $a \tan^2 x = bu$

$2a \tan x \sec^2 x \, dx = b \, du$

x	0	$\pi/2$
u	0	∞

$$\therefore \quad I = \int_0^{\infty} \frac{\left(\frac{bu}{a}\right)^{m-1} \cdot \frac{b}{2a} \, du}{(bu + b)^{m+n}}$$

$$= \frac{1}{2 b^n a^m} \int_0^{\infty} \frac{u^{m-1} \, du}{(1 + u)^{m+n}}$$

$$= \frac{1}{2 a^m b^n} \beta(m, n)$$

Example 5.25 : *Evaluate* $\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \frac{1}{2} \sin^2 \theta}}$

(May-1991, Dec.-1998)

Solution : Put $\cos^2 \theta = \sqrt{t}$, $\cos\theta = t^{1/4}$, $\theta = \cos^{-1}\left(t^{1/4}\right)$

θ	0	$\pi/2$
t	1	0

$$\therefore \quad d\theta = \frac{-1}{\sqrt{1 - t^{1/2}}} \cdot \left(\frac{1}{4} t^{-3/4}\right) dt$$

$$d\theta = \frac{-\frac{1}{4}t^{-3/4}}{\sqrt{1-\sqrt{t}}}\,dt$$

Also $\qquad \sin^2\theta = 1 - \cos^2\theta = 1 - \sqrt{t}$

$$\therefore \qquad 1 - \frac{1}{2}\sin^2\theta = 1 - \frac{1}{2}\left(1 - \sqrt{t}\right)$$

$$= \frac{1}{2}\left(1 + \sqrt{t}\right)$$

Thus the integral becomes

$$I = \int_0^1 \frac{1}{\sqrt{\frac{1}{2}\left(1+\sqrt{t}\right)}} \cdot \frac{-\frac{1}{4}t^{-3/4}dt}{\sqrt{1-\sqrt{t}}}$$

$$= \frac{\sqrt{2}}{4}\int_0^1 \frac{t^{-3/4}\,dt}{\sqrt{1-t}}$$

$$= \frac{1}{2\sqrt{2}}\int_0^1 t^{-3/4}(1-t)^{-1/2}\,dt$$

$$= \frac{1}{2\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

$$= \frac{1}{2\sqrt{2}}\frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(1/4+1/2)} \qquad \left(\text{As } \Gamma(1/2) = \sqrt{\pi}\right)$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}}\frac{\Gamma(1/4)}{\Gamma(3/4)}$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}} \cdot \frac{\left(\Gamma(1/4)\right)^2}{\Gamma(1/4)\,\Gamma(1-1/4)}$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}}\frac{\left(\Gamma(1/4)\right)^2}{\left(\frac{\pi}{\sin\pi/4}\right)}$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}}\frac{\left(\Gamma(1/4)\right)^2}{\left(\pi\sqrt{2}\right)} = \frac{\left(\Gamma(1/4)\right)^2}{4\sqrt{\pi}}$$

Example 5.26 : *Evaluate* $\int_1^\infty \frac{dx}{x^{p+1}(x-1)^q}$

Solution : Put $x = \frac{1}{t}$ $\quad dx = \frac{-1}{t^2}\,dt$

Limits :

x	1	∞
t	1	0

$$I = \int_1^0 \frac{-\frac{1}{t^2}\,dt}{\left(\frac{1}{t}\right)^{p+1}\left(\frac{1}{t}-1\right)^q}$$

$$= \int_0^1 t^{p+q-1}(1-t)^{-q}\,dt$$

$$= \beta(p+q,\ 1-q)$$

Exercise 5.9

1) *Show that* $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\,dx = \frac{\beta(m,n)}{a^n(1+a)^m}$ **(May-1991, Dec.-1995)**

Hint : *Put* $b = 1$ *in Example No. 5.22*

2) *Show that* $\int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\,dx = \frac{1}{9^{1/3}}\beta\left(\frac{2}{3},\frac{1}{3}\right)$ **(May-2003)**

Hint : *Put* $a = 1,\ b = 2,\ m = \frac{2}{3},\ n = \frac{1}{3}$ *in Example No. 5.22.*

3) *Prove that* $\beta(m,n) = \beta(m, n+1) + \beta(m+1, n)$

4) *Prove that* $\Gamma\!\left(1/4\right)\Gamma\!\left(3/4\right) = \pi\sqrt{2}$ **(May-1997)**

Hint : *In duplication formula put* $m = \frac{1}{4}$

5) *Prove that* $\beta(x+1, y) = \frac{x}{x+y}\beta(x, y)$

6) *Prove that* $\beta(m, n) \times \beta(m+n, p) = \frac{\Gamma m\ \Gamma n\ \Gamma p}{\Gamma(m+n+p)}$

7) *Prove that* $\beta(m, m) = 2^{1-2m}\beta\left(m, \frac{1}{2}\right)$

8) *Prove that* $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(1+x)^{m+n}}\,dx = \frac{1}{2^m}\beta(m,m)$

Hint : *Put* $\frac{x}{1+x} = \frac{t}{2}$

9) *Prove that* $\int_0^a \frac{dx}{(a^n - x^n)^{1/n}} = \frac{\frac{\pi}{2}}{\sin\frac{\pi}{n}}$

Hint : *Put* $\left(\frac{x}{a}\right)^n = t$

10) *Prove that* $\beta(n, n+1) = \frac{(\sqrt{n})^2}{2\sqrt{2n}}$

11) $\frac{\beta(m, n+1) + \beta(m+1, n)}{n} = \frac{\beta(m,n)}{m+n}$

12) $\beta(m, m) \cdot \beta\left(m + \frac{1}{2}, m + \frac{1}{2}\right) = \pi \cdot m^{-1} \cdot 2^{1-4m}$

University Questions

May - 2003

1. *Show that* $\int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\,dx = \frac{1}{9^{1/3}}\beta\left(\frac{2}{3}, \frac{1}{3}\right)$ **[4 Marks]**

2. *Evaluate* $\int_0^\infty \frac{x^8 - x^5}{(1+x^3)^5}\,dx$ **[3 Marks]**

Dec. - 2003

1. *Evaluate* $\int_0^1 (x \log x)^3\,dx$ **[5 Marks]**

2. *Show that* $\int_{-1}^1 (1+x)^m(1-x)^n\,dx = \frac{m!n!}{(m+n+1)!}(m > 0, n > 0)$ **[5 Marks]**

May - 2004

1. *Show that* $\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$ **[4 Marks]**

2. *Show that* $\int_0^\infty x^{m-1}\cos ax\,dx = \frac{\sqrt{m}}{a^m}\cos\frac{m\pi}{2}$ **[4 Marks]**

Dec. - 2004

1. *Prove that* $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta = \pi$ **[4 Marks]**

2. *Evaluate* $\int_0^\infty x^n e^{-x^m}\,dx$ **[4 Marks]**

May - 2005

1. *Prove that* $\int_0^1 \left(1-x^{1/n}\right)dx = \frac{m!n!}{(m+n)!}$ **[4 Marks]**

2. *Evaluate* $\int_0^1 (x\log x)^4\,dx$ **[4 Marks]**

Dec. - 2005

1. *Evaluate* $\int_0^1 x^m (\log x)^n\,dx$ **[4 Marks]**

2. *Establish the Legendre's duplication formula of Gamma functions* $\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\cdot\Gamma(2m)$ **[5 Marks]**

3. *Evaluate* $\int_0^\infty \frac{x^8\left(1-x^6\right)}{(1+x)^{24}}\,dx$ **[8 Marks]**

May - 2006

1. *Evaluate* $\int_0^\infty x^4\, e^{-x^4}\,dx$ **[4 Marks]**

2. *Prove that* $\int_1^\infty \frac{x^{\frac{n}{2}-1}}{(1+x)^n}\,dx = \frac{1}{2}\beta\left(\frac{n}{2},\frac{n}{2}\right)$ **[4 Marks]**

Dec. - 2006

1. *Show that* $\int_0^1 \frac{X^{m-1}(1-X)^{n-1}}{(1+X)^{m+n}}\,dx = \frac{B(m,n)}{2^m}$ **[5 Marks]**

2. *Evaluate* $\int_0^1 \frac{x\,dx}{\sqrt{\log\left(\frac{1}{x}\right)}}$ **[4 Marks]**

May - 2007

1. *Prove that* $\beta\,(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$. **[5 Marks]**

2. *Evaluate*

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$$

[5 Marks]

Dec. - 2007

1. c) *Prove that* $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{a^n b^m}\beta(m,n)$ **[4 Marks]**

2. *c) Evaluate* $\int_0^1 (x \log x)^4 \, dx.$

[4 Marks]

May - 2008

1. *Evaluate* $= \int_0^\infty \sqrt[8]{x}\, e^{-\sqrt{x}} dx$

[4 Marks]

2. *Show that* $\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

[4 Marks]

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6 Differentiation Under Integral Sign

6.1 Introduction

In real definite integrals some integrals are difficult to solve by our traditional methods of solving, here the rule of differentiation under integral sign can be effectively used in evaluating the integrals also it reduces the labour or steps of solving.

In these integrals in addition to the variable of integration one or more parameters are involved.

We consider only one parameter α then the integral will take the form

$$I(\alpha) = \int_a^b f(x, \alpha)\, dx$$

Here α = Parameter

x = Variable of integration

a and b = Limits of integration.

a and b may be constants or functions of α.

6.2 Rule 1 : Integrals with Constant Limits i.e. a and b are Constants

$$\frac{d}{d\alpha}\int_a^b f(x, \alpha)\, dx = \int_a^b \frac{\partial}{\partial \alpha} f(x, \alpha)\, dx$$

Proof : Let $I(\alpha) = \int_a^b f(x, \alpha)\, dx$

Now by using the definition of derivatives by first principle.

$$I'(\alpha) = \lim_{\delta\alpha \to 0} \frac{I(\alpha + \delta\alpha) - I(\alpha)}{\delta\alpha}$$

Substituting $I(\alpha)$ and $I(\alpha + \delta\alpha)$ we get,

$$I'(\alpha) = \lim_{\delta\alpha \to 0} \frac{I}{\delta\alpha}\left[\int_a^b f(x, \alpha + \delta\alpha)\, dx - \int_a^b f(x, \alpha)\, dx\right]$$

$$= \lim_{\delta\alpha \to 0} \int_a^b \frac{f(x, \alpha + \delta\alpha) - f(x, \alpha)}{\delta\alpha}\, dx$$

$$= \lim_{\delta\alpha \to 0} \int_a^b \frac{\delta}{\delta\alpha} f(x,\alpha)\ dx = \int_a^b \frac{\delta}{\delta\alpha} f(x,\alpha)\ dx$$

Thus

$$\boxed{\frac{d}{d\alpha}\int_a^b f(x,\alpha)\ dx = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\ dx}$$

i.e. if a and b are constants then derivative w.r.t. parameter α outside the definite integral becomes partial derivative inside the integral.

6.3 Illustrations on Rule 1

Example 6.1 *: Show that* $\int_0^1 \frac{x^a - 1}{\log x}\ dx = \log(a+1),\ (a \geq 0)$ **(May-2004)**

Solution :

Step 1 : Consider the integral as I (a).

$$I(a) = \int_0^1 \frac{x^a - 1}{\log x}\ dx$$

Step 2 : Differentiate both sides w.r.t. a.

$$I'(a) = \frac{d}{da}\int_0^1 \frac{x^a - 1}{\log x}\ dx$$

Step 3 : Apply DUIS.

$$I'(a) = \int_0^1 \frac{\partial}{\partial a} \frac{x^a - 1}{\log x}\ dx$$

Step 4 : Take the partial derivative.

$$I'(a) = \int_0^1 \frac{x^a \log x}{\log x}\ dx$$

Step 5 : Simplify.

$$I'(a) = \int_0^1 x^a\ dx$$

Step 6 : Integrate w.r.t. x.

$$I'(a) = \left[\frac{x^{a+1}}{a+1}\right]_0^1$$

Step 7 : Substitute the limits of x.

$$I'(a) = \frac{1}{a+1}$$

Step 8 : Integrate w.r.t. a.

$$I(a) = \log(a+1) + c$$

Step 9 : Substitute suitable value of a to find c.

Put a = 0.

$$I(0) = \log(1) + c$$

$$\int_0^1 \frac{x^0 - 1}{\log x}\, dx = \log 1 + c$$

$$0 = 0 + c$$

$$\Rightarrow \quad c = 0$$

Step 10 : Substituting c we get the value of the integral.

$$\therefore \quad I(a) = \log(a+1)$$

Example 6.2 : *Show that* $\int_0^\infty \frac{e^{-x} - e^{-ax}}{x \sec x} dx = \frac{1}{2}\log\left(\frac{a^2+1}{2}\right)$ **(May-2006)**

Solution :

Step 1 : Consider the integral as I (a).

$$I(a) = \int_0^\infty \frac{e^{-x} - e^{-ax}}{x \sec x}\, dx$$

Step 2 : Differentiate both sides w.r.t. a.

$$I'(a) = \frac{d}{da}\int_0^\infty \frac{e^{-x} - e^{-ax}}{x \sec x}\, dx$$

Step 3 : Apply DUIS.

$$I'(a) = \int_0^\infty \frac{\partial}{\partial a}\frac{e^{-x} - e^{-ax}}{x \sec x}\, dx$$

Step 4 : Take the partial derivative.

$$I'(a) = \int_0^\infty \frac{0 - e^{-ax}(-x)}{x \sec x}\, dx$$

Step 5 : Simplify.

$$I'(a) = \int_0^{\infty} e^{-ax} \cos x \, dx$$

Step 6 : Integrate w.r.t. x.

$$I'(a) = \left\{\frac{e^{-ax}}{a^2+1}[-a\cos x + \sin x]\right\}_0^{\infty}$$

Step 7 : Substitute the limits of x.

$$I'(a) = \left\{0 - \frac{1}{a^2+1}[-a+0]\right\}$$

$$I'(a) = \frac{a}{a^2+1}$$

Step 8 : Integrate w.r.t. a.

$$I(a) = \frac{1}{2}\log(a^2+1) + c$$

Step 9 : Substitute suitable value of a to find c.

Put a = 1.

$$I(1) = \frac{1}{2}\log 2 + c$$

$$\int_0^{\infty} \frac{e^{-x} - e^{-x}}{x \sec x} dx = \frac{1}{2}\log 2 + c$$

$$0 = \frac{1}{2}\log 2 + c$$

$$\Rightarrow \qquad c = \frac{-1}{2}\log 2$$

Step 10 : Substituting c we get the value of the integral.

$$\therefore \qquad I(a) = \frac{1}{2}\log(a^2+1) - \frac{1}{2}\log 2$$

$$I(a) = \frac{1}{2}\log\left(\frac{a^2+1}{2}\right)$$

Example 6.3 : *Show that* $\int_0^{\infty} \left(\frac{1 - e^{-ax}}{x}\right) e^{-x}\, dx = \log(a + 1)$.

Solution :

Step 1 : Consider the integral as I (a).

$$I(a) = \int_0^{\infty} \left(\frac{1 - e^{-ax}}{x}\right) e^{-x}\, dx$$

Step 2 : Differentiate w.r.t. a.

$$I'(a) = \frac{d}{da} \int_0^{\infty} \left(\frac{1 - e^{-ax}}{x}\right) e^{-x}\, dx$$

Step 3 : Apply DUIS.

$$I'(a) = \int_0^{\infty} \frac{\partial}{\partial a} \left(\frac{1 - e^{-ax}}{x}\right) e^{-x}\, dx$$

Step 4 : Take the partial derivative.

$$I'(a) = \int_0^{\infty} \left(\frac{0 - e^{-ax}(-x)}{x}\right) e^{-x}\, dx$$

Step 5 : Simplify.

$$I'(a) = \int_0^{\infty} e^{-(a+1)x}\, dx$$

Step 6 : Integrate w.r.t. x.

$$I'(a) = \left[\frac{e^{-(a+1)x}}{-(a+1)}\right]_0^{\infty}$$

Step 7 : Substitute the limits of x.

$$I'(a) = 0 + \frac{1}{a+1}$$

Step 8 : Integrate w.r.t. a.

$$I(a) = \log(a + 1) + c$$

Step 9 : Substituting suitable value of a to find c.

Put a = 0.

$$I(0) = \log(1) + c$$

$$0 = 0 + c$$

Step 10 : Substituting c we get the value of the integral.

$$I(a) = \log(a+1).$$

Note : Follow the same procedure for the remaining problems.

Example 6.4 : *Evaluate* $\int_0^\infty \frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x}e^{-ax}\right)dx$ **(May-2002)**

Solution : Let
$$I(a) = \int_0^\infty \frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x}e^{-ax}\right)dx$$

$$\therefore \quad I'(a) = \frac{d}{da}\int_0^\infty \frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x}e^{-ax}\right)dx$$

$\therefore$ By DUIS,

$$I'(a) = \int_0^\infty \frac{\partial}{\partial a}\frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x}e^{-ax}\right)dx$$

Differentiating partially w.r.t. a,

$$I'(a) = \int_0^\infty \frac{e^{-x}}{x}\left(1 - 0 + \frac{1}{x}e^{-ax}(-x)\right)dx$$

$$= \int_0^\infty \frac{e^{-x}}{x}\left(1 - e^{-ax}\right)dx$$

$$= I(a) \qquad \text{(Refer Example 6.3 and repeat the procedure)}$$

$$I'(a) = \log(a+1)$$

Integrating w.r.t. a, $(\because \int \log x \, dx = x\log x - x + C)$

$$I(a) = (a+1)\log(a+1) - (a+1) + c$$

Put a = 0.

$$0 = 0 - (0+1) + c$$

$$\Rightarrow \quad c = 1$$

$$\therefore \quad I(a) = (a+1)\log(a+1) - a$$

Note : We need the following integral.

$$I = \int_0^\infty e^{-x^2}\,dx$$

Put $x^2 = t$, $x = \sqrt{t}$, $dx = \frac{1}{2\sqrt{t}}\,dt$.

x	0	∞
t	0	∞

$$I = \int_0^\infty e^{-t} \frac{1}{2\sqrt{t}}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\, t^{-1/2}\,dt$$

$$= \frac{1}{2}\,\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

Example 6.5 : *Show that* $\int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)}dx = \frac{\sqrt{\pi}}{2}e^{-2a}$, $(a > 0)$ **(May-2000)**

Solution : Let

$$I(a) = \int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)}dx \qquad \dots (1)$$

$$\therefore \quad I'(a) = \int_0^\infty \frac{\partial}{\partial a} e^{-\left(x^2+\frac{a^2}{x^2}\right)}dx$$

$$= \int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)} \cdot \frac{-2a}{x^2}\,dx$$

Put $\frac{a}{x} = t$, $\therefore \frac{-a}{x^2}dx = dt$

x	0	∞
t	∞	0

$$I'(a) = \int_\infty^0 e^{-\left(\frac{a^2}{t^2}+t^2\right)} \cdot 2\,dt$$

$$= -2\int_0^\infty e^{-\left(t^2+\frac{a^2}{t^2}\right)}dt$$

$$= -2\,I(a) \qquad \dots \text{From equation (1)}$$

Thus

$$\frac{I'(a)}{I(a)} = -2$$

Integrate w.r.t. a,

$$\log I(a) = -2a + c$$

$$\therefore \quad I(a) = e^{-2a+c}$$

$$I(a) = e^{-2a} \cdot e^{c}$$

Let $e^{c} = A$ say

$$I(a) = A\, e^{-2a} \qquad \dots (2)$$

Put a = 0.

$$I(0) = A$$

Put a = 0 in equation (1).

$$\int_0^\infty e^{-x^2}\, dx = A = \frac{\sqrt{\pi}}{2} \qquad \left(\text{Since } \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}\right)$$

Thus from equation (2),

$$I(a) = \frac{\sqrt{\pi}}{2} e^{-2a}$$

Example 6.6 : *Prove that* $\int_0^\infty e^{-x^2} \cos 2\lambda x\, dx = \frac{\sqrt{\pi}}{2} e^{-\lambda^2}$ **(Dec.-1998, Dec.-2002)**

Solution : Let $I(\lambda) = \int_0^\infty e^{-x^2} \cos 2\lambda\, x\, dx$ (λ is a parameter.) ... (1)

Using DUIS,

$$I'(\lambda) = \int_0^\infty \frac{\partial}{\partial \lambda} e^{-x^2} \cos 2\lambda\, x\, dx$$

$$I'(\lambda) = \int_0^\infty e^{-x^2} (-2x) \cdot \sin 2\lambda\, x\, dx$$

[We know $\int e^{f(x)} f'(x)\, dx = e^{f(x)}$, $\int e^{-x^2}(-2x)\, dx = e^{-x^2}$]

$$\therefore \quad I'(\lambda) = \int_0^\infty \underset{U}{(\sin 2\lambda x)} \cdot \underset{V}{\left(e^{-x^2}(-2x)\right)} dx$$

Integrating by parts,

$$= \left[\sin 2\lambda x\; e^{-x^2}\right]_0^\infty - \int_0^\infty 2\lambda \cos 2\lambda x\; e^{-x^2}\, dx$$

$$= (0 - 0) - 2\lambda\, I(\lambda) \qquad \dots \text{From equation (1)}$$

$$\therefore \quad \frac{I'(\lambda)}{I(\lambda)} = -2\lambda$$

Integrating w.r.t. λ,

$$\log I(\lambda) = -\lambda^2 + c$$

$$I(\lambda) = e^{-\lambda^2 + c}$$

$$I(\lambda) = e^{-\lambda^2} \cdot e^c$$

Let $e^c = A$ say

$$I(\lambda) = A\, e^{-\lambda^2} \qquad \text{... (2)}$$

Put $\lambda = 0$.

$$I(0) = A$$

Put $\lambda = 0$ in equation (1).

$$\int_0^\infty e^{-x^2} \cdot dx = A = \frac{\sqrt{\pi}}{2} \qquad \left(\text{Since } \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}\right)$$

$\therefore$ From equation (2),

$$I(\lambda) = \frac{\sqrt{\pi}}{2} e^{-\lambda^2}$$

Example 6.7 : *Prove that* $\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\, dx = \frac{\pi}{2} \log(a+1)$

Solution :

$$I(a) = \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}$$

Differentiating w.r.t. a,

$$I'(a) = \frac{d}{da} \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\, dx$$

Apply DUIS,

$$I'(a) = \int_0^\infty \frac{\partial}{\partial a} \frac{\tan^{-1} ax}{x(1+x^2)}\, dx$$

Differentiate partially w.r.t. a,

$$I'(a) = \int_0^\infty \frac{1}{x(1+x^2)} \cdot \frac{x}{1+a^2x^2}\, dx$$

$$= \int_0^\infty \frac{1}{(1+x^2)\, a^2 \left(x^2 + \frac{1}{a^2}\right)}\, dx$$

$$= \frac{1}{a^2} \int_0^\infty \frac{1}{(x^2+1)\left(x^2+\frac{1}{a^2}\right)} dx$$

Note the following step.

$$= \frac{1}{a^2\left(\frac{1}{a^2}-1\right)} \int_0^\infty \left[\frac{1}{(x^2+1)} - \frac{1}{\left(x^2+\frac{1}{a^2}\right)}\right] dx$$

(By using Partial fraction)

$$= \frac{1}{1-a^2}\left[\tan^{-1}x - \frac{1}{\frac{1}{a}}\tan^{-1}\left(\frac{x}{\frac{1}{a}}\right)\right]_0^\infty$$

$$= \frac{1}{1-a^2}\left[\tan^{-1}x - a\tan^{-1}(ax)\right]_0^\infty$$

$$= \frac{1}{1-a^2}\left[\left(\frac{\pi}{2}-\frac{a\pi}{2}\right)-(0-0)\right]$$

$$= \frac{1}{(1-a)(1+a)}(1-a)\frac{\pi}{2}$$

$$I'(a) = \frac{\pi}{2}\,\frac{1}{1+a}$$

Integrate w.r.t. a,

$$I(a) = \frac{\pi}{2}\log(1+a)+c$$

Put a = 0.

$$0 = \frac{\pi}{2}\log 1 + c \Rightarrow c = 0$$

Thus $\quad I(a) = \frac{\pi}{2}\log a$

Example 6.8 : *Evaluate* $I(m) = \int_0^\infty e^{-ax}\frac{\sin mx}{x}dx$*. Hence determine* $\int_0^\infty \frac{\sin x}{x}dx$.

Solution : We have,

$$I(m) = \int_0^\infty e^{-ax}\frac{\sin mx}{x}dx$$

Differentiate w.r.t. m,

$$I'(m) = \frac{d}{dm}\int_0^\infty e^{-ax}\frac{\sin mx}{x}\,dx$$

Apply DUIS,

$$I'(m) = \int_0^\infty \frac{\partial}{\partial m} e^{-ax}\frac{\sin mx}{x}\,dx$$

$$= \int_0^\infty e^{-ax}\cdot\frac{x\cos mx}{x}\,dx$$

$$= \int_0^\infty e^{-ax}\cos mx\,dx$$

$$\left\{ \text{Use} : \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx] \right\}$$

$$= \left\{\frac{e^{-ax}}{a^2+m^2}[-a\cos mx + m\sin mx]\right\}_0^\infty$$

$$= 0 - \frac{1}{a^2+m^2}[-a+0]$$

$$I'(m) = \frac{-(-a)}{a^2+m^2} = \frac{a}{a^2+m^2}$$

Integrate w.r.t. m,

$$I(m) = \tan^{-1}\frac{m}{a} + c$$

Put m = 0.

$$0 = 0 + c$$

Thus

$$\int_0^\infty e^{-ax}\frac{\sin mx}{x}\,dx = \tan^{-1}\frac{m}{a}$$

Put a = 0, m = 1

$$\therefore \quad \int_0^\infty \frac{\sin x}{x}\,dx = \tan^{-1}\frac{1}{0} = \frac{\pi}{2}$$

Example 6.9 : *Prove that* $\int_0^1 x^m (\log x)^n \, dx = \dfrac{(-1)^n \, n!}{(m+1)^{n+1}}$ *where n is positive integer.* **(May-1998)**

Solution : Let $\quad I(m) = \int_0^1 x^m \, dx \quad$ (Note this step)

$$\therefore \quad \int_0^1 x^m \, dx = \left[\frac{x^{m+1}}{m+1}\right]_0^1$$

$$\int_0^1 x^m \, dx = \frac{1}{m+1}$$

Differentiate w.r.t. m and apply DUIS,

$$\int_0^1 \frac{\partial}{\partial m} x^m \, dx = \frac{-1}{(m+1)^2}$$

$$\int_0^1 x^m \log x \, dx = \frac{-1}{(m+1)^2}$$

Again differentiating w.r.t. m and DUIS,

$$\int_0^1 \frac{\partial}{\partial m} x^m \log x \, dx = \frac{(-1)(-2)}{(m+1)^3}$$

$$\int_0^1 x^m (\log x)^2 \, dx = \frac{(-1)^2 \cdot 2!}{(m+1)^3}$$

Hence applying DUIS n times we get,

$$\int_0^1 x^m (\log x)^n \, dx = \frac{(-1)^n \cdot (n)!}{(m+1)^{n+1}}$$

Example 6.10 : *Using rule of differentiation under integral sign, evaluate* $\int_0^\infty e^{-ax} \dfrac{\sin x}{x} \, dx$ *and deduce* $\int_0^\infty \dfrac{\sin x}{x} \, dx$. **(May-1998, May-2001)**

Solution : Let $\quad I(a) = \int_0^\infty e^{-ax} \dfrac{\sin x}{x} \, dx$

Differentiate w.r.t. a,

$$I'(a) = \frac{d}{da}\int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx$$

Apply DUIS,

$$= \int_0^\infty \frac{\partial}{\partial a} e^{-ax}\frac{\sin x}{x}\,dx$$

Taking partial derivative.

$$= \int_0^\infty e^{-ax}(-x)\frac{\sin x}{x}\,dx$$

$$= -\int_0^\infty e^{-ax}\sin x\,dx$$

$$= -\left[\frac{e^{-ax}}{a^2+1}(-a\sin x - \cos x)\right]_0^\infty$$

$$= -\left[0 - \frac{1}{a^2+1}(0-1)\right]$$

$$I'(a) = \frac{-1}{a^2+1}$$

Integrating w.r.t. a,

$$I(a) = -\tan^{-1} a + c$$

Let $a \to \infty$,

$$\lim_{a\to\infty} I(a) = 0 = -\frac{\pi}{2} + c$$

$$\therefore \quad \frac{\pi}{2} = c \qquad \left\{\text{As} \int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}\right\}$$

Thus

$$\int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx = \frac{\pi}{2} - \tan^{-1} a$$

Put a = 0.

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

Example 6.11 : *Show that* $\int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x}dx = \frac{\pi^2}{8} - \frac{\alpha^2}{2}$ **(May-2003)**

Solution : Let $I(\alpha) = \int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x}dx$

Differentiate w.r.t. α and apply DUIS,

$$I'(\alpha) = \int_0^{\pi/2} \frac{\partial}{\partial\alpha} \frac{\log(1+\cos\alpha\cos x)}{\cos x}dx$$

Take partial derivative.

$$I'(\alpha) = \int_0^{\pi/2} \frac{-\sin\alpha\cos x}{1+\cos\alpha\cos x}\left(\frac{1}{\cos x}\right)dx$$

$$= \int_0^{\pi/2} \frac{-\sin\alpha}{1+\cos\alpha\cos x}dx$$

Put $\tan\frac{x}{2} = t$, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2\,dt}{1+t^2}$

x	0	π/2
t	0	1

$$= \int_0^1 \frac{-\sin\alpha\frac{2\,dt}{1+t^2}}{1+\cos\alpha\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int_0^1 \frac{-2\sin\alpha\ dt}{1+t^2+\cos\alpha\ (1-t^2)}$$

$$= -2\sin\alpha\int_0^1 \frac{dt}{(1+\cos\alpha)+(1-\cos\alpha)\ t^2}$$

$$= \frac{-2\sin\alpha}{1-\cos\alpha}\int_0^1 \frac{dt}{\left(\frac{1+\cos\alpha}{1-\cos\alpha}\right)+t^2}$$

$$= \frac{-2\sin\alpha}{2\sin^2\alpha/2}\int_0^1 \frac{dt}{\frac{2\cos^2\alpha/2}{2\sin^2\alpha/2}+t^2}$$

$$= \frac{-2\cdot 2\cdot \sin\alpha/2\cos\alpha/2}{2\sin^2\alpha/2}\int_0^1 \frac{dt}{\cot^2\alpha/2 + t^2}$$

$$= -2\cot\frac{\alpha}{2}\,\frac{1}{\cot\frac{\alpha}{2}}\left[\tan^{-1}\left(\frac{t}{\cot\frac{\alpha}{2}}\right)\right]_0^1$$

$$= -2\left[\tan^{-1}\left(\tan\frac{\alpha}{2}\right) - 0\right]$$

$$= -2\left[\frac{\alpha}{2}\right]$$

$$I'(\alpha) = -\alpha$$

Integrate w.r.t. α,

$$I(\alpha) = \frac{-\alpha^2}{2} + c$$

Put $\alpha = \frac{\pi}{2}$.

$$0 = -\frac{\pi^2/4}{2} + c$$

$$\Rightarrow \qquad c = \frac{\pi^2}{8}$$

Thus $$I(\alpha) = \frac{\pi^2}{8} - \frac{\alpha^2}{2}$$

Example 6.12 : *Show that* $\int_0^\pi \log(1 - a\cos x)\,dx = \pi\log\left(\frac{1+\sqrt{1-a^2}}{2}\right)$ $|a| < 1.$ **(Dec.-1995)**

Solution : $$I(a) = \int_0^\pi \log(1 - a\cos x)\,dx$$

Using DUIS, we get,

$$I'(a) = \int_0^\pi \frac{\partial}{\partial a}\{\log(1 - a\cos x)\}\,dx$$

$$= \int_0^\pi \frac{(-\cos x)}{1 - a\cos x}\cdot dx = \frac{1}{a}\int_0^\pi \frac{-a\cos x}{1 - a\cos x}\,dx$$

$$= \frac{1}{a}\int_0^\pi \frac{1 - a\cos x - 1}{1 - a\cos x}\cdot dx = \frac{1}{a}\int_0^\pi\left(1 - \frac{1}{1 - a\cos x}\right)dx$$

$$= \frac{1}{a}[x]_0^\pi - \frac{1}{a}\int_0^\pi \frac{dx}{1 - a\cos x}$$

Put $t = \tan x/2$, $dx = \dfrac{2\,dt}{1+t^2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$

x	0	π
t	0	∞

$$I'(a) = \frac{1}{a}\pi - \frac{1}{a}\int_0^\infty \frac{\dfrac{2dt}{1+t^2}}{1-a\left[\dfrac{1-t^2}{1+t^2}\right]}$$

$$= \frac{\pi}{a} - \frac{2}{a}\int_0^\infty \frac{dt}{1-a\left(\dfrac{1-t^2}{1+t^2}\right)}$$

$$= \frac{\pi}{a} - \frac{2}{a}\int_0^\infty \frac{dt}{1+t^2-a+a\,t^2}$$

$$= \frac{\pi}{2} - \frac{2}{a}\int_0^\infty \frac{dt}{1-a+(1+a)\,t^2}$$

$$= \frac{\pi}{a} - \frac{2}{a(1+a)}\int_0^\infty \frac{dt}{\dfrac{1-a}{1+a}+t^2}$$

$$= \frac{\pi}{a} - \frac{2}{a(1+a)}\left[\frac{1}{\sqrt{\dfrac{1-a}{1+a}}}\tan^{-1}\left(\frac{t}{\sqrt{\dfrac{1-a}{1+a}}}\right)\right]_0^\infty$$

$$I'(a) = \frac{\pi}{a} - \frac{2}{a\sqrt{1-a^2}}\left(\frac{\pi}{2}-0\right)$$

$$= \frac{\pi}{a} - \frac{\pi}{a\sqrt{1-a^2}}$$

Integrating w.r.t. 'a',

$$I(a) = \pi\log a - \pi\int \frac{1}{a\sqrt{1-a^2}}\,da + c_1$$

$$= \pi\log a - \pi\log\left(\frac{1-\sqrt{1-a^2}}{a}\right) + c_1$$

$$\because \quad \int \frac{dx}{x\sqrt{1-x^2}} = \log\left(\frac{1-\sqrt{1-x^2}}{x}\right)$$

$$\therefore \quad I(a) = \pi \log\left(\frac{a^2}{1-\sqrt{1-a^2}}\right) + c_1$$

$$= \pi \log\left[\frac{a^2\left(1+\sqrt{1-a^2}\right)}{\left(1-\sqrt{1-a^2}\right)\left(1+\sqrt{1-a^2}\right)}\right] + c_1$$

$$= \pi \log\left[\frac{a^2\left(1+\sqrt{1-a^2}\right)}{1-1+a^2}\right] + c_1$$

$$= \pi \log\left(1+\sqrt{1-a^2}\right) + c_1$$

Put $a = 0$, $I(0) = \pi \log 2 + c_1$

But $\quad I(0) = 0$

$$c_1 = -\pi \log 2$$

$$I(a) = \pi \log\left(1+\sqrt{1-a^2}\right) - \pi \log 2$$

$$I(a) = \pi \log\left(\frac{1+\sqrt{1-a^2}}{2}\right)$$

Example 6.13 : *Show that* $\displaystyle\int_0^{\pi/2} \frac{\log\left(1+a\sin^2 x\right)}{\sin^2 x}\, dx = \pi\left[\sqrt{1+a}-1\right]$.

Solution : Let $\quad I(a) = \displaystyle\int_0^{\pi/2} \frac{\log\left(1+a\sin^2 x\right)}{\sin^2 x}\, dx$

Differentiate w.r.t. a by DUIS, we get,

$$I'(a) = \int_0^{\pi/2} \frac{\partial}{\partial a} \frac{\log\left(1+a\sin^2 x\right)}{\sin^2 x}\, dx$$

$$= \int_0^{\pi/2} \frac{\left(\sin^2 x\right)}{\left(1+a\sin^2 x\right)} \cdot \frac{1}{\sin^2 x}\, dx$$

$$= \int_0^{\pi/2} \frac{1}{1+a\sin^2 x}\, dx$$

$$= \int_0^{\pi/2} \frac{1}{1 + \frac{a}{\operatorname{cosec}^2 x}} \, dx$$

$$= \int_0^{\pi/2} \frac{\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x + a} \, dx$$

$$= \int_0^{\pi/2} \frac{\operatorname{cosec}^2 x}{1 + \cot^2 x + a} \, dx$$

Put $\cot x = t$, $\therefore \; -\operatorname{cosec}^2 x \; dx = dt$

x	0	$\pi/2$
t	∞	0

$$= \int_\infty^0 \frac{-dt}{(a+1) + t^2}$$

$$= \int_0^\infty \frac{dt}{(a+1) + t^2}$$

$$= \frac{1}{\sqrt{(a+1)}} \left[\tan^{-1} \frac{t}{\sqrt{a+1}} \right]_0^\infty$$

$$I'(a) = \frac{1}{\sqrt{a+1}} \left(\frac{\pi}{2} - 0 \right)$$

Integrate w.r.t. a,

$$I(a) = \frac{\pi}{2} \, 2\sqrt{a+1} + c$$

Put a = 0.

$$I(0) = \pi + c$$

$$c = -\pi$$

$\therefore$

$$I(a) = \pi\sqrt{a+1} - \pi$$

$$= \pi\left[\sqrt{a+1} - 1\right]$$

Exercise 6.1 : Problems on Type 1

1) *The Bessel function of integral order n is defined as* $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(nt - x\sin t)\, dt$ *show that* $\frac{d}{dx}J_n(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$.

2) *Evaluate* $\int_0^{\pi} \frac{dx}{a + b\cos x}$, $a > 0$, $|b| < a$ *and deduce* $\int_0^{\pi} \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{(a^2 + b^2)^{3/2}}$ *and* $\int_0^{\pi} \frac{\cos x\, dx}{(a + b\cos x)^2} = \frac{-\pi b}{(a^2 - b^2)^{3/2}}$.

3) *Evaluate* $\int_0^{\infty} \frac{e^{-\beta x}\sin \alpha x}{x}\, dx$ *and deduce that* $\int_0^{\infty} \frac{\sin \alpha x}{x} dx = \frac{-\pi}{2}$ *if* $\alpha < 0$.

$$= 0 \quad \text{if } \alpha = 0$$

$$= \frac{\pi}{2} \quad \text{if } \alpha > 0.$$

4) *Prove that* $a^2 f(y) - \frac{d^2 f}{dy^2} = \frac{\pi}{2}$ *where* $f(y) = \int_0^{\infty} \frac{\sin xy}{x(a^2 + x^2)} dx$. **(Dec.-2005)**

5) *Prove that* $\int_0^{\infty} \frac{e^{-\alpha x}\sin x}{x}\, dx = \cot^{-1}\alpha$.

6) *Prove that* $\int_0^{\infty} \frac{1 - \cos ax}{x^2}\, dx = \frac{\pi a}{2}$.

7) *Prove that* $\int_0^{\infty} \frac{\log(1 + ax^2)}{x^2} dx = \pi\sqrt{a}$, $(a > 0)$.

8) *Prove that* $\int_0^{\infty} e^{-x^2} \cos ax\, dx = \frac{\sqrt{\pi}}{2} e^{-a^2/4}$ *assuming that* $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

9) *Prove that* $\int_0^{\infty} \left(\frac{1 - \cos mx}{x}\right) e^{-x} dx = \log\sqrt{1 + m^2}$.

10) *Verify the rule of differentiation under integral sign for the integral* $\int_0^{\infty} e^{-at}\cos bt\, dt$ *where a is the parameter.*

11) *Verify the rule of differentiation under integral sign for the integral* $\int_0^{\pi/2} \sin ax\, dx$.

Note : The procedure of solving the problems on Type 2 is exactly same as that of Type 1.

Type 2 : Problems Involving Two Parameters

Example 6.14 : *Show that* $\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} dx = \log \frac{b}{a}$

(Dec.-2002, Dec.-2004, Dec.-2007)

Solution : Let $I(a) = \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} dx$

Differentiate w.r.t. a and apply DUIS,

$$I'(a) = \int_0^{\infty} \frac{\partial}{\partial a} \frac{e^{-ax} - e^{-bx}}{x} dx$$

$$= \int_0^{\infty} \frac{e^{-ax}(-x) - 0}{x} dx$$

$$= -\int_0^{\infty} e^{-ax} dx$$

$$= -\left[\frac{e^{-ax}}{-a}\right]_0^{\infty}$$

$$= -\left[0 + \frac{1}{a}\right]$$

$$I'(a) = \frac{-1}{a}$$

Integrate w.r.t. a,

$$I(a) = -\log a + c$$

Put a = b.

$$0 = -\log b + c$$

$$\Rightarrow \quad c = \log b$$

Thus $\quad I(a) = -\log a + \log b = \log \frac{b}{a}$

Example 6.15 : *Prove that* $\int_0^{\infty} \frac{\cos \lambda x}{x} \left(e^{-ax} - e^{-bx}\right) dx = \frac{1}{2} \log\left(\frac{b^2 + \lambda^2}{a^2 + \lambda^2}\right)$ *$(a > 0, b > 0)$.*

(Dec.-1999)

Solution : Let $I(a) = \int_0^{\infty} \frac{\cos \lambda x}{x} \left(e^{-ax} - e^{-bx}\right) dx$

Differentiate w.r.t. a and DUIS,

$$I'(a) = \int_0^\infty \frac{\partial}{\partial a} \frac{\cos\lambda x}{x} \left(e^{-ax} - e^{-bx}\right) dx$$

$$= \int_0^\infty \frac{\cos\lambda x}{x} \left[e^{-ax}(-x) - 0\right] dx$$

$$= -\int_0^\infty \frac{\cos\lambda x}{x} \left[e^{-ax}(x)\right] dx$$

$$= -\int_0^\infty e^{-ax} \cos\lambda x \, dx$$

$$= -\left[\frac{e^{-ax}}{a^2 + \lambda^2}(-a\cos\lambda x + \lambda \sin\lambda x)\right]_0^\infty$$

$$= -\left[0 - \frac{1}{a^2 + \lambda^2}(-a + 0)\right]$$

$$I'(a) = \frac{-a}{a^2 + \lambda^2}$$

Integrate w.r.t. a,

$$I(a) = \frac{-1}{2}\log\left(\lambda^2 + a^2\right) + c$$

Put a = b.

$$0 = \frac{-1}{2}\log\left(b^2 + \lambda^2\right) + c$$

$$\therefore \qquad c = \frac{1}{2}\log\left(b^2 + \lambda^2\right)$$

Substituting c we get,

$$I(a) = \frac{1}{2}\left[\log\left(b^2 + \lambda^2\right) - \log\left(a^2 + \lambda^2\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{b^2 + \lambda^2}{a^2 + \lambda^2}\right)$$

Example 6.16 : *Evaluate* $\int_0^{\pi/2} \log\left(a^2 \cos^2\theta + b^2 \sin^2\theta\right) d\theta$.

Solution :

$$I(a) = \int_0^{\pi/2} \log\left(a^2 \cos^2\theta + b^2 \sin^2\theta\right) d\theta \qquad \ldots (1)$$

$$\therefore \qquad I'(a) = \int_0^{\pi/2} \frac{\partial}{\partial a} \log\left(a^2 \cos^2\theta + b^2 \sin^2\theta\right) d\theta$$

$$= \int_0^{\pi/2} \frac{2a \cos^2\theta \; d\theta}{a^2\cos^2\theta + b^2\sin^2\theta}$$

$$= \int_0^{\pi/2} \frac{2a \cos^2\theta \; d\theta}{\left(a^2 - b^2\right)\cos^2\theta + b^2}$$

$$= \frac{2a}{a^2 - b^2} \int_0^{\pi/2} \frac{\left(a^2 - b^2\right)\cos^2\theta + b^2 - b^2}{\left(a^2 - b^2\right)\cos^2\theta + b^2}\, d\theta$$

$$= \frac{2a}{a^2 - b^2} \int_0^{\pi/2} 1 - \frac{b^2}{\left(a^2 - b^2\right)\cos^2\theta + b^2}\, d\theta$$

$$= \frac{2a}{a^2 - b^2} \left\{\frac{\pi}{2} - \int_0^{\pi/2} \frac{b^2 \; d\theta}{\left(a^2 - b^2\right)\cos^2\theta + b^2}\right\}$$

$$= \frac{2a}{a^2 - b^2} \left\{\frac{\pi}{2} - b^2 \int_0^{\pi/2} \frac{\sec^2\theta \; d\theta}{\left(a^2 - b^2\right) + b^2\sec^2\theta}\right\}$$

$$= \frac{2a}{a^2 - b^2} \left\{\frac{\pi}{2} - b^2 \int_0^{\pi/2} \frac{\sec^2\theta \; d\theta}{a^2 + b^2\tan^2\theta}\right\}$$

$$= \frac{2a}{a^2 - b^2} \left\{\frac{\pi}{2} - \int_0^{\pi/2} \frac{\sec^2\theta \; d\theta}{\frac{a^2}{b^2} + \tan^2\theta}\right\}$$

Put $\tan\theta = t$, $\sec^2\theta \; d\theta = dt$,

θ	0	π/2
t	0	∞

$$= \frac{2a}{a^2 - b^2} \left\{\frac{\pi}{2} - \int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}\right\}$$

$$= \frac{2a}{a^2 - b^2} \left\{\frac{\pi}{2} - \frac{b}{a}\left(\tan^{-1}\frac{bt}{a}\right)_0^{\infty}\right\}$$

$$= \frac{2a}{a^2 - b^2} \left\{\frac{\pi}{2} - \frac{b}{a}\left(\frac{\pi}{2} - 0\right)\right\}$$

$$= \frac{2a}{a^2 - b^2} \left\{1 - \frac{b}{a}\right\} \frac{\pi}{2}$$

$$= \frac{\pi (a - b)}{(a - b)(a + b)}$$

$$I'(a) = \frac{\pi}{a + b}$$

Integrate w.r.t. a,

$$I(a) = \pi \log(a + b) + c$$

Put a = b.

$$I(b) = \pi \log(2b) + c$$

Now from equation (1),

$$I(b) = \int_0^{\pi/2} \log\left(b^2 \cos^2\theta + b^2 \sin^2\theta\right) d\theta$$

$$= \int_0^{\pi/2} \log b^2 \cdot d\theta$$

$$= 2 \log b \left(\frac{\pi}{2}\right)$$

$$= \pi \log b$$

Thus $\quad \pi \log b = \pi \log 2b + c$

$\Rightarrow \quad c = \pi \log \frac{1}{2}$

Thus $\quad I(a) = \pi \log(a + b) + \pi \log \frac{1}{2}$

$$= \pi \log\left(\frac{a + b}{2}\right)$$

Exercise 6.2 : Problems on Type 2

1) *Show that* $\int_0^1 \frac{x^a - x^b}{\log x}\,dx = \log\left(\frac{a+1}{b+1}\right)$

2) *Show that* $\int_0^\infty \frac{\tan^{-1}\left(\frac{x}{a}\right) - \tan^{-1}\left(\frac{x}{b}\right)}{x}\,dx = \frac{\pi}{2}\log\frac{b}{a}$

3) *Show that* $\int_0^\infty \left(\frac{e^{-\alpha x} - e^{-\beta x}}{x}\right)\sin mx\,dx = \tan^{-1}\frac{\beta}{m} - \tan^{-1}\frac{\alpha}{m}$

4) *Show that* $\int_0^{\pi/2} \log\left(\frac{a + b\sin\theta}{a - b\sin\theta}\right)d\theta = \pi\,\sin^{-1}\left(\frac{b}{a}\right)$

Type 3 : Integral with Limits as Functions of the Parameter (Leibnitz Rule)

If $a = f_1(\alpha)$, $b = f_2(\alpha)$ then

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx + f(b,\alpha)\cdot\frac{db}{d\alpha} - f(a,\alpha)\cdot\frac{da}{d\alpha}$$

Proof : Let $I(\alpha) = \int_a^b f(x,\alpha)\,dx$

As a and b are functions of α.

$\therefore I \to (\alpha, a, b) \to \alpha$

$$\therefore \quad \frac{dI}{d\alpha} = \frac{\partial I}{\partial\alpha}\cdot\frac{d\alpha}{d\alpha} + \frac{\partial I}{\partial a}\cdot\frac{da}{d\alpha} + \frac{\partial I}{\partial b}\cdot\frac{db}{d\alpha} \quad \text{... (1)}$$

$\frac{\partial I}{\partial\alpha}$ is obtained by treating a, b constants.

Thus $$\frac{\partial I}{\partial\alpha} = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx$$

Let $\int f(x,\alpha)\,dx = \phi(x,\alpha)$

$$\therefore \quad \frac{\partial}{\partial x}\phi(x,\alpha) = f(x,\alpha) \quad \text{... (2)}$$

Thus $$I(\alpha) = \int_a^b f(x,\alpha)\,dx = [\phi(x,\alpha)]_a^b = \phi(b,\alpha) - \phi(a,\alpha)$$

Thus
$$\frac{\partial I}{\partial a} = \frac{\partial}{\partial a}\left[\phi(b,\alpha) - \phi(a,\alpha)\right]$$
$$= 0 - \frac{\partial}{\partial a}\phi(a,\alpha)$$
$$= -f(a,\alpha) \qquad \text{... From equation (2)}$$

and
$$\frac{\partial I}{\partial b} = \frac{\partial}{\partial b}\left[\phi(b,\alpha) - \phi(a,\alpha)\right]$$
$$= \frac{\partial}{\partial b}\phi(b,\alpha) - 0$$
$$= f(b,\alpha) \qquad \text{... From equation (2)}$$

Substituting the values of $I(\alpha), \frac{\partial I}{\partial a}, \frac{\partial I}{\partial b}$ in equation (1)

$$\boxed{\frac{dI}{d\alpha} = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\ dx + \frac{db}{d\alpha}\ f(a,\alpha) - \frac{da}{d\alpha}\ f(b,\alpha)}$$

Note : If a, b are functions of α then

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\ dx = \begin{pmatrix}\text{Partial derivative}\\ \text{inside the integral}\end{pmatrix} + \begin{pmatrix}\text{Derivative of}\\ \text{upper limit}\end{pmatrix}\begin{pmatrix}\text{Value of the function}\\ \text{at the upper limit}\end{pmatrix}$$
$$- \begin{pmatrix}\text{Derivative of}\\ \text{lower limit}\end{pmatrix}\begin{pmatrix}\text{Value of the function}\\ \text{at the lower limit}\end{pmatrix}$$

Illustrations on Rule 2

Example 6.17 : *Verify Leibnitz rule of DUIS for the integral* $\int_a^{a^2} \frac{dx}{x+a}$. **(Dec.-2001)**

Solution : Let
$$I(a) = \int_a^{a^2} \frac{dx}{x+a}$$
$$= \left[\log(x+a)\right]_a^{a^2}$$
$$= \log(a + a^2) - \log 2a$$
$$= \log\frac{a(a+1)}{2a}$$

$$I(a) = \log\frac{a+1}{2}$$

Thus $$I'(a) = \frac{1}{a+1} \qquad \dots (1)$$

Now again $$I(a) = \int_a^{a^2} \frac{dx}{x+a}$$

By DUIS, $$I'(a) = \int_a^{a^2} \frac{\partial}{\partial a}\frac{1}{x+a}\, dx + \left(\frac{d}{da}a^2\right)\cdot\left(\frac{1}{a+a^2}\right) - \left(\frac{d}{da}a\right)\cdot\left(\frac{1}{a+a}\right)$$

$$= \int_a^{a^2} \frac{-1}{(x+a)^2}\, dx + \frac{2a}{a+a^2} - \frac{1}{2a}$$

$$= \left(\frac{1}{x+a}\right)_a^{a^2} + \frac{2}{a+1} - \frac{1}{2a}$$

$$= \frac{1}{a^2+a} - \frac{1}{2a} + \frac{2}{a+1} - \frac{1}{2a}$$

Simplifying we get,

$$= \frac{1}{a+1} \qquad \dots (2)$$

From equation (1) and (2) DUIS is verified.

Example 6.18 : *Verify the rule of DUIS for the integral* $\int_a^{a^2} \log(ax)\, dx$. **(Dec.-2003)**

Solution : Let $$I(a) = \int_a^{a^2} \log(ax)\, dx$$

Differentiate w.r.t. a,

$$I'(a) = \frac{d}{da}\int_a^{a^2} \log(ax)\, dx$$

Apply DUIS,

$$= \int_a^{a^2} \frac{\partial}{\partial a}\log ax\, dx + \left(\frac{d}{da}a^2\right)\cdot\log(a\cdot a^2) - \left(\frac{d}{da}a\right)\cdot\log(a\cdot a)$$

$$= \int_a^{a^2} \frac{1}{ax}\cdot x\cdot dx + (2a)\cdot\log a^3 - (1)\log a^2$$

$$= \int_a^{a^2} \frac{dx}{a} + 6a \log a - 2 \log a$$

$$= \left(\frac{x}{a}\right)_a^{a^2} + (6a - 2) \log a$$

$$= \frac{a^2 - a}{a} + (6a - 2) \log a$$

$$= a - 1 + (6a - 2) \log a \qquad \dots (1)$$

Consider I (a) again.

$$I(a) = \int_a^{a^2} \log ax \cdot 1 \cdot dx$$

Apply integration by parts,

$$= [x \log ax]_a^{a^2} - \int_a^{a^2} \frac{1}{ax} \cdot a \cdot x \, dx$$

$$= \left[a^2 \log a^3 - a \log a^2\right] - [x]_a^{a^2}$$

$$I(a) = \left(3a^2 - 2a\right) \log a - a^2 + a$$

Differentiate w.r.t. a,

$$I'(a) = (6a - 2) \log a + \left(3a^2 - 2a\right)\frac{1}{a} - 2a + 1$$

$$= (6a - 2) \log a + a - 1 \qquad \dots (2)$$

Thus from equations (1) and (2) the rule of differentiation under integral sign is verified.

Example 6.19 : *Verify the rule of differentiation under integral sign for the integral* $\int_0^{a^2} \tan^{-1} \frac{x}{a} \cdot dx$.

Solution : Let $\quad I(a) = \int_0^{a^2} \tan^{-1} \frac{x}{a} \, dx$

Apply DUIS,

$$I'(a) = \int_0^{a^2} \frac{\partial}{\partial a} \tan^{-1}\left(\frac{x}{a}\right) dx + \frac{d}{da}\left(a^2\right) \cdot \tan^{-1}\left(\frac{a^2}{a}\right) - 0$$

$$= \int_0^{a^2} \frac{1}{1 + \left(\frac{x}{a}\right)^2} \left(\frac{-x}{a^2}\right) dx + 2a \tan^{-1} a$$

$$= -\int_0^{a^2} \frac{x}{x^2 + a^2} dx + 2a \tan^{-1} a$$

$$= \frac{-1}{2}\left[\log\left(x^2 + a^2\right)\right]_0^{a^2} + 2a \tan^{-1} a$$

$$= \frac{-1}{2}\left[\log\left(\frac{a^4 + a^2}{a^2}\right)\right] + 2a \tan^{-1} a$$

Thus $$I'(a) = \frac{-1}{2} \log\left(a^2 + 1\right) + 2a \tan^{-1} a \qquad \text{... (1)}$$

Again $$I(a) = \int_0^{a^2} \tan^{-1}\left(\frac{x}{a}\right) \cdot 1 \cdot dx$$

Use integration by parts.

$$= \left[x \cdot \tan^{-1} \frac{x}{a}\right]_0^{a^2} - \int_0^{a^2} \frac{a}{x^2 + a^2} \cdot x \, dx$$

$$= \left(a^2 \tan^{-1} a - 0\right) - \frac{a}{2}\left[\log\left(x^2 + a^2\right)\right]_0^{a^2}$$

$$= a^2 \tan^{-1} a - \frac{a}{2} \log\left(\frac{a^4 + a^2}{a^2}\right)$$

$$I(a) = a^2 \tan^{-1} a - \frac{a}{2} \log\left(a^2 + 1\right)$$

Differentiating w.r.t. a we get,

$$I'(a) = \frac{-1}{2} \log\left(a^2 + 1\right) + 2a \tan^{-1} a \qquad \text{... (2)}$$

From equations (1) and (2) the rule of differentiation under integral sign is verified.

Example 6.20 : *If* $f(x) = \int_0^x (x - t)^2 G(t) \, dt$ *then prove that* $\frac{d^3 f}{dx^3} = 2G(x)$. **(May-2005)**

Solution : $$f(x) = \int_0^x (x - t)^2 G(t) \, dt$$

$$\frac{df}{dx} = \frac{d}{dx} \int_0^x (x - t)^2 G(t) \, dt$$

By DUIS,

$$= \int_0^x \frac{\partial}{\partial x} (x - t)^2 G(t) \, dt + \frac{dx}{dx}(0) - \frac{d}{dx}(0) \cdot t^2 G(t) \, dt$$

$$\frac{df}{dx} = \int_0^x 2(x-t)\,G(t)\,dt$$

Again by DUIS,

$$\frac{d^2f}{dx^2} = \int_0^x \frac{\partial}{\partial x} 2(x-t)\,G(t)\,dt + 0 - 0$$

$$\frac{d^2f}{dt^2} = \int_0^x 2\,G(t)\,dt$$

Again by DUIS,

$$\frac{d^3f}{dx^3} = \int_0^x \frac{\partial}{\partial x} 2\,G(t)\,dt + \frac{dx}{dx}\cdot 2\,G(x) - 0$$

$$= 0 + 2\,G(x) - 0$$

$$\therefore \quad \frac{d^3f}{dx^3} = 2\,G(x)$$

Example 6.21 : *If* $y = \int_0^x f(t)\,\sin a(x-t)\,dt$. *Show that* $\frac{d^2y}{dx^2} + a^2y = a\cdot f(x)$.

(May-1999, Dec.-2000)

Solution :

$$\frac{dy}{dx} = \frac{d}{dx}\int_0^x f(t)\cdot \sin a(x-t)\,dt$$

Applying DUIS,

$$= \int_0^x \frac{\partial}{\partial x} f(t)\sin a(x-t)\,dt + \left(\frac{dx}{dx}\right)\cdot f(x)\sin a(x-x) -$$

$$\left(\frac{d0}{dx}\right)\cdot f(0)\cdot \sin a(x-0)$$

$$\frac{dy}{dx} = \int_0^x a\cdot f(t)\,\cos a(x-t)\,dt + 0 - 0$$

Differentiating again and applying DUIS,

$$\frac{d^2y}{dx^2} = \int_0^x \frac{\partial}{\partial x} a\cdot f(t)\cdot \cos a(x-t)\,dt + \left(\frac{dx}{dx}\right)\cdot a\cdot f(x)\cdot \cos a\,(x-x)$$

$$-\left(\frac{d0}{dx}\right)\cdot a\cdot f(0)\cdot \cos a\,(x-0)$$

$$= \int_0^x -a^2 f(t)\sin a(x-t)\,dt + a\;f(x) - 0$$

i.e. $$\frac{d^2y}{dx^2} = -a^2y + a\,f(x)$$

i.e. $$\frac{d^2y}{dx^2} + a^2y = a\,f(x)$$

Example 6.22 : *Show that* $\int_{\pi/6a}^{\pi/2a} \frac{\sin ax}{x}\,dx$ *is independent of a.*

Solution : To show that $I'(a) = 0$.

$$I'(a) = \frac{d}{da}\int_{\pi/6a}^{\pi/2a} \frac{\sin ax}{x}\,dx$$

Apply DUIS,

$$= \int_{\pi/6a}^{\pi/2a} \frac{\partial}{\partial a}\frac{\sin ax}{x}\,dx + \left(\frac{-\pi}{2a^2}\right)\cdot\frac{\sin\frac{\pi}{2}}{\left(\frac{\pi}{2a}\right)} - \left(\frac{-\pi}{6a^2}\right)\cdot\frac{\sin\frac{\pi}{6}}{\left(\frac{\pi}{6a}\right)}$$

$$= \int_{\pi/6a}^{\pi/2a} \cos ax\,dx - \frac{(1)}{a} + \frac{\left(\frac{1}{2}\right)}{a}$$

$$= \left[\frac{\sin ax}{a}\right]_{\pi/6a}^{\pi/2a} - \frac{1}{a} + \frac{1}{2a}$$

$$= \frac{1}{a} - \frac{1}{2a} - \frac{1}{a} + \frac{1}{2a} = 0$$

$\Rightarrow I(a)$ is independent of a.

Example 6.23 : *Evaluate* $\int_0^a \frac{\log(ax+1)}{x^2+1}dx$ *and show that* $\int_0^1 \frac{\log(1+x)}{x^2+1}\,dx = \frac{\pi}{8}\log 2.$

Solution : $$I'(a) = \frac{d}{da}\int_0^a \frac{\log(ax+1)}{x^2+1}\,dx \quad \text{(a is parameter.)}$$

Apply DUIS,

$$= \int_0^a \frac{\partial}{\partial a}\frac{\log(1+ax)}{1+x^2}\,dx + \left(\frac{d}{da}a\right)\cdot\frac{\log(1+a^2)}{1+a^2} - 0$$

$$= \int_0^a \frac{x}{(1+ax)(1+x^2)}\,dx + \frac{\log(1+a^2)}{1+a^2}$$

Now using partial fractions for,

$$\frac{x}{(1+ax)\left(1+x^2\right)} = \frac{A}{1+ax} + \frac{Bx+c}{1+x^2}$$

$$\therefore \qquad x = A\left(1+x^2\right) + (1+ax)(Bx+c)$$

Comparing the coefficients of x^2, x and constants we get,

$A + aB = 0$

$B + aC = 1$

$A + C = 0$

Solving we get,

$$A = \frac{-a}{a^2+1}$$

$$B = \frac{1}{a^2+1}$$

$$C = \frac{a}{a^2+1}$$

Substituting A, B, C we get,

$$\frac{x}{(1+ax)\left(1+x^2\right)} = \frac{-a}{a^2+1}\cdot\frac{1}{1+ax} + \frac{1}{a^2+1}\cdot\frac{x+a}{x^2+1}$$

$$\therefore \qquad I'(a) = \frac{-1}{a^2+1}\int_0^a \frac{adx}{1+ax} + \frac{1}{a^2+1}\int_0^a \frac{x+a}{1+x^2}dx + \frac{\log\left(1+a^2\right)}{1+a^2}$$

$$= \frac{-1}{a^2+1}\left[\log(1+ax)\right]_0^a + \frac{1}{a^2+1}\left[\frac{1}{2}\log\left(1+x^2\right) + a\cdot\tan^{-1}x\right]_0^a$$

$$+\frac{\log\left(1+a^2\right)}{1+a^2}$$

$$= \frac{-1}{a^2+1}\left[\log\left(1+a^2\right) - \log 1\right] + \frac{1}{a^2+1}\left[\frac{1}{2}\log\left(1+a^2\right) + a\cdot\tan^{-1}a - 0\right]$$

$$+\frac{\log\left(1+a^2\right)}{1+a^2}$$

$$= \frac{-1}{a^2+1}\log\left(1+a^2\right) + \frac{1}{a^2+1}\left[\frac{1}{2}\log\left(1+a^2\right) + a\tan^{-1}a\right]$$

$$+\frac{\log\left(1+a^2\right)}{1+a^2}$$

$$= \frac{1}{a^2+1}\left[\frac{1}{2}\log\left(1+a^2\right) + a\tan^{-1}a\right]$$

$$I'(a) = \frac{1}{2}\frac{\log\left(1+a^2\right)}{1+a^2} + \frac{a}{1+a^2}\tan^{-1}a$$

To find I (a) integrate w.r.t. a.

$$I(a) = \frac{1}{2}\int \frac{\log\left(1+a^2\right)}{1+a^2}\,da + \int \frac{a}{1+a^2}\tan^{-1}a\,da + c$$

Integrate the second integral using by parts.

$$= \frac{1}{2}\int \frac{\log\left(1+a^2\right)}{1+a^2}\,da + \tan^{-1}a\left(\frac{1}{2}\log\left(1+a^2\right)\right)$$

$$-\int \frac{1}{1+a^2}\left(\frac{1}{2}\log\left(1+a^2\right)\right)da + c$$

$$I(a) = \frac{1}{2}\tan^{-1}a\cdot\log\left(1+a^2\right) + c$$

Put a = 0.

$$I(a) = 0 + c$$

But $I(0) = 0$

$\Rightarrow$ $c = 0$

$$\therefore \int_0^a \frac{\log(1+ax)}{1+x^2}\,dx = \frac{1}{2}\tan^{-1}a\cdot\log\left(1+a^2\right)$$

Put a = 1.

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{1}{2}\cdot\frac{\pi}{4}\cdot\log 2 = \frac{\pi}{8}\log 2$$

Exercise 6.3

1) *If* $y = \int_0^x f(t)\left[\sin x\cos t - \cos x\sin t\right]dt$ *find the value of* $\frac{d^2y}{dx^2} + y$. **(Dec.-2000)**

2) *If* $f(t) = \int_t^{t^2} e^{tx^2}\,dx$ *then show that* $\frac{dF}{dt} = \frac{1}{2t}\left[5t^2e^{t^5} - 3t\,e^{t^3} - f(t)\right]$.

3) *Find* $\frac{dI}{da}$ *if* $I(a) = \int_a^{a^2} \frac{\sin ax}{x}\,dx$.

4) *Show that* $\dfrac{d}{da}\int_{\sqrt{a}}^{1/a} \cos ax^2\, dx = -\int_{\sqrt{a}}^{1/a} x^2 \sin ax^2 dx - \dfrac{1}{a^2}\cos\dfrac{1}{a} - \dfrac{1}{2\sqrt{a}}\cos a^2.$

5) *Prove that* $\int_0^x \dfrac{dx}{\left(x^2+a^2\right)^2} = \dfrac{1}{2a^3}\tan^{-1}\left(\dfrac{x}{a}\right) + \dfrac{x}{2a^2\left(x^2+a^2\right)}.$

Hint : $\int_0^x \dfrac{dx}{x^2+a^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$ *Differentiate w.r.t. a.*

University Questions

May - 2003

1. *Show that* $\int_0^{\pi/2} \dfrac{\log(1+\cos a\cos x)}{\cos x} dx = \dfrac{\pi^2}{8} - \dfrac{\alpha^2}{2}.$ **[6 Marks]**

Dec. - 2003

1. *Verify the rule of differentiation under integral sign for the integral* $\int_a^{a^2} \log(ax)dx.$ **[5 Marks]**

May - 2004

1. *Prove that* $\int_0^1 \dfrac{x^a-1}{\log x} dx = \log(a+1) a \geq 0.$ **[5 Marks]**

Dec. - 2004

1. *Prove that* $\int_0^\infty \dfrac{e^{-ax}-e^{-bx}}{x} dx = \log\dfrac{b}{a}\ \begin{matrix} a>0 \\ b>0 \end{matrix}$ **[5 Marks]**

May - 2005

1. *If* $f(x) = \int_a^x (x-t)^2 G(t)\, dt$, *then prove that* $\dfrac{d^3 f}{dx^3} - 2G(x) = 0.$ **[5 Marks]**

Dec. - 2005

1. *Assuming* $\int_0^\infty \dfrac{\sin x}{x} dx = \dfrac{\pi}{2}$, *prove that* $k^2 F(y) - \dfrac{d^2F}{dy^2} = \dfrac{\pi}{2}$ *where* $F(y) = \int_0^\infty \dfrac{\sin xy}{x\left(k^2+x^2\right)} dx.$ **[5 Marks]**

May - 2006

1. *Prove that* $\int_0^\infty \dfrac{e^{-x}-e^{-ax}}{x\sec x} dx = \dfrac{1}{2}\log\left(\dfrac{a^2+1}{2}\right),\ a > 0.$ **[5 Marks]**

Dec. - 2006

1. *Evaluate* $\int_0^\infty \dfrac{1}{x^2} \log\left(1+ax^2\right)dx.$

[5 Marks]

May - 2007

1. *a) Show that*

$$\int_0^{\pi/2} \frac{\log(1+a\sin^2 x)}{\sin^2 x}\, dx = \pi(\sqrt{1+a}-1)$$

[5 Marks]

Dec. - 2007

1. *Show that* $\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} dx = \log (b/a)$

[5 Marks]

May - 2008

1. *If* $y = \int_0^{x} f(t) \sin a (x - t)\, dt$, *show that*

$$\frac{d^2y}{dx^2} + a^2 y = af(x)$$

[5 Marks]

2. *Prove that* $\int_0^1 x^p (\log x)^n dx = \frac{(-1)^n n!}{(p+1)^{n+1}}$

where n is a positive integer and $p > -1$. **[4 Marks]**

Dec. - 2008

1. *Evaluate* $\int_0^1 x^m (\log x)^n dx$.

[4 Marks]

2. *Evaluate* $\int_0^{\infty} \frac{1}{x^2} \log(1 + ax^2)\, dx$

[5 Marks]

7 Error Function

7.1 Definition

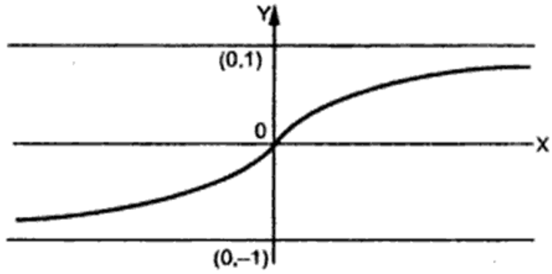
[Dec.-2005]

Error function of x or Probability Integral defined as the integral

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\,du \quad \text{... (1)}$$

which is used in normal probability distribution.

Graph :

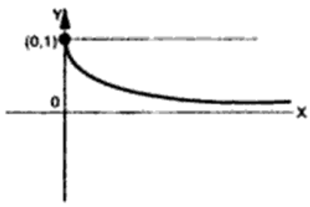


7.2 Complementary Error Function

Complementary error function of x is defined as the integral

$$\operatorname{erf}_c(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\,du \quad \text{... (2)}$$

Graph :



7.3 Alternative Forms of erf (x) and erf$_c$(x)

If we substitute $u^2 = t$ in equation (1) and (2) then $u = \sqrt{t}$, $du = \frac{1}{2\sqrt{t}}\,dt$

u	0	x
t	0	x^2

Substituting in equation (1) and (2) we get

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \int_0^{x^2} e^{-t} \frac{1}{\sqrt{t}}\,dt$$

$$\operatorname{erf}_c(x) = \frac{1}{\sqrt{\pi}} \int_{x^2}^{\infty} e^{-t} \frac{1}{\sqrt{t}}\,dt$$

7.4 Properties

1) Property 1 : erf (0) = 0

2) Property 1 : erf (∞) = 1

Proof :

$$\text{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} du$$

Put $u^2 = t, \; u = \sqrt{t}$

u	0	∞
t	0	∞

$$du = \frac{1}{2} t^{-1/2} \; dt$$

$$\therefore \quad e(\infty) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t} \cdot \frac{1}{2} t^{-1/2} \; dt$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{-1/2} dt$$

$$= \frac{1}{\sqrt{\pi}} \Gamma(1/2)$$

$$= \frac{\sqrt{\pi}}{\sqrt{\pi}}$$

$$= 1$$

3) Property 3 : erf (x) + $\text{erf}_c(x) = 1$ **[Dec.-2000]**

Proof :

$$\text{L.H.S.} = \text{erf}(x) + \text{erf}_c(x)$$

$$= \frac{2}{\sqrt{\pi}} \left[\int_0^{x} e^{-u^2} du + \int_x^{\infty} e^{-u^2} du \right]$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} du$$

$$= \text{erf}(\infty) = 1$$

4) Property 4 : erf (x) is an odd function of x.

i.e $\text{erf}(-x) = -\text{erf}(x)$ **[May-2000, Dec.-2000, May-2006]**

Proof :

$$\text{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-u^2} du$$

Put $u = -v, \; du = -dv$

u	0	– x
v	0	x

$$= \frac{2}{\sqrt{\pi}} \int_0^x e^{-v^2}(-\,dv)$$

$$= \frac{-2}{\sqrt{\pi}} \int_0^x e^{-v^2} dv$$

$$= -\operatorname{erf}(x)$$

Thus erf (x) is an odd function of x.

7.5 Illustration on Error Function

Example 7.1 : *Express erf (x) in power of x*

Solution : We know that

$$e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} \cdots$$

$$\therefore \quad e^{-u^2} = 1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \frac{u^8}{4!} \cdots$$

Now

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$

$$= \frac{2}{\sqrt{\pi}} \int_0^x \left[1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \frac{u^8}{4!} \cdots\right] du$$

$$= \frac{2}{\sqrt{\pi}} \left[u - \frac{u^3}{3} + \frac{u^5}{5 \cdot 2!} - \frac{u^7}{7 \cdot 3!} + \cdots\right]_0^x$$

$$= \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} \cdots\right]$$

Note :

1) The above series is uniformly convergent.

2) erf (x) is a continuous function of x.

3) The values of erf (x) can be calculated using above series.

Example 7.2 : *Show that* $\int_a^b e^{-x^2} dx = \frac{\sqrt{\pi}}{2} [erf(b) - erf(a)]$

[Dec.-1997, May-1991, Dec.-2001]

Solution : We know that erf (∞) = 1

$$\therefore \quad 1 = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx$$

$$\therefore \quad 1 = \frac{2}{\sqrt{\pi}} \left[\int_0^\infty e^{-x^2} dx \right]$$

Splitting the above integral in three parts we get

$$1 = \frac{2}{\sqrt{\pi}} \left[\int_0^a e^{-x^2} dx + \int_a^b e^{-x^2} dx + \int_b^\infty e^{-x^2} dx \right]$$

Using the definitions, we get,

$$1 = \text{erf}(a) + \frac{2}{\sqrt{\pi}} \int_a^b e^{-x^2} dx + \text{erf}_c(b)$$

$$[1 - \text{erf}_c(b)] = \text{erf}(a) + \frac{2}{\sqrt{\pi}} \int_a^b e^{-x^2} dx$$

$$\text{erf}(b) = \text{erf}(a) + \frac{2}{\sqrt{\pi}} \int_a^b e^{-x^2} dx$$

$$\therefore \quad \text{erf}(b) - \text{erf}(a) = \frac{2}{\sqrt{\pi}} \int_a^b e^{-x^2} dx$$

$$\therefore \quad \int_a^b e^{-x^2} dx = \frac{\sqrt{\pi}}{2} [\text{erf}(b) - \text{erf}(a)]$$

Example 7.3 : *Show that* $\int_0^\infty e^{-x^2 - 2ax} dx = \frac{\sqrt{\pi}}{2} e^{a^2} [1 - erf(a)]$

[Dec.-1990, May-1992, May-1994, Dec.-1995, May-1998, May-2001, Dec.-2002, May-2004, May-2005]

Solution :

$$\text{L.H.S.} = \int_0^\infty e^{-x^2 - 2ax - a^2 + a^2} dx$$

$$= e^{a^2} \int_0^\infty e^{-(x+a)^2} dx$$

$$\therefore \qquad \text{R.H.S.} = \frac{-1}{a}\frac{d}{dx}\,\text{erf}\,(ax)$$

$$= \frac{-1}{a}\cdot\left[\frac{2a}{\sqrt{\pi}}\,e^{-a^2x^2}\right]$$

$$-\frac{1}{a}\frac{d}{dx}\,\text{erf}\,(ax) = \frac{-2}{\sqrt{\pi}}\,e^{-a^2x^2} \qquad \text{... (2)}$$

From equation (1) and (2) L.H.S. = R.H.S.

Example 7.7 : *Find* $\frac{d}{dx}\,erf\left(ax^n\right)$ **[May-2004]**

Solution : $\frac{d}{dx}\,\text{erf}\left(ax^n\right)$

$$= \frac{2}{\sqrt{\pi}}\frac{d}{dx}\int_0^{ax^n} e^{-u^2}du$$

By rule of DUIS,

$$= \frac{2}{\sqrt{\pi}}\left\{\int_0^{ax^n}\frac{\partial}{\partial x}\,e^{-u^2}du + \left(\frac{d}{dx}\,ax^n\right)e^{-a^2x^{2n}} - \left(\frac{d}{dx}\,0\right)e^{-0}\right\}$$

$$= \frac{2}{\sqrt{\pi}}\left\{0 + n\,a\,x^{n-1}\,e^{-a^2x^{2n}} - 0\right\}$$

$$= \frac{2an\;x^{n-1}}{\sqrt{\pi}}\,e^{-a^2x^{2n}}$$

Example 7.8 : *Show that* $\frac{d}{dx}\,erf\,(ax) = \frac{2a}{\sqrt{\pi}}\,e^{-a^2x^2}$ *and evaluate* $\int_0^t erf(ax)\;dx$ **[May-2002]**

Solution : $\qquad \text{erf}\,(ax) = \frac{2}{\sqrt{\pi}}\frac{d}{dx}\int_0^{ax} e^{-u^2}du$

$$\frac{d}{dx}\,\text{erf}\,(ax) = \frac{2}{\sqrt{\pi}}\frac{d}{dx}\int_0^{ax} e^{-u^2}du, \text{ (using DUIS)}$$

$$= \frac{2}{\sqrt{\pi}}\left\{\int_0^{ax}\frac{\partial}{\partial x}\,e^{-u^2}du + \left(\frac{d}{dx}\,ax\right)e^{-a^2x^2} - \left(\frac{d}{dx}\,0\right)e^{-0}\right\}$$

$$= \frac{2}{\sqrt{\pi}}\left\{0 + a\cdot e^{-a^2x^2} - 0\right\}$$

$$= \frac{2a}{\sqrt{\pi}}\,e^{-a^2x^2} \qquad \text{... (1)}$$

May - 2004

1. *Show that* $\int_0^{\infty} e^{-x^2-2bx}dx = \frac{\sqrt{\pi}}{2}e^{b^2}[1-erf(b)]$. **[4 Marks]**

2. *Find* $\frac{d}{dx}erf(ax^n)$. **[4 Marks]**

Dec. - 2004

1. *Prove that* $erf_c(-x)+erf_c(x)=2$ **[4 Marks]**

May - 2005

1. *Prove that* $\int_0^{\infty} e^{-x^2-2bx}dx = \frac{\sqrt{\pi}}{2}e^{b^2}[1-erf(b)]$. **[4 Marks]**

2. *Prove that* $\int_0^{\infty} e^{-st}erf(\sqrt{t})\,dt = \frac{1}{s\sqrt{s+1}}$. **[4 Marks]**

Dec. - 2005

1. *Plot the graphs of* $erf(x)$ *and* $erf_c(x)$. **[4 Marks]**

May - 2006

1. *Show that* $\frac{d}{dt}(erf(\sqrt{t})) = \frac{e^{-t}}{\sqrt{\pi t}}$, *and hence evaluate*

   $\int_0^{\infty} e^{-t}\,erf(\sqrt{t})\,dt$ **[4 Marks]**

2. *Prove that erf(x) is an odd function.*
   Deduce $erf_c(-x)-erf(x)=1$. **[4 Marks]**

Dec. - 2006

1. *If* $\alpha(x)=\sqrt{\frac{2}{\pi}}\int_0^{x} e^{-t^2/2}\,dt$, *show that*

   $erf(x)=\alpha(x\sqrt{2})$ **[4 Marks]**

2. *Prove that* $\frac{d}{dx}(erf(ax)) = \frac{2a\,e^{-a^2x^2}}{\sqrt{\pi}}$

   Hence show that $\frac{d}{dx}erf(ax^n) = \frac{2an}{\sqrt{\pi}}x^{n-1}e^{-a^2x^{2n}}$. **[4 Marks]**

May - 2007

1. *Show that* $\frac{d}{dx}erf(ax) = \frac{2a}{\sqrt{\pi}}e^{-a^2x^2}$ *and hence to find* $\int_0^{t} erf(ax)\,dx$. **[4 Marks]**

2. *Show that* $\int_a^{b} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}[erf(b)-erf(a)]$ **[4 Marks]**

8

Curve Tracing

Introduction

In this chapter we shall deal with tracing of curves which means finding approximate shape of the curves using different features viz. symmetry, intercepts, tangents, asymtotes, region of existance etc. Knowledge of tracing of curves is useful in applications of integrations in finding area, mass, centre of gravity, volume etc. Here we are going to trace some standard curves and other curves in (i) Cartesian (ii) Polar (iii) Parametric.

8.1 Concavity

i) Concavity upward (Convex downward)

The curve is said to be concave upward at A. If the portion of the curve on both sides of A lies above the tangent to the curve at A.

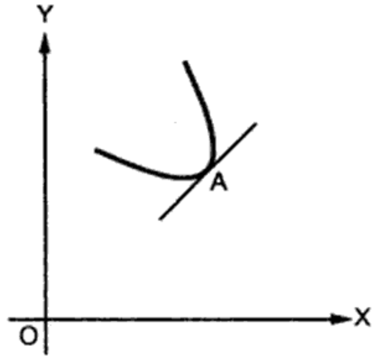


Fig. 8.1

ii) Concavity downward (Convex upward)

If the portion of the curve on both sides of A lies below the tangent to the curve at A.

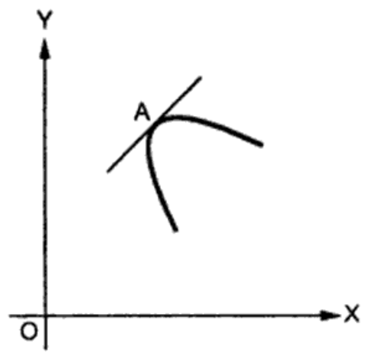


Fig. 8.2

8.2 Singular Points

Following points are called singular points.

i) Double point

A point through which two branches of curve passes.

ii) Multiple point

A point through which more than one branch passes.

iii) Node

A double point is called as node if distinct branches have distinct tangents.

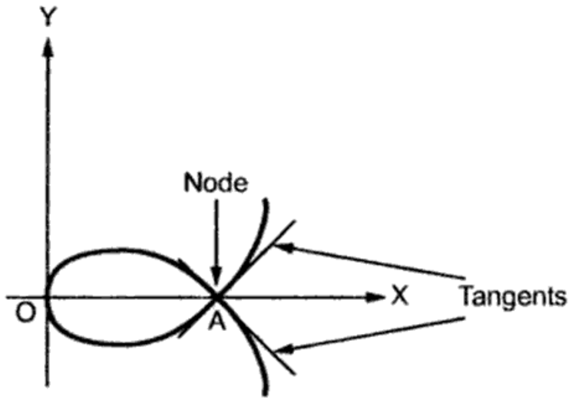


Fig. 8.3

iv) Cusp

A double point is called a cusp if two branches have a common tangent.

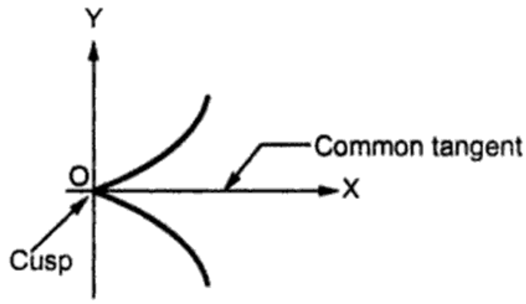


Fig. 8.4

v) Point of inflexion

A curve has inflexion at P if it changes from concavity upward to concavity downwards, or vice versa, as a point moving along the curve passes through P.

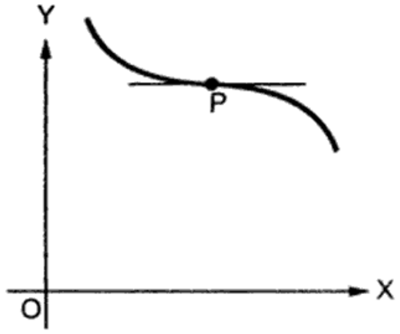


Fig. 8.5

vi) Isolated point

A point 'P' is called a isolated point or conjugate point if the co-ordinates of P satisfies the equation of curve, but no branches pass through P.

The curve y = f (x).

a	Is increasing in [a, b] if $f'(x) > 0$ for all $x \in [a, b]$	Concave upward in [a, b] if $f''(x) > 0$ for all $x \in [a, b]$.
b	Is decreasing in [a, b] if $f'(x) < 0$ for $x \in [a, b]$	Is concave downward if $f''(x) < 0$ for all $x \in [a, b]$.
c	Has extreme point if $f'(x) = 0$ for some $x \in [a, b]$.	Has a point of inflexion if $f''(x) = 0$ for some $x \in [a, b]$.

8.3 Rules for Tracing of Cartesian Curves

Rule 1 : Symmetry

a) About X-axis

On changing y by – y, if the equation remains unchanged i.e. all the terms y are even degree, then the curve is symmetric about X-axis.

e.g. i) $y^2 = 4ax$ ii) $y^2 = x^3$

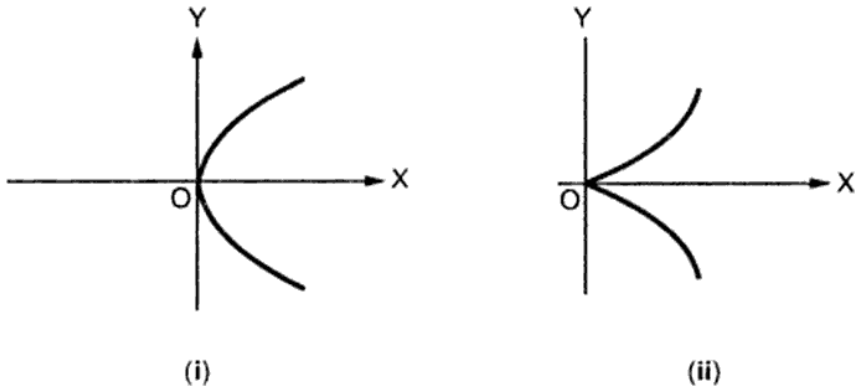


Fig. 8.6

b) About Y-axis

On changing x by $-$ x, if the equation remains unchanged i.e. all the terms of x are even degree then the curve is symmetric about Y-axis.

e.g. $x^2 = 4ay$.

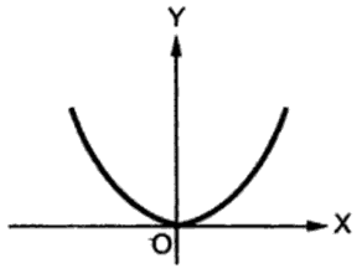


Fig. 8.7

c) About X and Y-axis

If all the terms x and y are both even degree, then symmetry about both axis.

e.g. $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

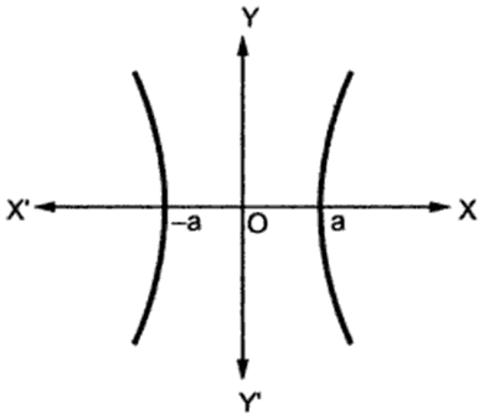


Fig. 8.8

d) Symmetry about origin (Opposite quadrants)

A curve is symmetrical in opposite quadrants, if the equation of the curve remains unchanged if x and y are replaced by $-$ x and $-$ y respectively.

e.g. i) $xy = C$

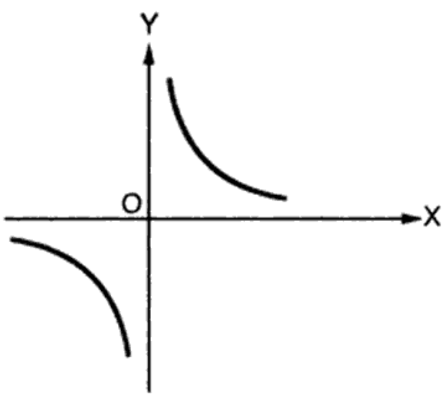


Fig. 8.9

Rule 2 : Points of Intersection

i) Intersection with X-axis : Put $y = 0$ to find intersections with X-axis.

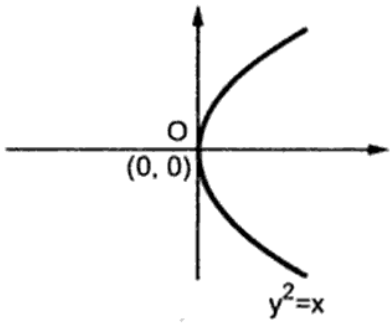
ii) Intersection with Y-axis : Put $x = 0$ to find intersections with Y-axis.

iii) Points on line of symmetry : If $y = x$ is the line of symmetry then put $y = x$ to find the points on line of symmetry.

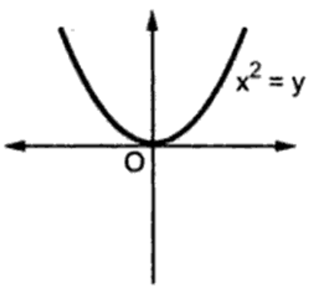
Rule 3 : Tangents :

i) Tangent at origin : If (0, 0) is the point on the curve then only we can find tangent at origin.

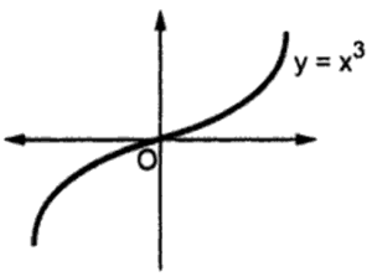
Newton's method : For the rational integral algebraic equation tangents at origin can be obtained by equating to zero, the lowest degree terms from the equation. Observe the following figures to understand tangents at origin.



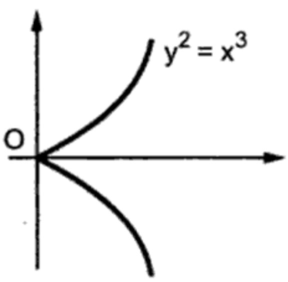
Tangent at (0, 0) is $x = 0$ i.e. Y-axis



Tangent at (0, 0) is $y = 0$ i.e. X-axis



Tangent at (0, 0) is $y = 0$ i.e. X-axis



Tangent at (0, 0) is

$y^2 = 0$

i.e. $y = 0$

i.e. X-axis

Substituting $m = -1$ we get,

$$c = -a$$

Thus for the curve $x^3 + y^3 = 3axy$, the oblique asymptote is $y = mx + c$.

i.e. $y = -x - a$

i.e. $x + y = -a$

Rule 5 : Region of absence of the curve

i) If it is possible to express the equation as $y^2 = f(x)$ i.e. the symmetry is about X-axis then we can find some values of x where y^2 becomes negative which is not possible, hence the curve does not exist in that region. For example : For the curve $y^2 = x$

If x is negative like $-1, -2, -3...$ then y^2 becomes negative which is not possible thus the parabola $y^2 = x$ does not exist in the region where x is negative i.e. the parabola $y^2 = x$ exists only for positive values of x i.e. For $y = f(x)$ if y becomes imaginary for some values of x then the curve does not exists in that part or for those values of x.

ii) If it is possible to express the equation as $x^2 = f(y)$ i.e. the symmetry is about Y-axis then we can find some values of y for which x^2 becomes negative which is not possible, hence the curve does not exist in that region.

For example : For the curve

$$x^2 = y^3 (a - y)$$

If y is negative then the term $(a - y)$ is positive but y^3 is negative thus x^2 negative which is not possible.

Also if $y > a$ like $y = 2a, 3a, 4a, ...$ then the term $a - y$ is negative and y^3 is positive thus x^2 is negative which is not possible.

Hence if $y < 0$ and $y > a$ then x^2 becomes negative which is not possible thus the curve will not exist for the values of $y < 0$ and $y > 0$ is the curve exists only for $0 < y < a$,

i.e. for $x = f(y)$ if x becomes imaginary for some values of y then the curve does not exists in that part or for those values of y.

Note : i) If there are two points on the line of symmetry and the curve exists in between then there is a loop.

ii) If the curve is symmetric about X-axis then check for asymptote parallel to Y-axis and vice versa.

3) Oblique asymptote : If the equation of the curve is

$x^3 + y^3 = 3a\,xy$, then let $y = mx+c$ be oblique asymptote.

$x^3 + (mx+c)^3 = 3a\,x\,(mx + c)$

$x^3 + m^3 x^3 + 3cm^2x^2 + 3c^2mx + c^3 = 3amx^2 + 3acx$

Equating coefficients, we get,

Coefficient of $x^3 \Rightarrow 1 + m^3 = 0 \Rightarrow m = -1$.

Coefficient of $x^2 \Rightarrow 3\,cm^2 = 3am \Rightarrow c = -a$.

Thus, $y = = -x - a$, is the oblique asymptote.

E) Region of absence of the curve

1) For $y = f(x)$, if y becomes imaginary for some value of $x > a$ (say), then curve does not exists beyond $x = a$.

2) For $x = f(y)$, if x becomes imaginary for some value of $y > a$ (say), then curve does not exists beyond $y = a$.

Type 1

8.4 Illustrations on Type 1: Curves Given by Cartesian Co-ordinates

Trace the following curves.

Example 8.1 : *Trace the curve $x = (y-1)(y-2)(y-3)$.*

Solution :

Symmetry	No symmetry
Points of intersection : Put x = 0, y = 0 to find intersections with Y & X-axis respectively.	(0,1) (2,0) (0,3) are the three points on Y-axis and (– 6,0) is the point on X-axis.
Tangents at origin : Tangent at origin is obtained by equating to zero the lowest degree term from the equation.	(0,0) is not the point on the curve.
Tangents at any other point : To find the nature of the tangent at any point, find $\frac{dy}{dx}$ at that point.	Here, $\frac{dy}{dx} = (y-2)(y-3) + (y-3)(y-3) + (y-1)$ $(y-2)$ Substituting the points, we get $\left(\frac{dy}{dx}\right)_{(-6,0)} = +\text{ve}$ $\left(\frac{dy}{dx}\right)_{(0,1)} = +\text{ve}$ $\left(\frac{dy}{dx}\right)_{(0,2)} = -\text{ve}$ $\left(\frac{dy}{dx}\right)_{(0,3)} = +\text{ve}$ Thus tangent makes acute angle with X-axis at (– 6, 0), (0, 1), (0,3) and obtuse angle at (0, 2).
Asymptotes : Asymptote parallel to Y(or X) axis is obtained by equation to zero the coefficient of the highest degree term in y (or x).	As the coefficients of the highest degree term in y (or x) are constants thus no asymptotes parallel to axes.
Region : Find the values of x where y becomes imaginary, then curve does not exists in that region.	As the given equation involves odd powers of x and y thus the curve exists for all values of x and y.

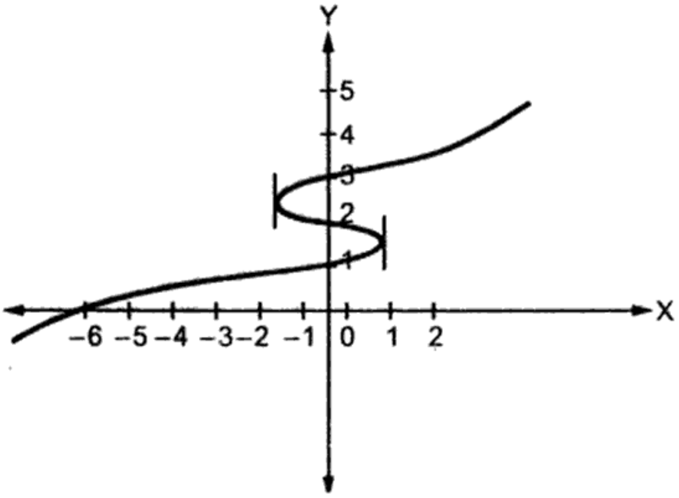


Fig. 8.13

Example 8.2 : *Trace the curve* $y(x^2 + 4a^2) = 8a^3$.

Solution : The given curve can be written as $y = \dfrac{8a^3}{(x^2 + 4a^2)}$

Symmetry :	Even powers of x $\Rightarrow$ symmetry about Y-axis.
Points of intersection : Put x = 0, y = 0 to find intersections with Y & X-axis respectively.	Here (0, 2a) is the point on the curve.
Tangents at origin : Tangent at origin is obtained by equating to zero the lowest degree term from the equation.	(0, 0) is not the point on the curve.
Tangents at any other point : To find the nature of the tangent at any point, find $\frac{dy}{dx}$ at that point.	$\frac{dy}{dx} = \frac{-16a^3 x}{(x^2 + 4a^2)^2}$. $\left(\frac{dy}{dx}\right)_{(0,\,2a)} = 0$ Thus the tangent at (0, 2a) is parallel to X-axis.
Asymptotes : Asymptote parallel to Y (or X)-axis is obtained by equating to zero the coefficient of the highest degree term in y (or x).	Here the highest degree term in y is y And the coefficient of y is $(4a^2 + x^2)$. Thus $(4a^2 + x^2) = 0$ which is not possible. Thus no asymptote parallel to Y-axis. And the highest degree term in x is x^2 & the coefficient of x^2 is y. Thus y = 0 is the asymptote parallel to Xaxis.
Region : Find the values of x where y becomes imaginary, then curve does not exists in that region.	The given curve can be written as $(x^2 + 4a^2) = \frac{8a^3}{y}$ $(x^2) = \frac{8a^3}{y} - 4a^2$ At y= − 2a, x^2 = − ve At y = 3a, x^2 = − ve For y < 0 and y > 2a, x becomes imaginary therefore the curve does not exist for y < 0 and y > 2a. Thus the curve exists for 0 < y < 2a & the curve exists for all the values of x.

Example 8.4 : *Trace the curve $y^2(2a-x)=x^3$.*

Solution : The given curve can be written as $y^2 = \dfrac{x^3}{2a - x}$

Symmetry :	Even powers of y $\Rightarrow$ symmetry about X-axis.
Points of intersection: Put x = 0, y = 0 to find intersections with Y & X axis respectively.	Here (0, 0) is the point on the curve.
Tangents at origin: Tangent at origin is obtained by equating to zero the lowest degree term from the equation.	From the equation $y^2(2a - x) = x^3$, Lowest degree term = $2ay^2$ Thus $2ay^2 = 0$ i.e. $y^2 = 0$ i.e. X-axis is the tangent at origin.
Tangents at any other point: To find the nature of the tangent at any point, find $\frac{dy}{dx}$ at that point.	No other points.
Asymptotes : Asymptote parallel to Y(or X) axis is obtained by equating to zero the coefficient of the highest degree term in y (or x).	Here the highest degree term in y is y^2. As the coefficient of y^2 is (2a – x). Thus (2a – x) = 0 i.e. x = 2a is the asymptote parallel to Y-axis.
Region : Find the values of x where y becomes imaginary, then curve does not exists in that region.	The given curve can be written as $y^2 = \frac{x^3}{2a - x}$. At x = – a, y^2 = – ve At x = 3a, y^2 = – ve For x < 0 and x > a, y becomes imaginary therefore the curve does not exist for x < 0 and x > a. Thus the curve exists for 0 < x < 2a.

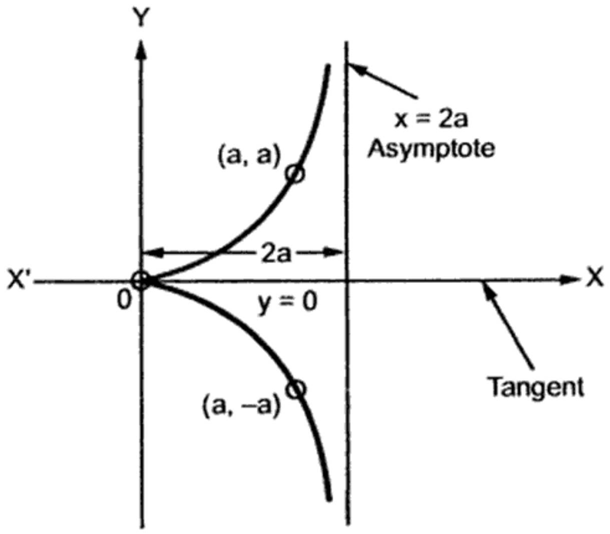


Fig. 8.16

Example 8.5 : *Trace the curve 'Strophoid $x\left(x^2+y^2\right)=a\left(x^2-y^2\right) a>0$.*

Solution : The given curve can be written as $y^2 = \dfrac{x^2(a-x)}{(a+x)}$

Symmetry :	Even powers of y $\Rightarrow$ symmetry about X-axis.
Points of intersection: Put x = 0, y = 0 to find intersections with Y & X axis respectively.	Here (0, 0) and (a, 0) are the points on the curve.
Tangents at origin: Tangent at origin is obtained by equating to zero the lowest degree term from the equation.	From the equation $x\left(x^2+y^2\right)=a\left(x^2-y^2\right)$, Lowest degree term = $a\left(x^2-y^2\right)$ Thus $a\left(x^2-y^2\right)=0$ i.e. $x^2=y^2$ i.e. $x = \pm y$ are the tangents at origin.
Tangents at any other point: To find the nature of the tangent at any point, find $\dfrac{dy}{dx}$ at that point.	Here $\left(\dfrac{dy}{dx}\right)_{(a,0)} = \infty$ Thus tangent at (a, 0) is parallel to Y-axis.
Asymptotes : Asymptote parallel to Y (or X) axis is obtained by equating to zero the coefficient of the highest degree term in y (or x).	Here the highest degree term in y is y^2. And the coefficient of y^2 is (a + x). Thus (a + x) = 0 i.e. x = – a is the asymptote parallel to Y-axis.
Region : Find the values of x where y becomes imaginary, then curve does not exists in that region.	The given curve can be written as $y^2 = \dfrac{x^2(a-x)}{(a+x)}$ At x = – 2a, y^2 = – ve At x = 3a, y^2 = – ve For x < – a and x > a, y becomes imaginary therefore the curve does not exist for x < – a and x > a. Thus the curve exists for – a < x < a

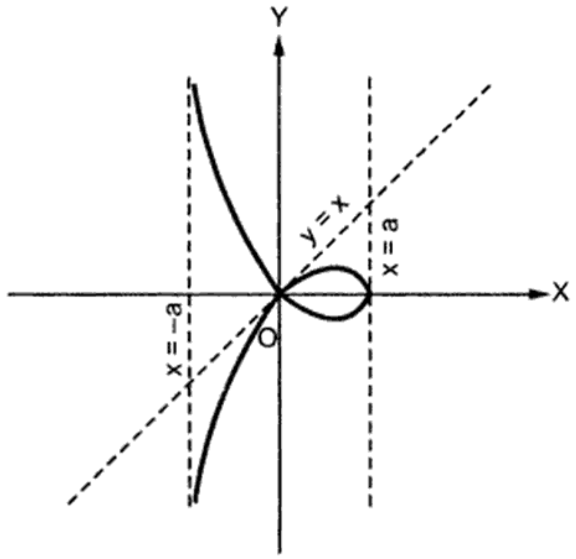


Fig. 8.17

Example 8.6 : *Trace the curve $y^2(x-a) = x^2(2a-x)$.* [Dec.-2006]

Solution : The given curve can be written as $y^2 = \frac{x^2(2a-x)}{(x-a)}$.

Symmetry :	Even powers of y $\Rightarrow$ symmetry about X-axis.
Points of intersection: Put x = 0, y = 0 to find intersections with Y & X-axis respectively.	Here (0, 0) and (2a, 0) are the points on the curve.
Tangents at origin: Tangent at origin is obtained by equating to zero the lowest degree term from the equation.	From the equation $y^2(x-a) = x^2(2a-x)$, Lowest degree term = $a(2x^2+y^2)$ Thus $a(2x^2+y^2) = 0$ i.e. $2x^2 = -y^2$ i.e. No tangent at origin.
Tangents at any other point: To find the nature of the tangent at any point, find $\frac{dy}{dx}$ at that point.	Here $\left(\frac{dy}{dx}\right)_{(2a,0)} = \infty$ Thus the tangent at (2a, 0) is parallel to Y-axis.
Asymptotes : Asymptote parallel to Y(or X) axis is obtained by equating to zero the coefficient of the highest degree term in y (or x).	Here the highest degree term in y is y^2. And the coefficient of y^2 is (x − a). Thus (x–a) = 0 i.e. x = a is the asymptote parallel to Y-axis.
Region : Find the values of x where y becomes imaginary, then curve does not exists in that region.	The given curve can be written as $y^2 = \frac{x^2(2a-x)}{(x-a)}$ At x = − a, y^2 = − ve At x = a/2, y^2 = − ve At x = 3a, y^2 = − ve At x = 3a/2, y^2 = +ve Thus the curve exists for 0 < x < 2a

Note that x = 0 is the point on the curve but no branches passes through (0, 0), such a point is called as isolated point.

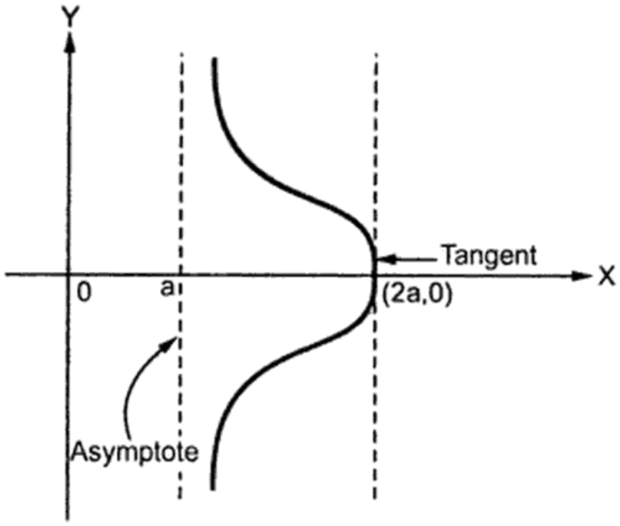


Fig. 8.18

Example 8.7 : *Trace the curve* $xy^2 = a^2(a-x)$. **[May-2001, Dec.-2005]**

Solution : The given curve can be written as $y^2 = \dfrac{a^2(a-x)}{x}$.

Symmetry :	Even powers of y ⇒ symmetry about X-axis.
Points of intersection: Put x = 0, y = 0 to find intersections with Y & X-axis respectively.	Here (a, 0) is the point on the curve.
Tangents at origin: Tangent at origin is obtained by equating to zero the lowest degree term from the equation.	Origin is not the point on the curve.
Tangents at any other point: To find the nature of the tangent at any point, find $\dfrac{dy}{dx}$ at that point.	Here $\left(\dfrac{dy}{dx}\right)_{(a,0)} = \infty$ Thus the tangent at (a, 0) is parallel to Y-axis.
Asymptotes : Asymptote parallel to Y(or X) axis is obtained by equating to zero the coefficient of the highest degree term in y (or x).	Here the highest degree term in y is y^2. And the coefficient of y^2 is (x). Thus (x) = 0 i.e. Y-axis is the asymptote parallel to Y-axis.
Region : Find the values of x where y becomes imaginary, then curve does not exists in that region.	The given curve can be written as $y^2 = \dfrac{a^2(a-x)}{x}$. At x = – a, y^2 = – ve At x = 2a, y^2 = – ve For x < 0 and x > a, y becomes imaginary therefore the curve does not exist for x < 0 and x > a. Thus the curve exists for 0 < x < a

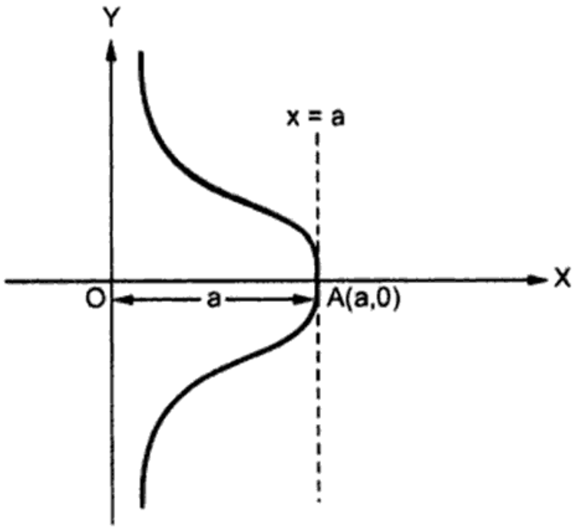


Fig. 8.19

Example 8.8 : *Trace the curve* $x^2y^2 = a^2\left(y^2 - x^2\right)$. [Dec.-2004]

Solution : The given curve can be written as $y^2 = \frac{a^2 x^2}{\left(a^2 - x^2\right)}$.

Symmetry :	Even powers of x, y $\Rightarrow$ symmetry about X and Y-axis.
Points of intersection: Put x = 0, y = 0 to find intersections with Y & X-axis respectively.	Here (0, 0) is the point on the curve.
Tangents at origin: Tangent at origin is obtained by equating to zero the lowest degree term from the equation.	From the equation $x^2y^2 = a^2\left(y^2 - x^2\right)$, Lowest degree term = $a^2\left(x^2 - y^2\right)$ Thus $a^2\left(x^2 - y^2\right) = 0$ i.e. $x^2 = y^2$ i.e. $x = \pm y$ are the tangents at origin.
Tangents at any other point: To find the nature of the tangent at any point, find $\frac{dy}{dx}$ at that point.	No other points.
Asymptotes : Asymptote parallel to Y (or X) axis is obtained by equating to zero the coefficient of the highest degree term in y (or x).	Here the highest degree term in y is y^2. And the coefficient of y^2 is $\left(a^2 - x^2\right)$. Thus $\left(a^2 - x^2\right) = 0$ i.e. $x = \pm a$ are the asymptote parallel to Y-axis.
Region : Find the values of x where y becomes imaginary, then curve does not exists in that region.	The given curve can be written as $y^2 = \frac{a^2 x^2}{\left(a^2 - x^2\right)}$. At $x = -2a$, $y^2 = -$ ve At $x = 3a$, $y^2 = -$ ve For $x < -a$ and $x > a$, y becomes imaginary therefore the curve does not exist for $x < -a$ and $x > a$. Thus the curve exists for $-a < x < a$.

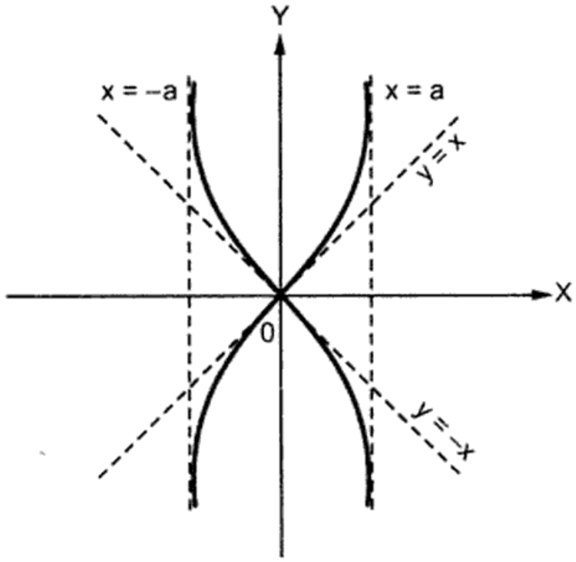


Fig. 8.20

Example 8.9 : *Trace the curve $y^2(a^2 + x^2) = a^2 x^2$.*

Solution : The given curve can be written as $y^2 = \dfrac{a^2 x^2}{(a^2 + x^2)}$.

Symmetry :	Even powers of x, y ⇒ symmetry about X and Y-axis.
Points of intersection: Put x = 0, y = 0 to find intersections with Y & X-axis respectively.	Here (0, 0) is the point on the curve.
Tangents at origin: Tangent at origin is obtained by equating to zero the lowest degree term from the equation.	From the equation $y^2(a^2 + x^2) = a^2 x^2$ $x^2 y^2 = - a^2 (y^2 - x^2)$, Lowest degree term = $- a^2 (x^2 - y^2)$ Thus $a^2(x^2 - y^2) = 0$ i.e. $x^2 = y^2$ i.e. $x = \pm y$ are the tangents at origin.
Tangents at any other point: To find the nature of the tangent at any point, find $\frac{dy}{dx}$ at that point.	No other points.
Asymptotes : Asymptote parallel to Y (or X) axis is obtained by equating to zero the coefficient of the highest degree term in y (or x).	Here the highest degree term in y is y^2. And the coefficient of y^2 is $(a^2 + x^2)$. Thus $(a^2 + x^2) = 0$ which is not possible. The given equation can be written as $x^2 (a^2 - y^2) = a^2 y^2$ Here the highest degree term in x is x^2. And the coefficient of x^2 is $(a^2 - y^2)$ Thus $(a^2 - y^2) = 0$ i.e. $y = \pm a$ are the asymptote parallel to X-axis.
Region : Find the values of x where y becomes imaginary, then curve does not exists in that region.	The given curve can be written as $x^2 = \dfrac{a^2 y^2}{(a^2 - y^2)}$ At y = – 2a, x^2 = – ve At y = 3a, x^2 = – ve For y < – a and y > a, x becomes imaginary therefore the curve does not exist for y < – a and y > a. Thus the curve exists for – a < y < a.

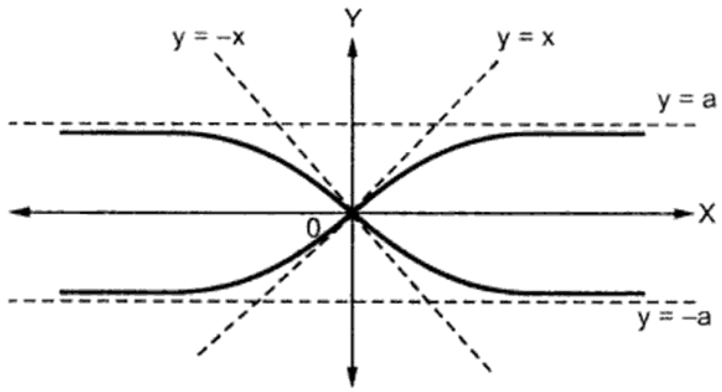


Fig. 8.21

Example 8.10 : *Trace the curve $xy^2 = a(x^2 - a^2)$.*

Solution : The given curve can be written as $y^2 = \frac{a(x^2 - a^2)}{x}$.

Symmetry :	Even powers of y $\Rightarrow$ symmetry about X-axis.
Points of intersection: Put x = 0, y = 0 to find intersections with Y & X-axis respectively.	Here (a, 0), (– a, 0) are the points on the curve.
Tangents at origin: Tangent at origin is obtained by equating to zero the lowest degree term from the equation.	Origin is not the point on the curve.
Tangents at any other point: To find the nature of the tangent at any point, find $\frac{dy}{dx}$ at that point.	Here $\left(\frac{dy}{dx}\right)_{(a,0)} = \infty$ $\left(\frac{dy}{dx}\right)_{(-a,0)} = \infty$ Thus the tangents at (a, 0) (–a, 0) are parallel to Y-axis.
Asymptotes : Asymptote parallel to Y(or X) axis is obtained by equating to zero the coefficient of the highest degree term in y (or x).	Here the highest degree term in y is y^2. And the coefficient of y^2 is (x). Thus (x) = 0 i.e. Y-axis is the asymptote parallel to Y-axis.
Region : Find the values of x where y becomes imaginary, then curve does not exists in that region.	The given curve can be written as $y^2 = \frac{a(x^2 - a^2)}{x}$ At x = – 2a, y^2 = – ve At x = – a/2, y^2 = +ve At x = +a/2, y^2 = – ve At x = 2a, y^2 = +ve For x < –a and 0 < x < a. y becomes imaginary therefore the curve does not exist for x < – a and 0 < x < a. Thus the curve exists for – a < x < 0 & x > a.

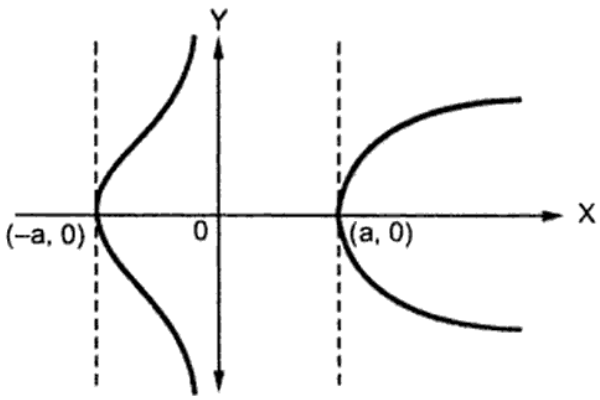


Fig. 8.22

Example 8.12 : *Trace the curve* $x^2\left(x^2-4a^2\right)=y^2\left(x^2-a^2\right)$.

Solution : The given curve can be written as $y^2=\dfrac{x^2\left(x^2-4a^2\right)}{\left(x^2-a^2\right)}$

Symmetry :	Even powers of x, y $\Rightarrow$ symmetry about X and Y-axis.
Points of intersection: Put x = 0, y = 0 to find intersections with Y & X-axis respectively.	Here (0, 0), (– 2a, 0) and (2a, 0) are the points on the curve.
Tangents at origin: Tangent at origin is obtaining by equating to zero the lowest degree term from the equation.	From the equation $x^2\left(x^2-4a^2\right)=y^2\left(x^2-a^2\right)$, Lowest degree term $=x^2\ 4a^2-y^2\ a^2$ Thus $a^2\left(4x^2-y^2\right)=0$ i.e. i.e. $y=\pm 2\ x$ are the tangents at origin.
Tangents at any other point: To find the nature of the tangent at any point, find $\frac{dy}{dx}$ at that point.	Here $\left(\frac{dy}{dx}\right)_{(2a,0)}=\infty$ $\left(\frac{dy}{dx}\right)_{(-2a,0)}=\infty$ Thus tangents at (2a, 0) (– 2a, 0) are parallel to Y-axis.
Asymptotes : Asymptote parallel to Y(or X) axis is obtained by equating to zero the coefficient of the highest degree term in y (or x).	Here the highest degree term in y is y^2. And the coefficient of y^2 is $\left(x^2-a^2\right)$. Thus $\left(x^2-a^2\right)=0$ i.e. $x=\pm a$ are the asymptote parallel to Y-axis. No asymptote parallel to X-axis.
Region : Find the values of x where y becomes imaginary, then curve does not exists in that region.	The given curve can be written as $y^2=\dfrac{x^2\left(x^2-4a^2\right)}{\left(x^2-a^2\right)}$ At x = +a/2, y^2 = +ve At x = 3a/2, y^2 = – ve At x = 3a, y^2 = +ve Thus the curve exists for –a < x < a & x < –2a & x > 2a.

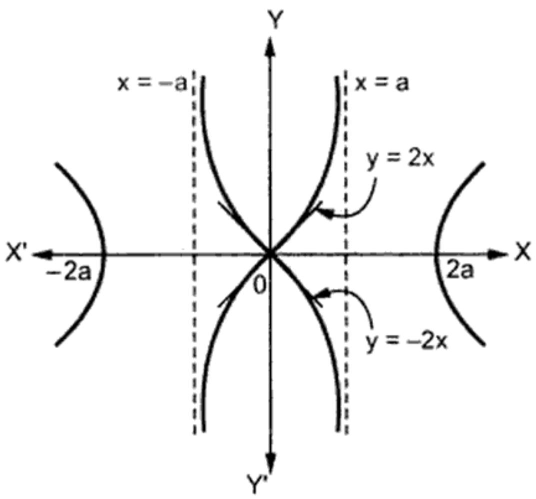


Fig. 8.24

Exercise 8.1

Trace the following curves.

1) $a^2 y^2 = x^2(2a - x)(x - a)$.

Ans. :

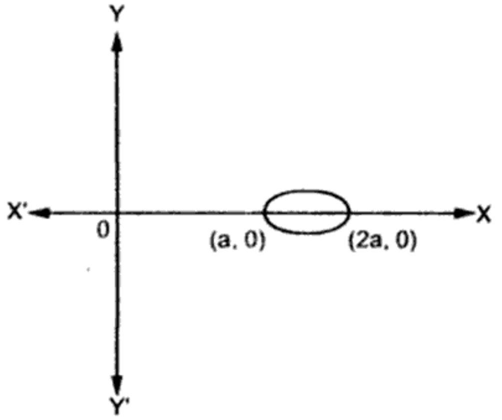


Fig. 8.26

2) $x^{1/2} + y^{1/2} = a^{1/2}$

Ans. :

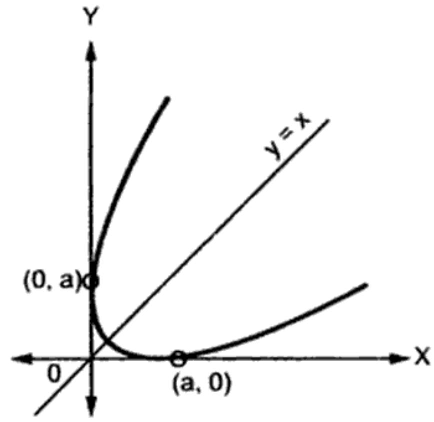


Fig. 8.27

3) $a^2 x^2 = y^3(2a - y)$

Ans. :

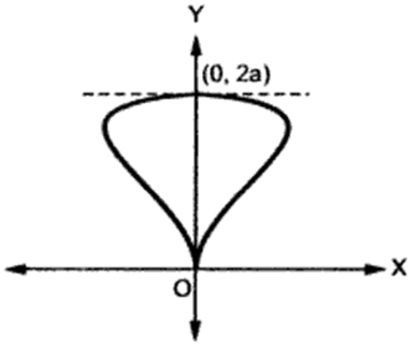


Fig. 8.28

4) $y = x^3$

Ans. :

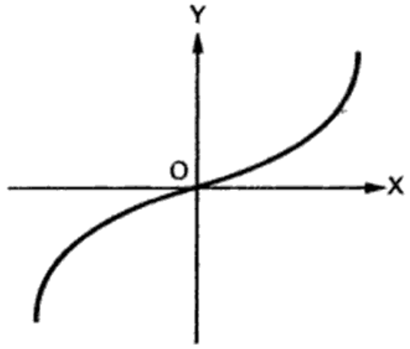


Fig. 8.29

5) $27ay^2 = 4(x - 2a)^3$

Ans. :

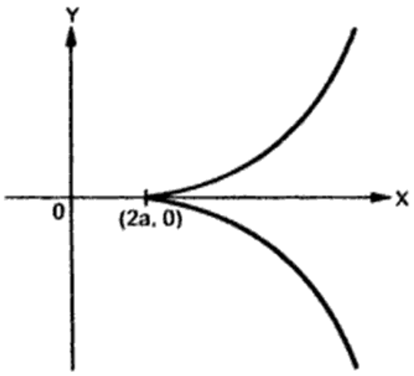


Fig. 8.30

6) $y = x(x^2 - 1)$

Ans. :

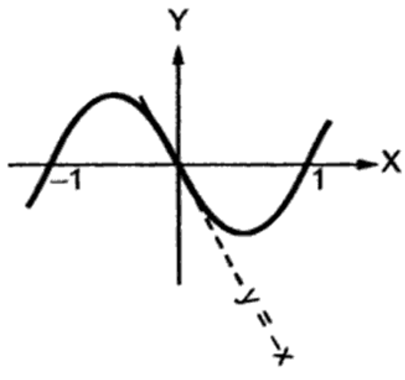


Fig. 8.31

7) $y^2(a - x) = x^3$

Ans. :

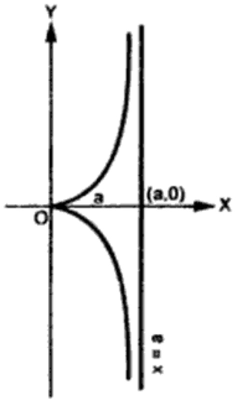


Fig. 8.32

8) $3ay^2 = x(x - a)^2$

Ans. :

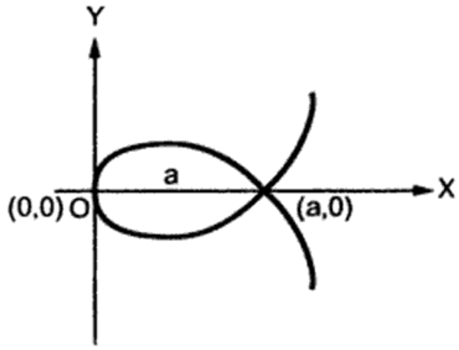


Fig. 8.33

9) $ay^2 = x(a^2 + x^2)$

Ans. :

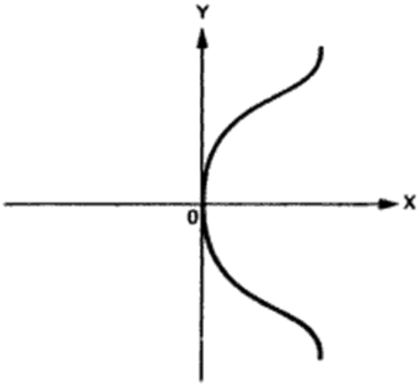


Fig. 8.34

10) $y^2(a + x) = (x - a)^3$

Ans. :

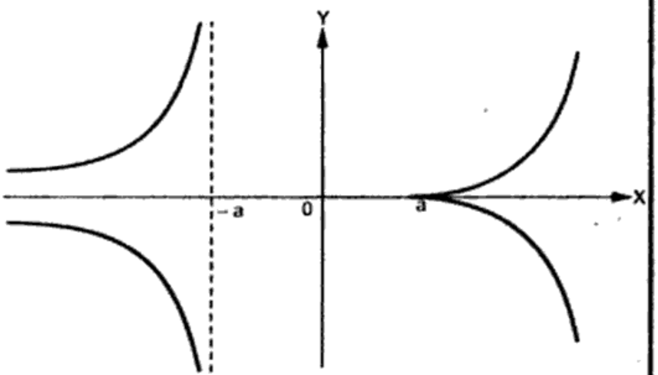


Fig. 8.35

11) $a^2 y^2 = x^2(a^2 - x^2)$

Ans. :

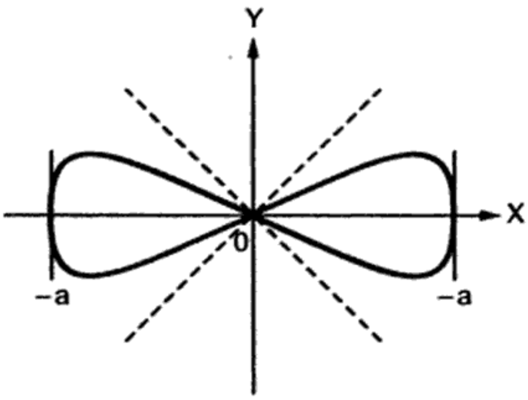


Fig. 8.36

12) $a^2y^2 = x^2(x + 2a)(x - a)$

Ans. :

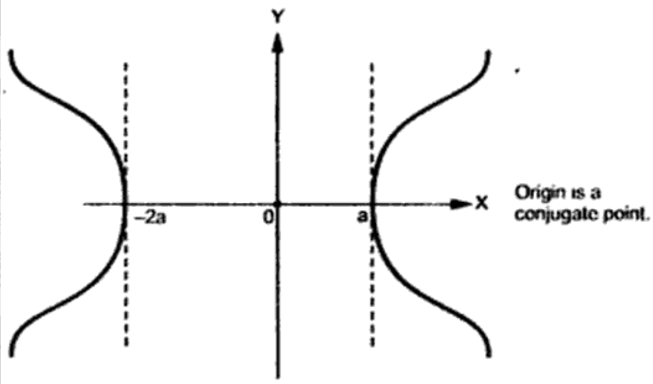


Fig. 8.37

13) $a^2y^2 = x^2(x^2 - a^2)$.

Ans. :

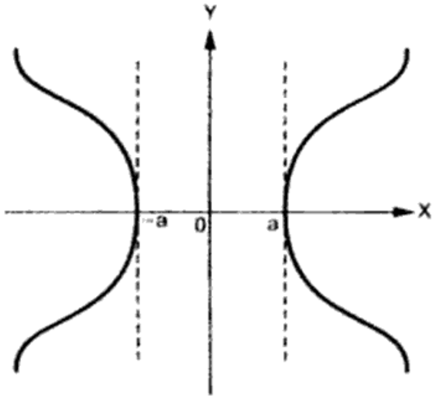


Fig. 8.38

14) $a^2\, y^2 = x^2(a-x)(x-b)$ *where* $b < a$.

Ans. :

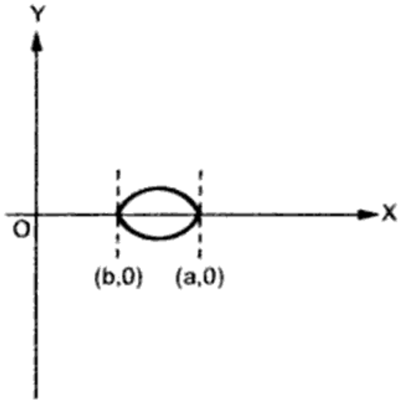


Fig. 8.39

15) $y^2 = (x-1)(x-2)(x-3)$.

(May-2003, Dec-2006)

Ans. :

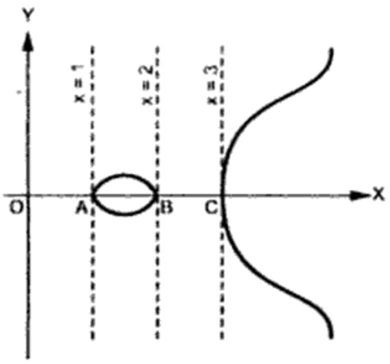


Fig. 8.40

16) $y^2(x^2 - 1) = x$.

Ans. :

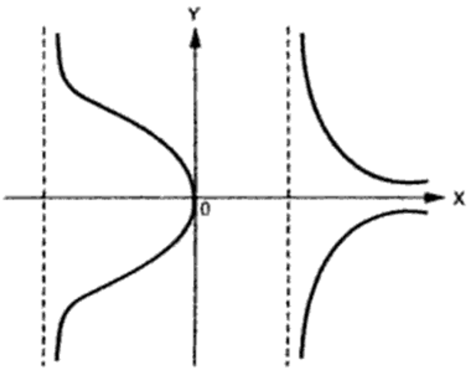


Fig. 8.41

17) $y(x^2 - 1) = x^2 + 1$.

Ans. :

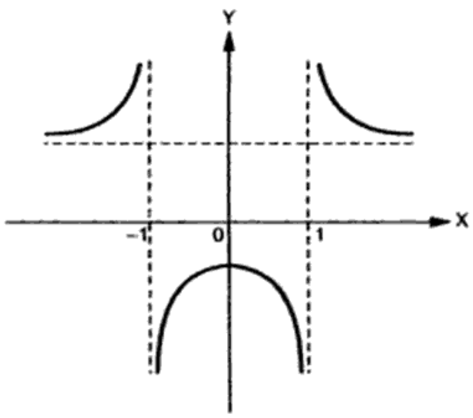


Fig. 8.42

18) $x^2y^2 = x^2 + 1$.

Ans. :

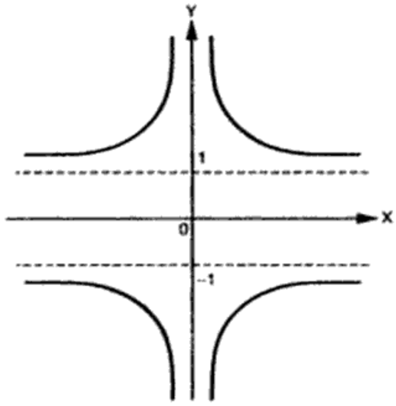


Fig. 8.43

19) $ay^2 = x(a^2 - x^2)$.

Ans. :

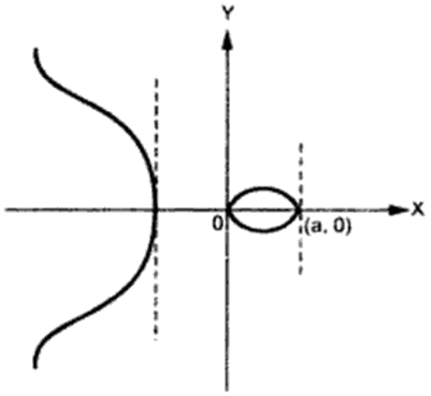


Fig. 8.44

20) $y^2 = x^5(2a - x)$.

Ans. :

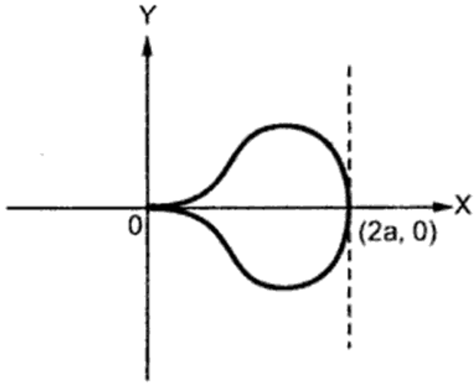


Fig. 8.45

21) $y^2(x^2 + y^2) + a^2(x^2 - y^2) = 0$.

Ans. :

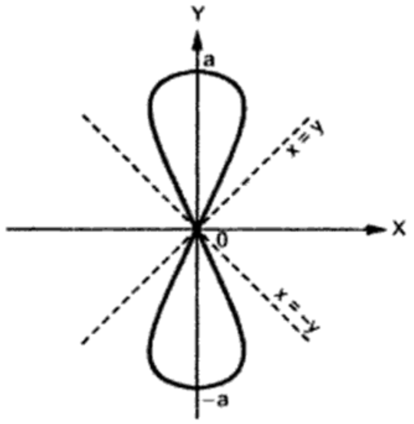


Fig. 8.46

22) $y^2(4 - x) = x(x - 2)^2$. **[May-2006]**

Ans. :

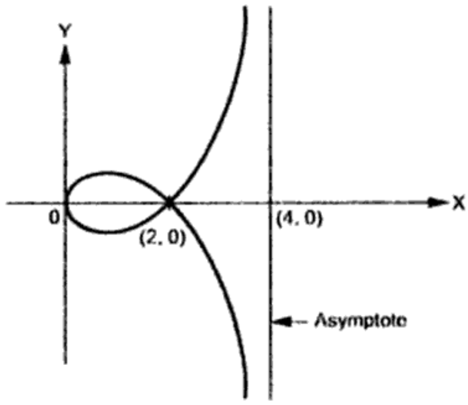


Fig. 8.47

23) $(x^2 - a^2)(y^2 - b^2) = a^2b^2$.

Ans. :

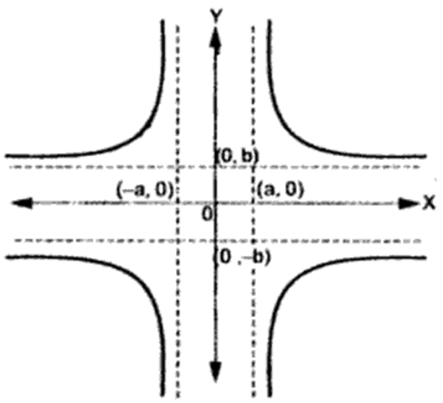


Fig. 8.48

24) $ay^2 = (x - a)(x - 5a)^2$.

Ans. :

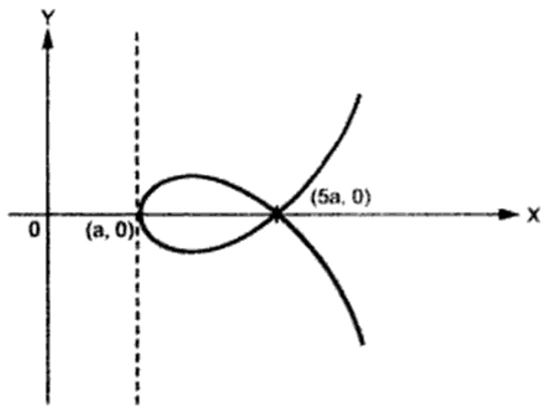


Fig. 8.49

Type 2 : Implicit Relations

8.5 Illustrations on Type 2: Curves Given Implicit Relations

Example 8.14 : *Trace the curve $x^3 + y^3 = 3axy$ (Foulium of descarts).*

[May-2003, Dec.-2006, May-2007]

Solution :

Symmetry :	If the equation remains same after the replacement of (x, y) by (y, x) then symmetry about the line y = x.
Points of intersection: Put x = 0, y = 0 to find intersections with Y & X-axis respectively.	Here (0, 0) and (3a/2, 3a/2) are the points on the curve.
Tangents at origin: Tangent at origin is obtained by equating to zero the lowest degree term from the equation.	From the equation $x^3 + y^3 = 3a\,xy,$ Lowest degree term = 3axy Thus i.e. x = 0, y = 0 i.e. Two tangents at origin.
Tangents at any other point: To find the nature of the tangent at any point, find $\frac{dy}{dx}$ at that point.	Here $\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = -1$ Thus tangent at (3a/2, 3a/2) makes angle $\frac{3\pi}{4}$ with X-axis.
Oblique Asymptotes:	The equation of the curve is $x^3 + y^3 = 3axy$, let y=mx+c be the oblique asymptote. $x^3 + (mx + c)^3 = 3ax\,(mx + c)$ $x^3 + m^3x^3 + 3cm^2x^2 + 3c^2\,mx + c^3 = 3amx^2 + 3acx$ Equating the coefficients, we get, Coeff. of $x^3 \Rightarrow 1 + m^3 = 0 \Rightarrow m = -1.$ Coeff. of $x^2 \Rightarrow 3\,cm^2 = 3\,am \Rightarrow c = -a.$ Thus, y = = – x – a, is the oblique asymptote.
Region : Find the values of x where y becomes imaginary, then curve does not exists in that region.	The given curve exists for all values of x, y.

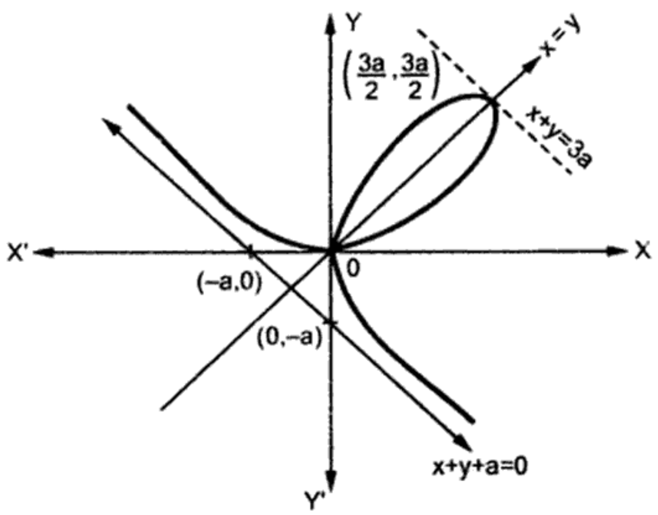


Fig. 8.50

Exercise 8.2

Trace the following curves.

1) *Trace the curve* $x^4 + y^4 = 4axy^2$.

Ans. :

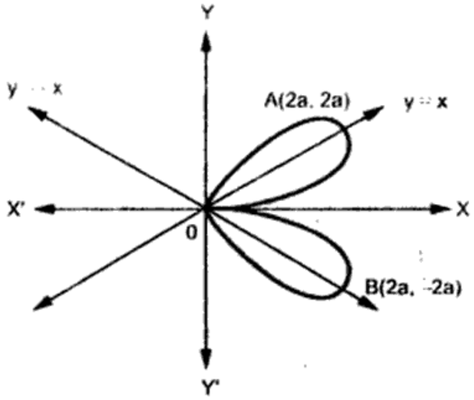


Fig. 8.52

2) *Trace the curve* $x^4 + y^4 = a^2(x^2 - y^2)$.

Ans. :

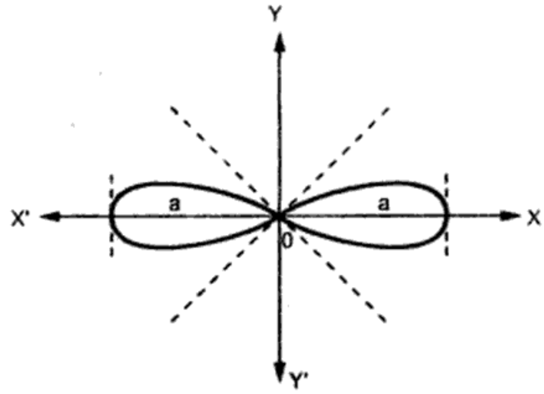


Fig. 8.53

3) *Trace the curve* $x^6 + y^6 = a^2 x^2 y^2$.

Ans. :

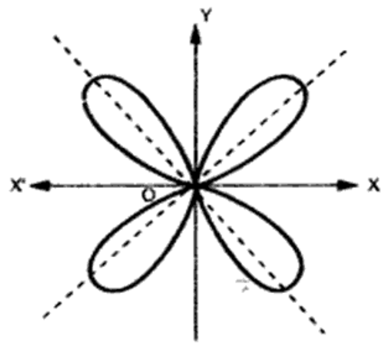


Fig. 8.54

4) *Trace the curve* $x^4 - y^4 + xy = 0$.

Ans. :

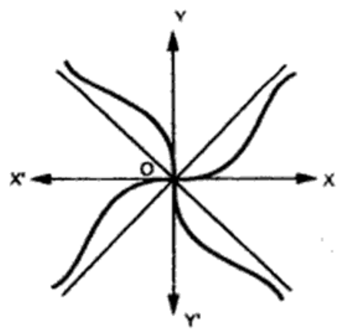


Fig. 8.55

5) *Trace the curve* $x^4 + y^4 = 2a^2 xy$.

Ans. :

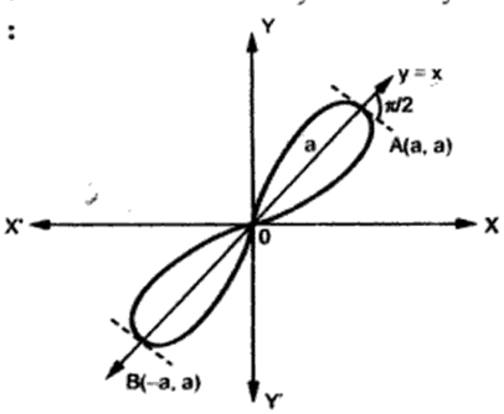


Fig. 8.56

6) *Trace the curve* $x^5 + y^5 = 5a^2 x^2 y$

Ans. :

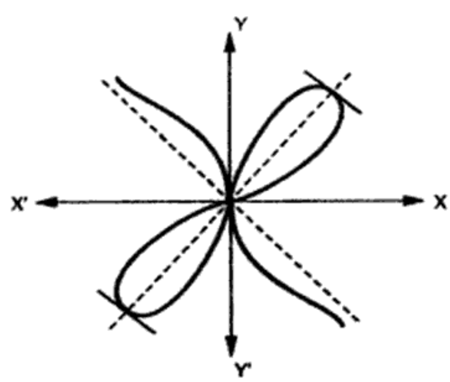


Fig. 8.57

Rule 2 : Points of intersections

1) If some values of t, both x and y becomes zero, then the curve passes through origin.
2) Find x and y intercepts if any.

Rule 3 : Nature of tangents

1) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

2) Form the table of values of t, x, y, $\frac{dy}{dx}$.

Rule 4 : Asymptotes and region

1) Find asymptotes if any.

2) Find region of absence.

Thus for tracing PARAMETRIC curves note the following.

Note : If possible change to Cartesian and trace.
1) If f(–t) = f(t) & g(–t) = – g(t) $\Rightarrow$ symmetry about X-axis.
2) If f(–t) = – f(t) & g(–t) = g(t) $\Rightarrow$ symmetry about Y-axis.
3) If f(–t) = –f(t) & g(–t) = –g(t) $\Rightarrow$ symmetry about opposite quadrants.
4) If $f(\pi - t) = -f(t)$ & $g(\pi - t) = g(t)$ $\Rightarrow$ symmetry about Y-axis.
5) To find the nature of tangent find $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.
6) Find x, y, $\frac{dy}{dx}$ for different values of 't'.

8.7 Illustrations on Type 3: Curves Given by Parametric Co-ordinates

Astroid

➡ **Example 8.16** : *Trace the curve* $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$. ***(Hypocycloid).***

Solution : Its parametric equation is

$$x = a \cos^3 \theta$$

$$y = b \sin^3 \theta$$

5) Table

t	0	1	$\sqrt{3}$
x	0	1	3
y	0	$\frac{2}{3}$	2a
dy/dx	∞	0	$-1/\sqrt{3}$

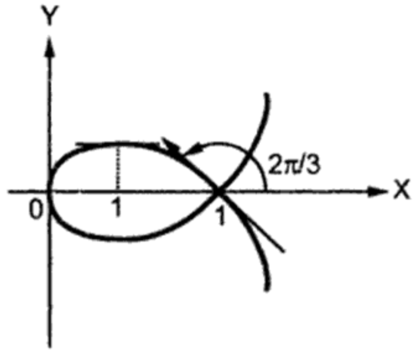


Fig. 8.64

Exercise 8.3

Trace the curves

1) $x^{2/3} + y^{2/3} = a^{2/3}$.

Ans. :

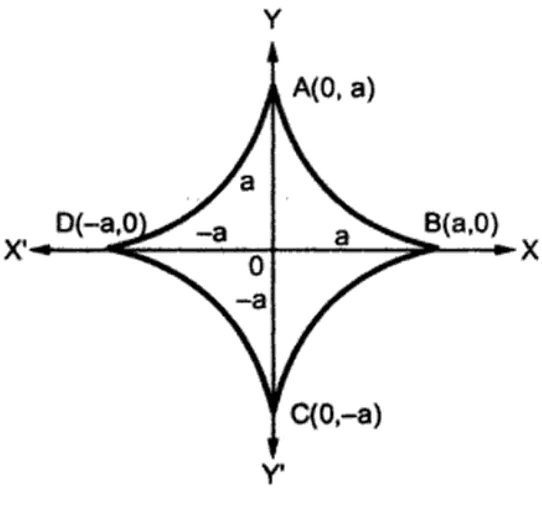


Fig. 8.65

2) $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.

Ans. :

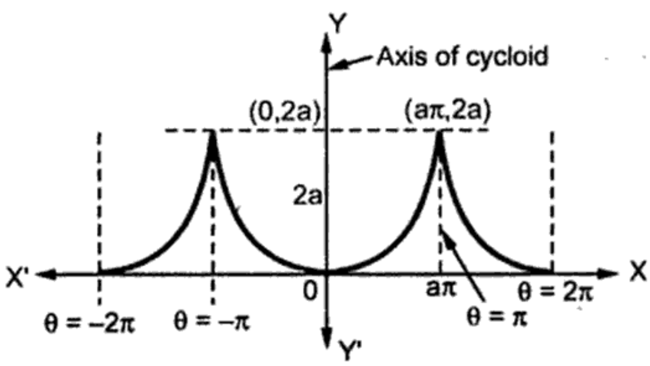


Fig. 8.66

3) $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$.

Ans. :

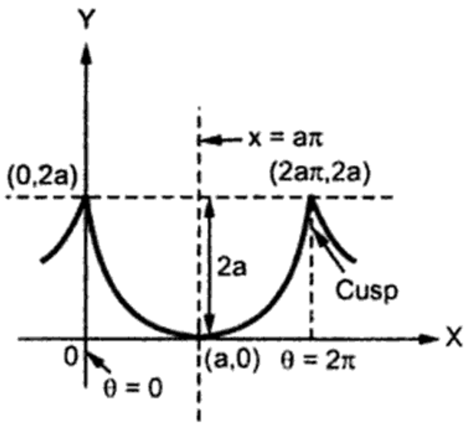


Fig. 8.67

4) $x = a\left(\cos t + \frac{1}{2}\log \tan^2 \frac{t}{2}\right)$, $y = a \sin t$ *(Tractrix)*

Ans. :

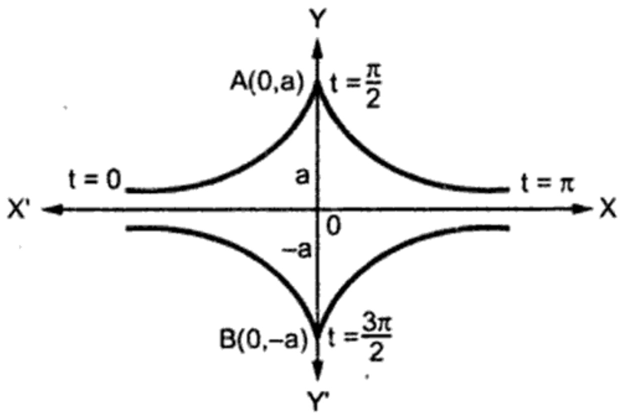


Fig. 8.68

5) $x = at$, $y = \frac{a}{t^2}$ *rectangular hyperbola.*

Ans. :

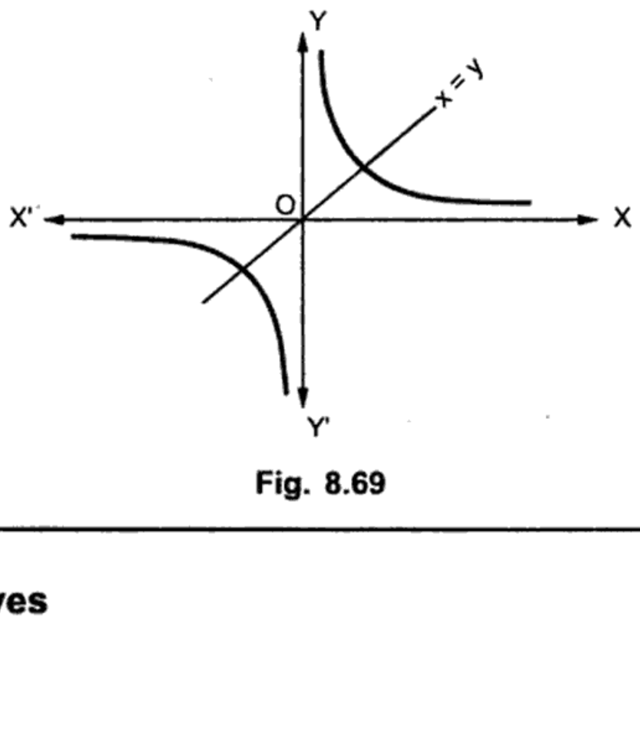


Fig. 8.69

Type 4: Polar curves

8.8 Polar Curves

If it is not possible to trace a curve in cartesian co-ordinates change the equation into polar co-ordinates by using the transformation $x = r\cos\theta$, $y = r\sin\theta$.

where,
$$r^2 = x^2 + y^2$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Rule 5

Find the angle between radius vector and the tangent (ϕ).

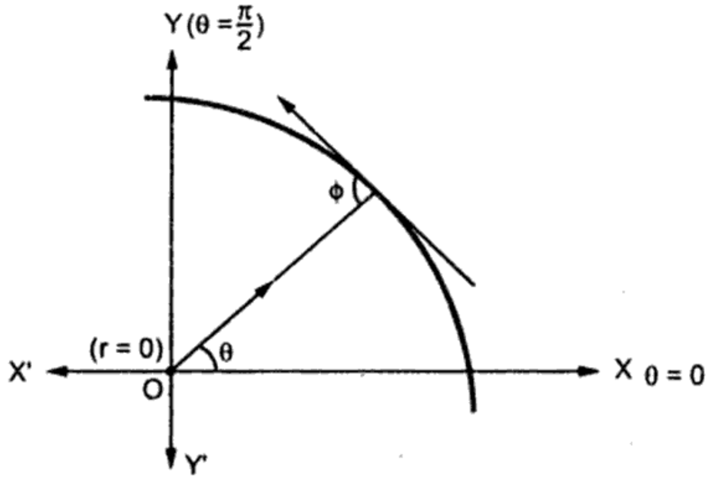


Fig. 8.75

$$\tan \phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}}$$

Find the values of θ for which $\phi = 0$ or ∞.

The values of θ for which $\phi = 0$, tangent will coincide with radius vector and the values of θ for which $\phi = \frac{\pi}{2}$ the tangent will be perpendicular to radius vector.

Rule 6 - Asymptote

Find asymptote if any.

Note : In most of the cases of Implicit curves in cartesian form have oblique asymptote. Refer the procedure as discussed in oblique asymptotes.

Rule 7 - Region of absence

a) If for $\alpha < \theta < \beta$, r^2 is negative then there is no branch between $\theta = \alpha$ to $\theta = \beta$.

b) We know that $|\sin \theta| \leq 1$ and $|\cos \theta| \leq 1$

$\therefore$ For the curves $r = a \cos n\theta$ or $r = a \sin n\theta$,

$r^n = a^n \cos n\theta, n > 0 \; |r| \leq a$,

$\therefore$ No curves lies outside the circle of radius 'a'.

Note :

1) If for the curves having $\sin n\theta$ or $\cos n\theta$ each of the quadrant should be divided into n-parts.

8.9 Illustrations on Type 4: Curves Given by Polar Co-ordinates

Example 8.20 : *Cardioide*

$$r = \frac{a}{2}(1+\cos\theta)$$

Solution :

$$r = \frac{a}{2}(1 + \cos\theta)$$

i) Symmetry - About initial line.

ii) Passes through pole.

iii) $\theta = \pi$ is tangent to the curve at pole. $r = 0 \;\therefore r = \frac{a}{2}(1+\cos\theta) \;\therefore \cos\theta = -1$

$\therefore \boxed{\theta = \pi}$

iv) Table

θ	0	$\frac{\pi}{2}$	π
r	a	$\frac{a}{2}$	0

v) $|r| \leq a$, curve lies within a circle of radius 'a'.

vi) Angle ϕ

$$\tan\phi = r\frac{d\theta}{dr} = \frac{\frac{a}{2}(1+\cos\theta)}{-\frac{a}{2}\sin\theta} = -\cot\frac{\theta}{2} = \tan(\pi/2+\theta/2)$$

$$\therefore \qquad \phi = \frac{\pi}{2}+\frac{\theta}{2}$$

θ	0	$\frac{\pi}{2}$	π
ϕ	$\frac{\pi}{2}$	$\frac{\pi}{4}$	π

$\therefore$ Curve have cusp at $\theta = \pi$.

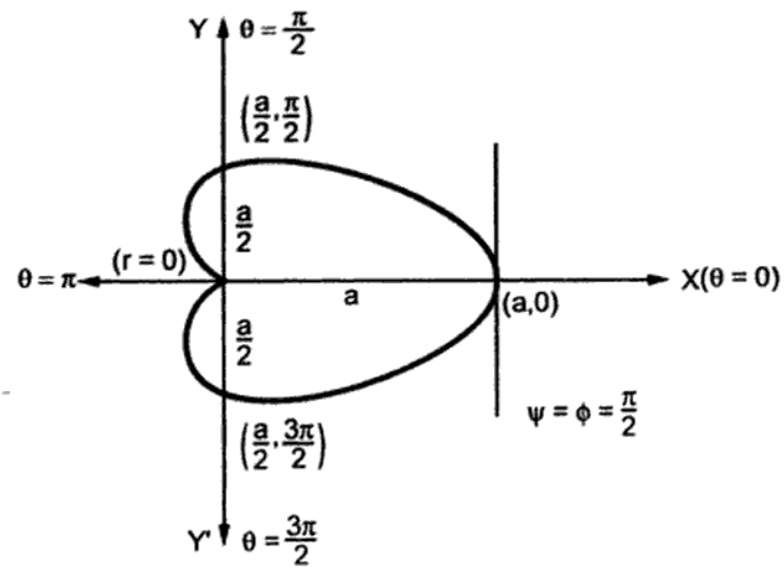


Fig. 8.79

Note :

$$\because \qquad r = \frac{a}{2}(1 + \sin\theta) = \frac{a}{2}\left[1 + \cos\left(\frac{\pi}{2} - \theta\right)\right]$$

can be obtained by rotating $r = \frac{a}{2}(1 + \cos\theta)$ through $\frac{\pi}{2}$ in anticlockwise direction.

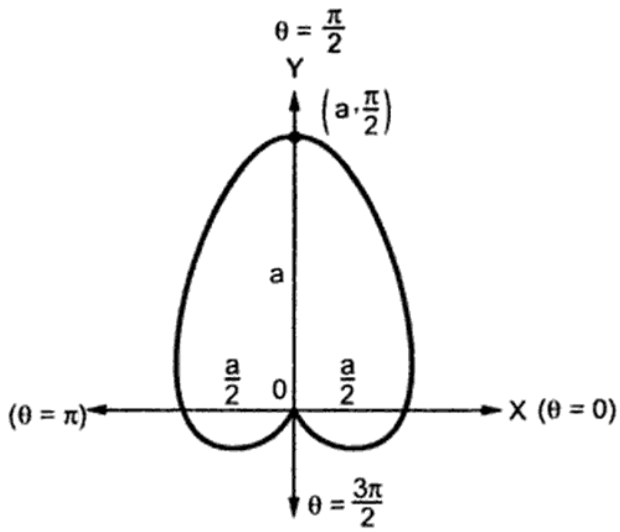


Fig. 8.80

Example 8.21 : *Parabola,* $r = \frac{2a}{1+\cos\theta}$.

Solution : $$r = \frac{2a}{1+\cos\theta}$$

i) Symmetry - About initial line.

ii) Not passes through origin.

iii) Table

θ	0	$\frac{\pi}{2}$	π
r	a	2a	∞

i.e. as θ increases, r increases.

iv) Angle ϕ

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = \frac{a\sec^2\left(\frac{1}{2}\theta\right)}{\frac{2a}{2}\sec\frac{1}{2}\theta\sec\frac{1}{2}\theta\tan\frac{1}{2}\theta}$$

$$= \cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

θ	0	$\frac{\pi}{2}$	π
ϕ	$\frac{\pi}{2}$	$\frac{\pi}{4}$	0

iv) Variation of r w.r.t. θ.

(Here we divide each quadrant in two parts).

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	a	0	Imaginary	0	a

v) For $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, $r^2 < 0$ i.e. r is imaginary.

$\therefore$ There is no curve between $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$.

vi) Angle ϕ

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}}$$

$$= \frac{a\sqrt{\cos 2\theta}}{\frac{a}{2\sqrt{\cos 2\theta}}(-2\sin 2\theta)}$$

$$= \frac{\cos 2\theta}{-\sin 2\theta}$$

$$= -\cot 2\theta$$

$$= \tan\left(\frac{\pi}{2}+2\theta\right)$$

$$\therefore \quad \phi = \frac{\pi}{2}+2\theta$$

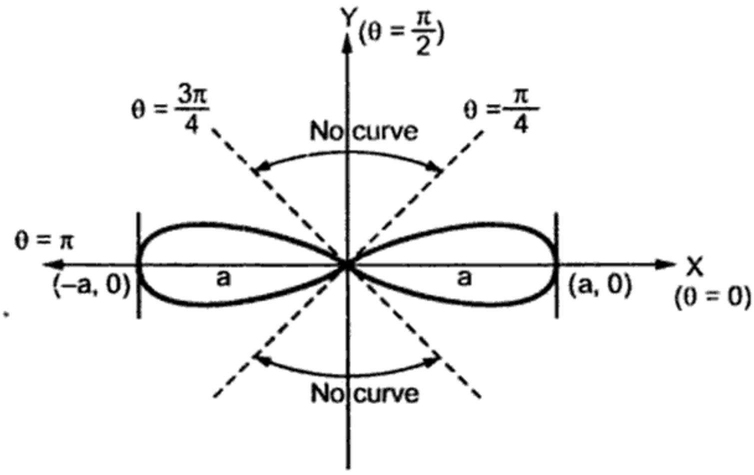


Fig. 8.83

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
ϕ	$\frac{\pi}{2}$	π	–	2π	$\frac{5\pi}{2}$

Note :

$$r^2 = a^2 \sin 2\theta = a^2 \cos 2\left(\frac{\pi}{4} - \theta\right)$$

$\therefore$ Its value is obtained by rotating the curve $r^2 = a^2 \cos 2\theta$ through $\frac{\pi}{4}$.

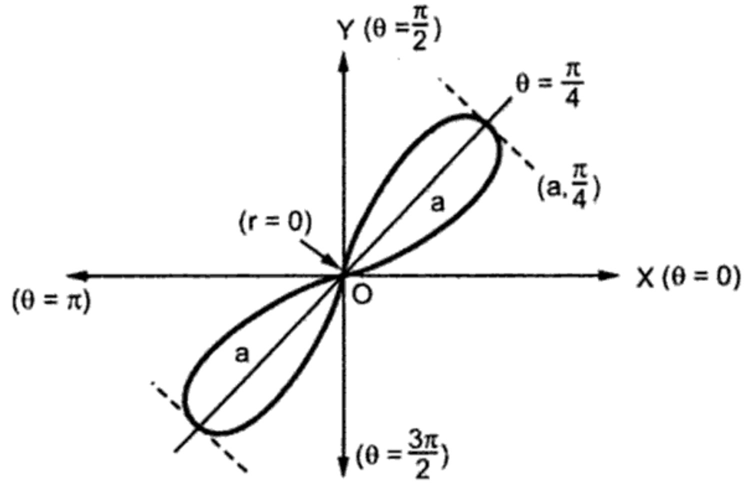


Fig. 8.84

(n = 1, 2, 3, ...)

Example 8.23 : *Rose curves, $r = a \sin 2\theta$*

Solution : $r = a \sin 2\theta$

1) Symmetry - Symmetry about Y-axis.

2) Passes through origin.

3) Tangents at pole :

Put $r = 0 \Rightarrow \sin 2\theta = 0$

$$\Rightarrow \qquad 2\theta = 0, \pi, 2\pi, 3\pi, 4\pi$$

$$\Rightarrow \qquad \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

4) Variation of r w.r.t. θ.

(Here we divide each quadrant in two parts).

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	0	a	0	– a	0

We observed that r is negative for $\frac{\pi}{2} < \theta < \pi$, therefore the curve is reflected through the origin in opposite quadrant.

5) Angle ϕ

$$\tan \phi = r\frac{d\theta}{dt} = \frac{r}{\frac{dr}{d\theta}} = \frac{a \sin 2\theta}{2a \cos 2\theta}$$

$$= \infty, \text{ at } \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$$

$$\therefore \qquad \phi = \frac{\pi}{2} \text{ at } \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$$

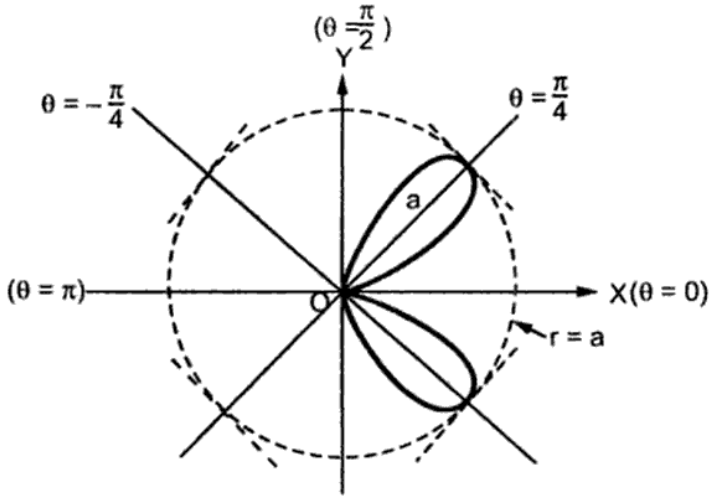


Fig. 8.85

Now taking reflection about Y-axis we get,

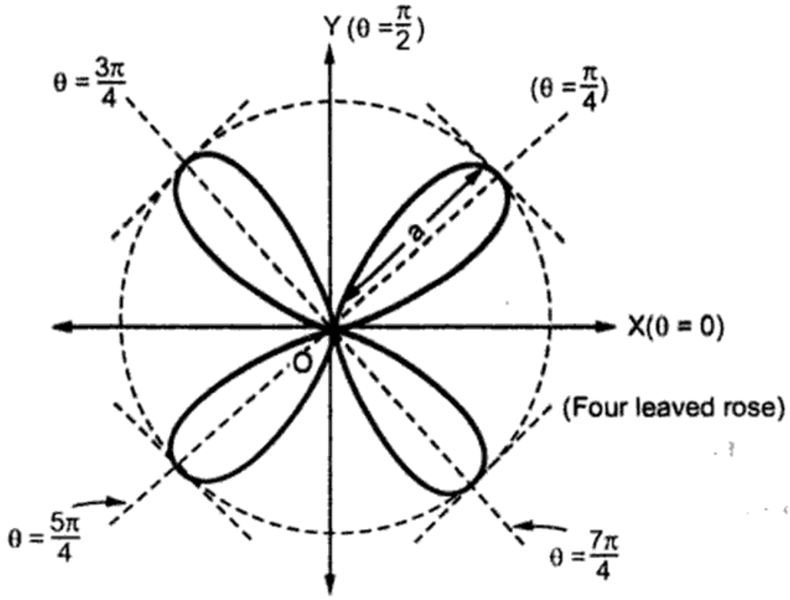


Fig. 8.86

Note : The curve r = a cos 2θ can be obtained from r = a sin 2θ.

We know that,

$$r = a \cos 2\theta = a \sin 2\left(\frac{\pi}{4} - \theta\right)$$

i.e. just rotate the plane through $\frac{\pi}{4}$ in anticlockwise direction.

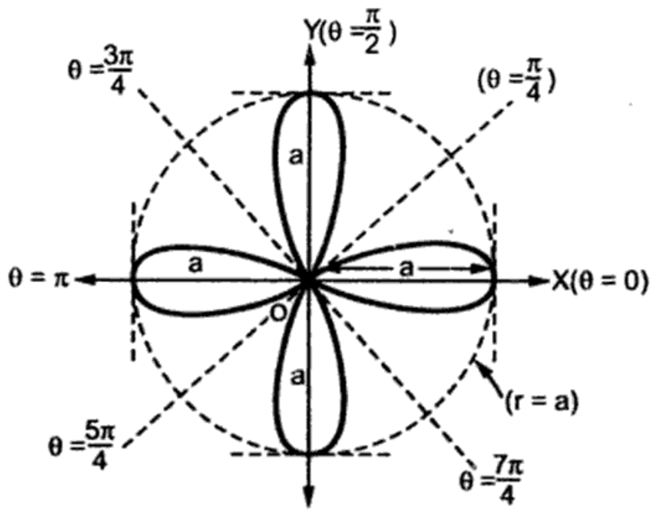


Fig. 8.87

Example 8.24 : *Trace the curve r = a cos 3θ.*

[Dec.-1997, May-1999, May-2000, Dec.-2000, Dec.-2004, May-2006, Dec.-2006]

Solution : 1) Symmetry - About initial line.

2) Passes through the pole.

3) Tangents at origin are

$$\cos 3\theta = 0 \Rightarrow \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \ldots$$

4) Variation of r w.r.t θ is

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{4\pi}{3}$	$\frac{5\pi}{6}$
r	a	0	– a	0	a	0

We observe that r is negative for $\frac{\pi}{6} < \theta < \frac{\pi}{2}$.

$\therefore$ Curves reflects through the origin in opposite quadrant.

5) Angle ϕ

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}}$$

$$= \frac{a\cos 3\theta}{-3a\sin 3\theta}$$

$$= \infty \text{ at } \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$$

$$\therefore \quad \phi = \frac{\pi}{2} \text{ at } \phi = 0, \frac{\pi}{3}, \frac{2\pi}{3}$$

Note : The curve r = a sin 3 θ can be obtained by rotating r = a cos 3 θ through an angle $\frac{\pi}{6}$ in anticlockwise direction because

$$r = a\sin 3\theta = a\cos 3\left(\frac{\pi}{6} - \theta\right)$$

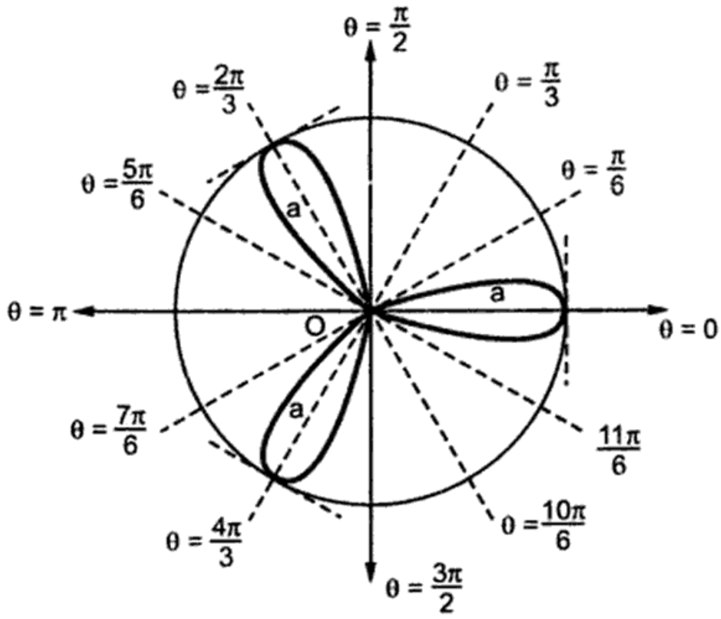


Fig. 8.88

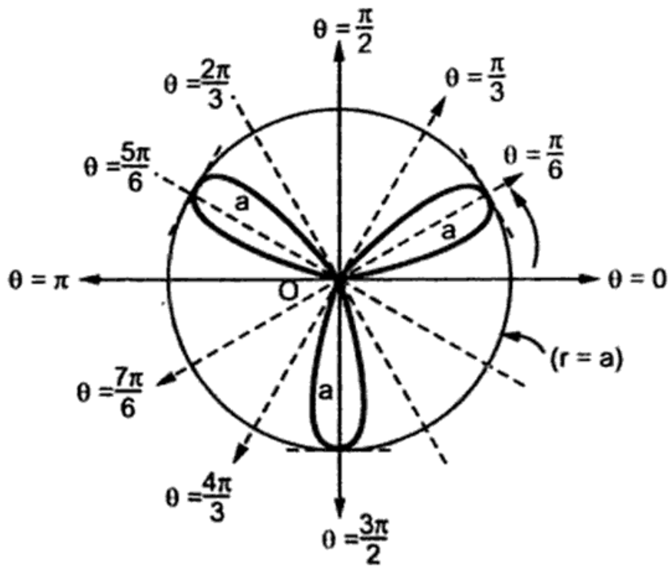


Fig. 8.89

Example 8.25 : ***Pascal's limacon*** $r = a + b\cos\theta$.
If (i) a = b (ii) a > b (iii) a < b. **[May-1998, May-2000, May-2001, May-2006]**

Solution :

Case I :

a = b, cardioide (Refer Fig. 8.79)

Case II : a > b

For example : $r = \sqrt{2} + \cos\theta$ (Pascal's Limacon).

1) Symmetry - Initial line.

2) Not passes through the pole.

3) Variation of r w.r.t. θ is

θ	0	$\frac{\pi}{2}$	π
r	a + b	a	a − b

i.e. as θ increases from 0 to π, r is continuously decreases.

4) $$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = \frac{a + b\cos\theta}{-b\sin\theta}$$

$$= \infty \text{ at } \theta = 0, \pi$$

i.e. $$\phi = \frac{\pi}{2} \text{ at } \theta = 0 \text{ and } \theta = \pi$$

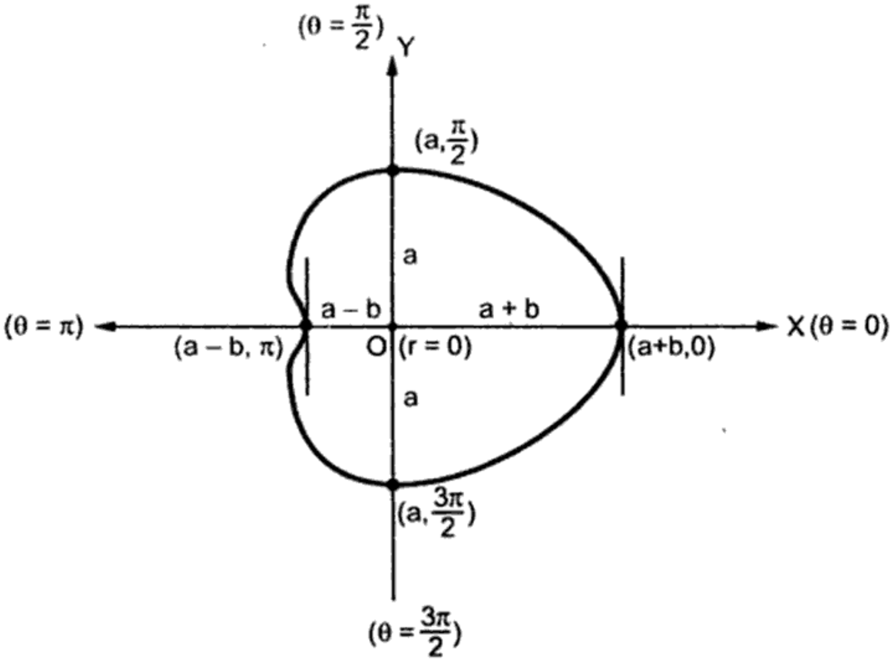


Fig. 8.90

Case III : a < b

$$r = a + b\cos\theta \qquad \text{where } a < b$$

For convenience consider the curve

$$r = a\left(1 + \sqrt{2}\cos\theta\right)$$

1) Symmetry - About initial line.

2) Passes through pole.

3) Tangent at the pole are

$$1 + \sqrt{2}\cos\theta = 0 \Rightarrow \cos\theta = -\frac{1}{\sqrt{2}}$$

$\Rightarrow \qquad \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}, \frac{5\pi}{4}$

4) Table

θ	0	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	$a\left(1+\sqrt{2}\right)$	a	0	$-a\left(\sqrt{2}-1\right)$

i) We observe that as θ increases, from $\theta = 0$ to $\theta = \frac{3\pi}{4}$, r continuously decreases.

ii) r is negative for $\frac{3\pi}{4} < \theta < \pi$.

$\therefore$ Curve reflects through the origin in opposite quadrants.

5) $$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = \frac{a\left(1+\sqrt{2}\cos\theta\right)}{-\sqrt{2}\,a\sin\theta}$$

$$= \infty \text{ at } \infty = 0, \pi$$

$\therefore \qquad \phi = \frac{\pi}{2}$ for $\theta = 0$ and $\theta = \pi$.

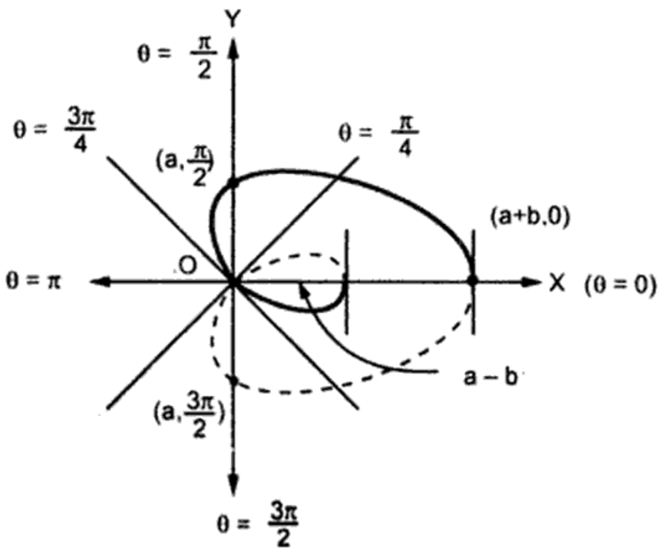


Fig. 8.91

Example 8.26 : *Trace the curve* $r = a\left(\frac{\sqrt{3}}{2} + \cos\frac{\theta}{2}\right)$.

Solution : 1) Symmetry - About initial line.

2) Passes through pole.

3) Tangents at the pole is

4) $\tan \phi = r\dfrac{d\theta}{dr}$

$$\frac{1}{m} = \text{Constant}$$

$$\therefore \quad \phi = \text{Constant}$$

i.e. at every point on the curve then angle between the radius vector and tangent at the end of radius vector is constant (Hence it is called as equiangular spiral).

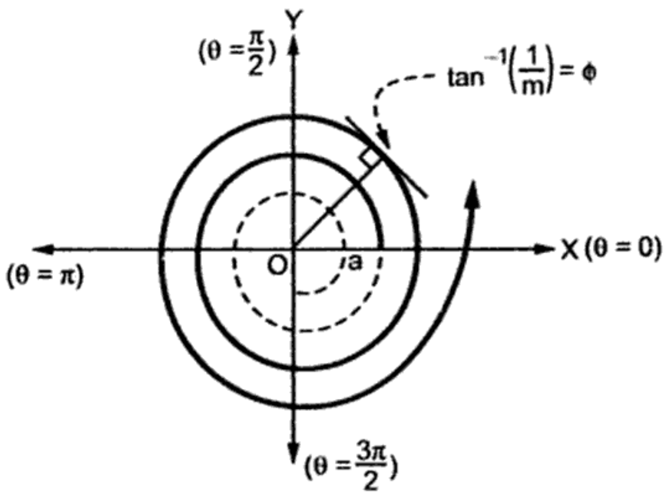


Fig. 8.93

Achimidies spiral

Example 8.28 : $r = a\theta.$

Solution : 1) Symmetry - Symmetric about Y-axis.

2) Pole - Passes through the pole.

3) $\theta = 0$, is tangents to the curve at pole.

4) As θ increases from 0 to ∞, r increases continuously.

5) $\tan \phi = r\dfrac{d\theta}{dr} = \dfrac{a\theta}{a} = \theta \Rightarrow \phi$ is proportional to θ.

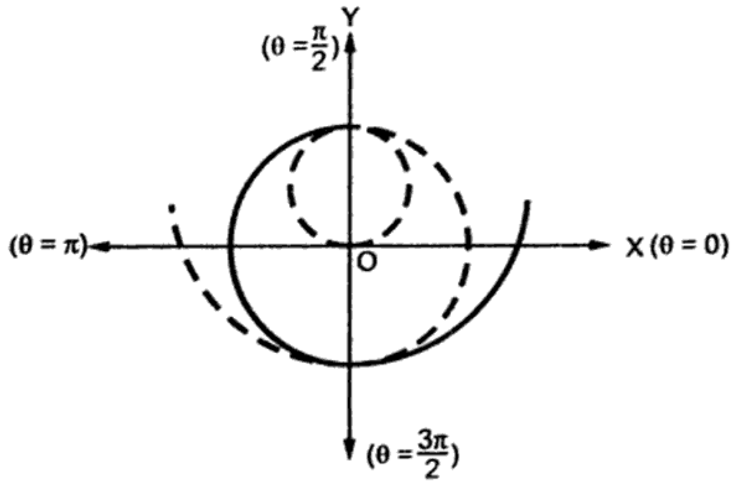


Fig. 8.94

Reciprocal spiral

Example 8.29 : $r = \dfrac{a}{\theta}.$

Solution : 1) Symmetry - Y-axis.

2) Not passes through pole.

3) $\theta = 0$ is asymptote.

4) As θ increases from 0 to ∞, r is continuously decreases from ∞ to 0.

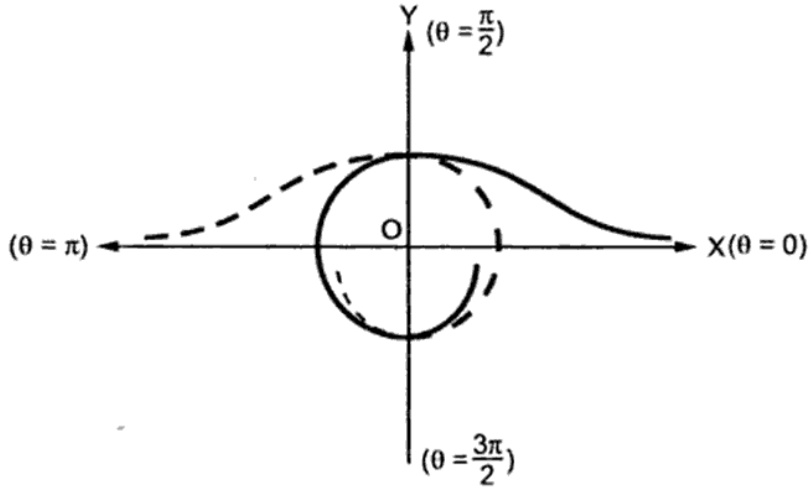


Fig. 8.95

Exercise 8.4

1) Trace the curve

i) $r = a \sin \theta$ *ii)* $r = a(1 + \sin\theta)$ *iii)* $r = \dfrac{2a}{1+\sin\theta}$ *iv)* $r^2 = a^2 \sin 2\theta$

Ans. :

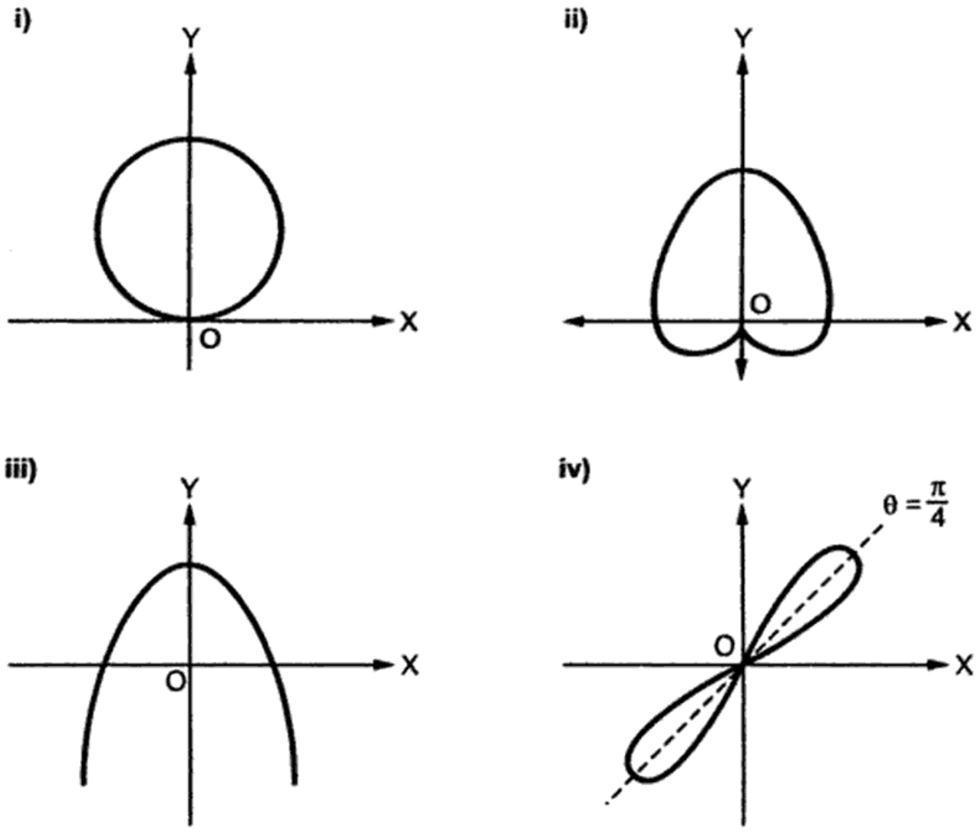


Fig. 8.96

2) *Trace the curve* i) $r = a\cos 2\theta$ ii) $r = a\sin 3\theta$.

Ans. : i)

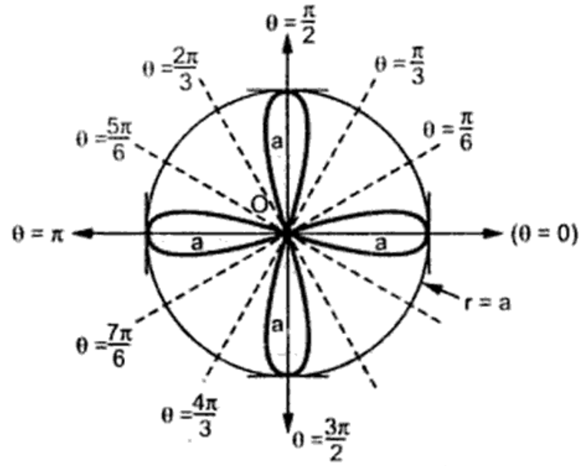


Fig. 8.97

ii)

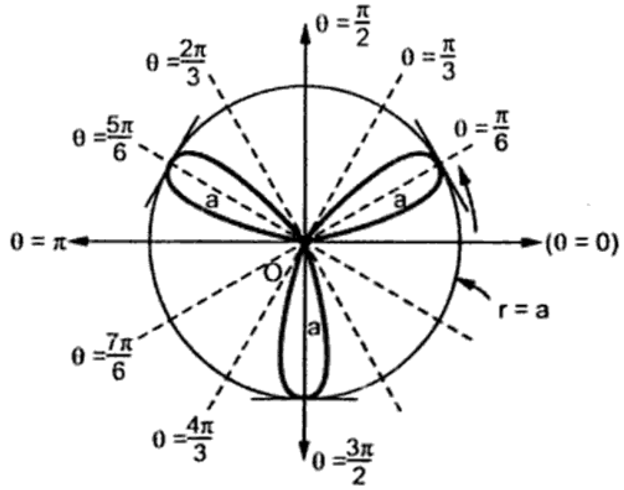


Fig. 8.98

3) *Trace the curve* $r = a + b\sin\theta$ *i)* $r = a(1+\sin\theta)$ *ii)* $r = a(\sqrt{2} + \sin\theta)$ *iii)* $r = a(\sqrt{3} + 2\sin\theta)$.

[Dec.-2006]

Ans. :

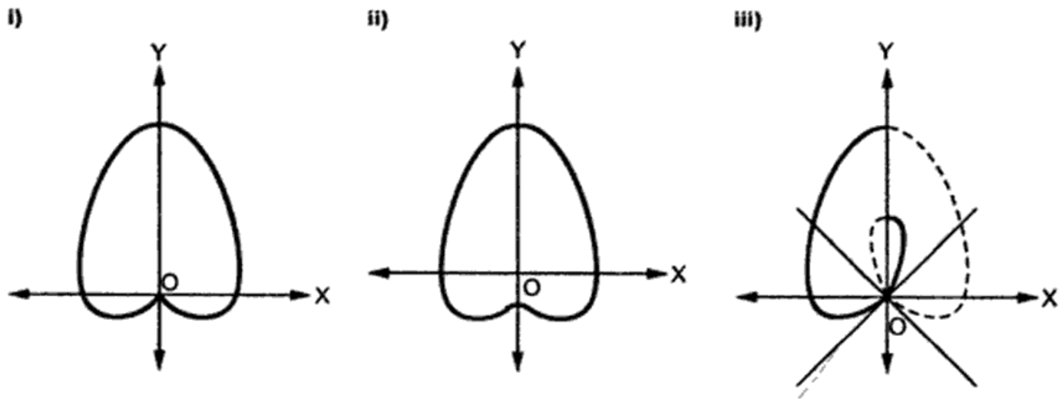


Fig. 8.99

Summary of Polar curves

Cardiods :
1) $r = a(1 + \cos\theta)$, 2) $r = a(1 - \cos\theta)$
3) $r = a(1 + \sin\theta)$, 4) $r = a(1 - \sin\theta)$,
Parabola :
5) $r(1 + \cos\theta) = 2a$, 6) $r(1 - \cos\theta) = 2a$
7) $r(1 + \sin\theta) = 2a$, 8) $r(1 - \sin\theta) = 2a$
Pascal's Limacon :
9) $r = a + b \cos\theta$ for $a > b$, $a < b$, and $a = b$
10) $r = a(\sqrt{3} + 2\cos\theta)$, 11) $r = a(1 + 2\cos\theta)$
Rose Curves :
12) $r = a \sin 2\theta$ 13) $r = a \cos 2\theta$
14) $r = a \sin 4\theta$ 15) $r = a \cos 4\theta$
16) $r = a \sin 3\theta$ 17) $r = a \cos 3\theta$
Lemiscates of Bernoulli's :
18) $r^2 = a^2 \cos 2\theta$
Spirals :
19) $r = a e^{m\theta}$ where a and m are +ve (Equiangular Spiral)
20) $r = a\theta$ 21) $r^2 \theta = a^2$ 22) $r\theta = a$, $a > 0$.

University Questions

May - 2003

1. *Trace any two curves.* **[10 Marks]**
   (i) $y^2 = (x-a)(x-b)(x-c)$ *where a, b, c are positive and* $a < b < c$.
   (ii) $ay^2 = x(a^2 - x^2)$.
   (iii) $r = a\sin 2\theta$.

May - 2004

1. *Trace the curves (any two).*
   i) $r = 1 + 2\cos\theta$.
   ii) $y^2(a^2 - x^2) = a^3 x$.
   iii) $r^2 = a^2 \cos 2\theta$. **[8 Marks]**
2. *Trace the following curves (any two).*
   i) $r(1 + \sin\theta) = 2a$.
   ii) $x^3 + y^3 = 3axy$
   iii) $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ **[8 Marks]**

Dec. - 2004

1. *Trace the curves (any 2).*
   i) $x^2y^2 = a^2(y^2 - x^2)$
   ii) $r = a\cos 3\theta$
   iii) $x = t^2$, $y = t - \frac{t^3}{3}$ **[8 Marks]**

2. *Trace the curves (any 2).*
   i) $r = \sqrt{2} + \cos\theta$
   ii) $y^2(2a - x) = x^3$
   iii) $x^4 + y^4 = 2a^2xy$ **[8 Marks]**

May - 2005

1. *Trace the curves (any 2).*
   i) $x(x^2 + y^2) = a(x^2 - y^2)$ *where* $a > 0$
   ii) $r(1 + \sin\theta) = 2a$
   iii) $x^3 + y^3 = 3axy$ **[8 Marks]**

Dec. - 2005

1) *Trace the following curves (any two).* **[8 Marks]**
   i) $xy^2 = a^2(a - x)$
   ii) $x = a(t - \sin t),\ y = a(1 - \cos t)$
   iii) $r^2 = b^2 \cos 2\theta$
2) *Trace the following curves (any two).* **[8 Marsks]**
   i) $r = b(1 + \cos\theta)$
   ii) $x = t^2,\ y = t - \dfrac{t^3}{3}$
   iii) $x^{2/3} + y^{2/3} = a^{2/3}$

May - 2006

1) *Trace the following curves : (Any two).* **[8 Marks]**
   i) $ay^2 = x^2(a - x)$
   ii) $r = a\cos 3\theta$
   iii) $x = a(t + \sin t),\ \ y = a(1 + \cos t)$
2) *Trace the following curves : (Any two).* **[8 Marks]**
   i) $y^2(a^2 + x^2) = a^2x^2$
   ii) $y^2(4 - x) = x(x - 2)^2$
   iii) $r = a + b\cos\theta,\ a < b$

Dec. - 2006

1. *Trace the following curves : (Any two).* **[8 Marks]**
   1) $y^2(x - a) = x^2(2a - x)$
   2) $r = a(1 + \sin\theta)$
   3) $x^{2/3} + y^{2/3} = a^{2/3}$
2. *Trace the following curves : (Any two).* **[8 Marks]**
   1) $y^2 = (x - 1)(x - 2)(x - 3)$
   2) $x^3 + y^3 = 3axy$
   3) $r = a\sin 3\theta$

May - 2007

1. *Trace the following curves : (Any Two)*

   1) $\sqrt{x}+\sqrt{y}=\sqrt{a}$ 2) $x = a\,(t - \sin t),\ y = a\,(1 + \cos t)$

   3) $r\,(1 + \sin\theta) = 2a$ **[8 Marks]**

2. *Trace the following curves : (Any two)*

   1) $r = a\cos 3\theta$ 2) $x^3+y^3 = 3axy$ 3) $x = t^2,\ y = t - \dfrac{t^3}{3}$ **[8 Marks]**

Dec. - 2007

1. *Trace the following curves : (Any two)* **[8 Marks]**

   1) $x(x^2+y^2) = a\,(x^2-y^2),\ a > 0$ 2) $r = a\sin 2\theta$

   3) $\left(\dfrac{x}{a}\right)^{2/3}+\left(\dfrac{y}{b}\right)^{2/3}=1$

2. *Trace the following curves : (Any two)* **[8 Marks]**

   1) $y^2\,(2a - x) = x^3$ 2) $r = a\,(1 - \cos\theta)$

   3) $x = a\,(\theta + \sin\theta),\ y = a\,(1 + \cos\theta)$

May - 2008

1. *Trace the following curves : (Any two)* **[8 Marks]**

   1) $y^2(a^2-x^2)=a^3x$

   2) $x = a\,(t + \sin t),\ y = a\,(1 + \cos t)$

   3) $r = a + b\cos\theta$ *when* $a < b$

2. *Trace the following curves : (Any Two)* **[8 Marks]**

   1) $r = a\sin 4\theta$

   2) $y^2=\dfrac{x^2(x^2-4a^2)}{x^2-a^2}$

   3) $\left(\dfrac{x}{a}\right)^{2/3}+\left(\dfrac{y}{b}\right)^{2/3}=1$

Dec. - 2008

1. *Trace the following curves : (Any two).* **[8 Marks]**

   1) $y^2(x-a)=x^2(2a-x)$

   2) $r = a\,(1 + \sin\theta)$

   3) $x=a\cos^3 t,\ y=a\sin^3 t$

2. *Trace the following curves : (Any two)* **[8 Marks]**

   1) $x^3+y^3 = 3axy$

   2) $r = a\cos 3\theta$

   3) $y^2=(x-1)(x-2)(x-3)$

1) ARC. $S = \int ds$

2) C.G. of ARC

$$\therefore \qquad \bar{x} = \frac{\int x\rho ds}{\int \rho\, ds}$$

$$\bar{y} = \frac{\int y\rho ds}{\int \rho\, ds}$$

3) M.I. of ARC – M.I. = $\int p^2 \rho\, ds$

where ρ is the density at any point (x, y).

p = Perpendicular distance of any point (x, y) from the axis of M.I.

Type 1

Example 9.1 : *Show that the length of arc of parabola $y^2 = 4ax$ cut-off by the line $3y = 8x$ is a $\left(\log 2 + \frac{15}{16}\right)$ and that cut-off by the latus rectum is $2a[\sqrt{2} + \log(1 + \sqrt{2})]$*

Solution :

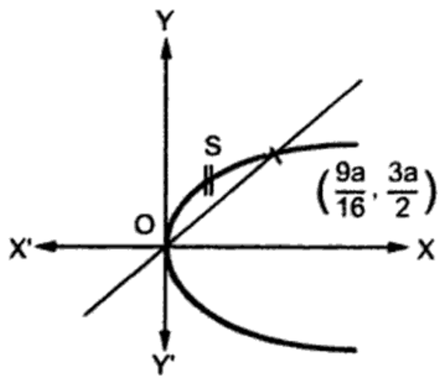


Fig. 9.1

Step 1 : $y^2 = 4ax$, $3y = 8x$

Solving these two equations we get two points (0, 0) and $\left(\frac{9a}{16}, \frac{3a}{2}\right)$.

Thus, $y_1 = 0$ and $y_2 = \frac{3a}{2}$

$$S = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example 9.3 : *Show that the whole length of the loop of the curve* $9y^2 = (x+7)(x+4)^2$ *is* $4\sqrt{3}$

Solution :

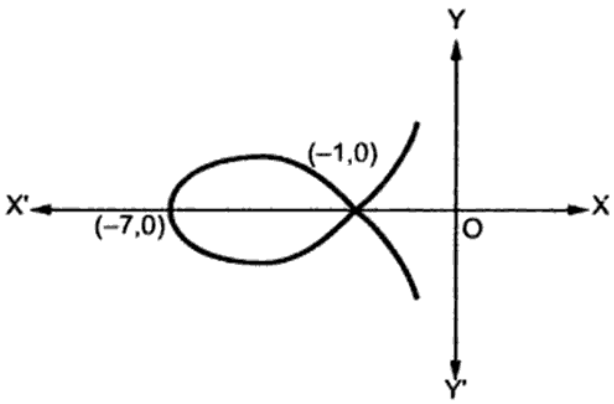


Fig. 9.3

Step 1 : From the equation of the curve it is clear that loop is around X-axis between $x = -7$ and $x = -4$. The curve is symmetrical about X-axis. Hence we integrate between the limits -7 and -4. We use formula given by,

$$S = 2\int_{-7}^{-4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

(We multiply the integrals by 2 to get the whole length of the loop).

Step 2 : Differentiating equation of the curve, w.r.t. 'x',

$$18y\frac{dy}{dx} = (x+4)^2 + 2(x+4)(x+7)$$

$$= (x+4)[x + 4 + 2x + 14]$$

$$\frac{dy}{dx} = \frac{3(x+4)(x+6)}{18y} = \frac{(x+4)(x+6)}{6y}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(x+4)^2(x+6)^2}{36y^2} = \frac{(x+4)^2(x+6)^2}{4(x+7)(x+4)^2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(x+6)^2}{4(x+7)}$$

$$= \frac{4x + 28 + x^2 + 12x + 36}{4(x+7)}$$

$$= \frac{x^2 + 16x + 64}{4(x+7)} = \frac{(x+8)^2}{4(x+7)}$$

Step 3 :

Thus,
$$S = \frac{2}{2}\int_{-7}^{-4} \frac{x+8}{\sqrt{x+7}}\,dx = \int_{-7}^{-4} \frac{x+7+1}{\sqrt{x+7}}\,dx$$

$$= \int_{-7}^{-4} \left\{\sqrt{x+7} + (x+7)^{-1/2}\right\} dx$$

$$= \left[\frac{(x+7)^{3/2}}{\frac{3}{2}} + 2(x+7)^{1/2}\right]_{-7}^{-4} = \frac{2}{3}3\sqrt{3} + 2\sqrt{3} = 4\sqrt{3}$$

Example 9.4 : *Show that for the curve $8a^2y^2 = x^2\left(a^2 - x^2\right)$, $S = \frac{a}{2\sqrt{2}}\left[2\theta + \sin\theta\cos\theta\right]$ and the perimeter of one of the loops is $\frac{\pi a}{\sqrt{2}}$.*

Solution :

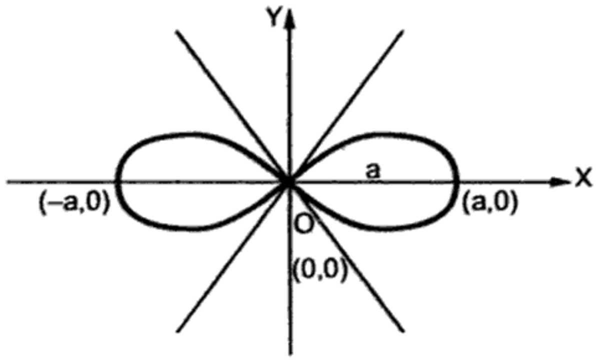


Fig. 9.4

Step 1 : Here the curve is symmetrical about X-axis and has two loops around X-axis between x = 0 and x = a.

We first integrate between x = 0 and x = x using formula i.e. we first find arc length of the curve from origin to any point (x, y) on the curve.

$$S = \int_{0}^{x} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Step 2 : Differentiating w.r.t. x, the equation of the curve, we get,

$$8a^2 \cdot 2y\frac{dy}{dx} = 2x(a^2 - x^2) - 2x^3$$

$$\therefore \qquad \frac{dy}{dx} = \frac{a^2x - 2x^3}{8a^2y}$$

$$\therefore \qquad \left(\frac{dy}{dx}\right)^2 = \frac{x^2 (a^2 - 2x^2)^2}{64 a^4 y^2}$$

$$= \frac{x^2 (a^2 - 2x^2)^2}{8a^2 \cdot x^2 (a^2 - x^2)}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(a^2 - 2x^2)^2}{8a^2 (a^2 - x^2)}$$

$$= \frac{8a^4 - 8a^2x^2 + a^4 - 4a^2x^2 + 4x^4}{8a^2(a^2 - x^2)}$$

$$= \frac{9a^4 - 12a^2x^2 + 4x^4}{8a^2 (a^2 - x^2)}$$

$$= \frac{(3a^2 - 2x^2)^2}{8a^2 (a^2 - x^2)}$$

Step 3 :

Now,
$$S = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^x \frac{3a^2 - 2x^2}{2\sqrt{2}\; a\sqrt{a^2 - x^2}}\, dx$$

$$= \frac{1}{2\sqrt{2}a} \int_0^x \frac{3a^2 - 2x^2}{\sqrt{a^2 - x^2}}\, dx$$

Putting $x = a \sin \theta$, $dx = a \cos \theta \, d\theta$.

x	0	x
θ	0	θ

As particular value of x is not given we take for x = x, upper limit as θ.

$$S = \frac{1}{2\sqrt{2}a} \int_0^\theta \frac{3a^2 - 2a^2 \sin^2\theta}{a \cos\theta} \cdot a \cos\theta \; d\theta$$

$$= \frac{1}{2\sqrt{2}a} \int_0^\theta [3a^2 - a^2 (1 - \cos 2\theta)]\, d\theta$$

$$= \frac{1}{2\sqrt{2}a} \int_0^\theta \left(2a^2 + a^2 \cos 2\theta\right) d\theta = \frac{a}{2\sqrt{2}} \left[2a + \frac{\sin 2\theta}{2}\right]_0^\theta$$

$$S = \frac{a}{2\sqrt{2}} [2\theta + \sin\theta \cdot \cos\theta]$$

which is the first required result.

Step 3 : Using the formula with limits of x from 0 to b, we get,

$$S = \int_0^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_0^b \sqrt{1+\frac{9x}{4a}}\, dx$$

$$= \frac{1}{2\sqrt{a}}\int_0^b \sqrt{4a+9x}\, dx$$

$$S = \frac{1}{2\sqrt{a}}\left[\frac{(4a+9x)^{3/2}}{\frac{3}{2}\cdot 9}\right]_0^b$$

$$= \frac{1}{27\sqrt{a}}\left[(4a+9b)^{3/2}-(4a)^{3/2}\right]$$

$$= \frac{(4a+9b)^{3/2}}{27\sqrt{a}} - \frac{8a^{3/2}}{27\sqrt{a}}$$

$$= \frac{(4a+9b)^{3/2}}{27\sqrt{a}} - \frac{8a}{27}$$

Example 9.6 : *Show that the length of the arc of the curve* $4ax = y^2 - 2a^2 \log\frac{y}{a} - a^2$ *measured from (0, a) to any point is given by* $\frac{y^2}{2a} - \frac{a}{2} - x$.

Solution :

Step 1 : Here the equation is x interms of y so we use formula,

$$S = \int_{y_1}^{y_2} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

Step 2 :

i.e. $$S = \int_a^y \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \quad \text{... (1)} \left[\because 4ax = y^2 - 2a^2\log\frac{y}{a} - a^2\right]$$

$$\therefore \quad 4a\frac{dx}{dy} = 2y - \frac{2a^2}{y}$$

$$\therefore \quad \frac{dx}{dy} = \frac{y^2-a^2}{2ay}$$

$$\therefore \quad 1+\left(\frac{dx}{dy}\right)^2 = 1+\left(\frac{y^2-a^2}{2ay}\right)^2 = \frac{(y^2+a^2)^2}{4a^2y^2}$$

Step 3 :

$$\therefore \text{From (1)} \qquad S = \int_a^y \sqrt{\frac{(y^2 + a^2)^2}{4a^2y^2}}\, dy = \frac{1}{2a}\int_a^y \left(y + \frac{a^2}{y}\right) dy$$

$$\therefore \qquad S = \frac{1}{2a}\left[\frac{y^2}{2} + a^2 \log\frac{y}{a} - \frac{a^2}{2}\right] \qquad \text{... (2)}$$

$$\text{Now, } \frac{y^2}{2a} - \frac{a}{2} - x = \frac{y^2}{2a} - \frac{a}{2} - \left(\frac{y^2 - 2a^2 \log\frac{y}{a} - a^2}{4a}\right) \qquad \text{... (3)}$$

$$= \frac{1}{2a}\left[\frac{y^2}{2} + a^2 \log\frac{y}{a} - \frac{a^2}{2}\right]$$

From (2) and (3)

$$S = \frac{y^2}{2a} - \frac{a}{2} - x$$

Example 9.7 : *Show that arc length of the curve $y = e^x$ from (0, 1) to (1 , e) is* $\sqrt{(e^2 + 1)} - \sqrt{2} + \log\left(1 + \sqrt{2}\right) - \log\left(1 + \sqrt{e^2 + 1}\right) + 1$

Solution :

Step 1 : We use

$$S = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad \text{... (1)}$$

$$\therefore y = e^x \quad \therefore \frac{dx}{dy} = \frac{1}{y}$$

Step 2 :

$$\therefore \qquad S = \int_1^e \sqrt{1 + \frac{1}{y^2}}\, dy$$

$$= \int_1^e \frac{\sqrt{y^2 + 1}}{y}\, dy = \int_1^e \frac{(y^2 + 1)}{y\sqrt{y^2 + 1}}\, dy$$

$$= \int_1^e \frac{y}{\sqrt{y^2 + 1}}\, dy + \int_1^e \frac{1}{y\sqrt{y^2 + 1}}\, dy \qquad \text{Put } y = \frac{1}{t}$$

$$S = I_1 + I_2$$

Step 3 : We evaluate I_1 and I_2 separately.

$$I_1 = \int_1^e \frac{y\,dy}{\sqrt{y^2+1}}$$

Put $1 + y^2 = t$, $2y\,dy = dt$

$$= \frac{1}{2}\int_1^{1+e^2} t^{-1/2}dt = \frac{1}{2}\left[\frac{t^{1/2}}{1/2}\right]_2^{1+e^2}$$

$$= \left[\sqrt{1+e^2} - \sqrt{2}\right]$$

Now, $$I_2 = \int_1^e \frac{dy}{y\sqrt{y^2+1}}$$

Put $y = \frac{1}{p} \Rightarrow dy = -\frac{1}{p^2}dp$

$\therefore$ $$I_2 = -\int_1^{1/e} \frac{1}{p^2}(p)\frac{dp}{\frac{\sqrt{1+p^2}}{p}} = \int_{1/e}^1 \frac{dp}{\sqrt{1+p^2}} = \left[\log\left[p+\sqrt{1+p^2}\right]\right]_{1/e}^1$$

$$= \log\left(1+\sqrt{2}\right) - \log\left[\frac{1}{e} + \sqrt{1+\frac{1}{e^2}}\right]$$

$$= \log\left(1+\sqrt{2}\right) - \log\left(1+\sqrt{1+e^2}\right) + \log e$$

$$= \log\left(1+\sqrt{2}\right) - \log\left(1+\sqrt{1+e^2}\right) + 1$$

Step 4 : Here $S = I_1 + I_2$

$$= \sqrt{(1+e^2)} - \sqrt{2} + \log\left(1+\sqrt{2}\right) - \log\left(1+\sqrt{1+e^2}\right) + 1$$

Example 9.8 : *Find the length of the loop of the given curve* $3ay^2 = x(a-x)^2$

[May-2006]

Solution :

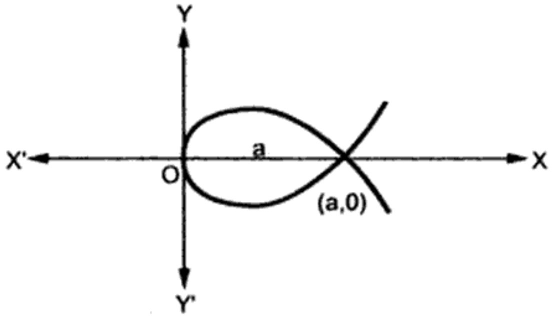


Fig. 9.6

Step 1 : The given curve is,

$$3ay^2 = x(a-x)^2 \qquad \dots (1)$$

Differentiating w.r.t. x,

$$6ay\frac{dy}{dx} = x \cdot 2(a-x)(-1) + (a-x)^2$$

$$= (a-x)(a-3x)$$

$$\therefore \quad \frac{dy}{dx} = \frac{(a-x)(a-3x)}{6ay}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(a-x)^2(a-3x)^2}{36a^2y^2}$$

$$= 1+\frac{(a-x)^2(a-3x)^2(3a)}{36a^2x(a-x)^2} \qquad \text{... Using (1)}$$

$$= 1+\frac{(a-3x)^2}{12ax}$$

$$= \frac{12ax+a^2-6ax+9x^2}{12ax}$$

$$= \frac{(a+3x)^2}{12ax} \qquad \dots (2)$$

Step 2 : Loop is symmetrical about X-axis with x = 0 to x = a hence length of the loop is given by (by symmetry).

$$S = 2\int_{x_1}^{x_2}\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx$$

$$S = 2\int_0^a\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx = 2\int_0^a\sqrt{\frac{(a+3x)^2}{12ax}}\,dx \qquad \text{... From (2)}$$

$$= 2\frac{1}{2\sqrt{3a}}\int_0^a\left(ax^{-1/2}+3x^{1/2}\right)dx$$

$$= \frac{1}{\sqrt{3a}}\left[\frac{ax^{1/2}}{1/2}+\frac{3x^{3/2}}{3/2}\right]_0^a$$

$$= \frac{2}{\sqrt{3a}}\left[a^{3/2}+a^{3/2}\right] = \frac{4}{\sqrt{3}}a$$

$$= \int_1^2 \coth x \, dx = (\log \sinh x)_1^2$$

$$= \left(\log \frac{e^x - e^{-x}}{2}\right)_1^2$$

$$= \log\left(\frac{e^2 - e^{-2}}{e - e^{-1}}\right)$$

$$= \log\left(\frac{e^4 - 1}{e^2} \times \frac{e}{e^2 - 1}\right)$$

$$= \log\left(\frac{e^2 + 1}{e}\right) = \log\left(e + \frac{1}{e}\right)$$

9.1 Catenary

Example 9.10 : *Find the length of the arc of the catenary $y = c\cosh\frac{x}{c}$ measured from its vertex to any point (x, y). And show that $S^2 = y^2 - c^2$.*

Solution :

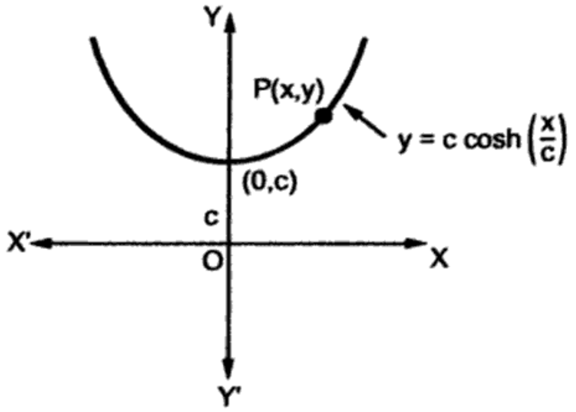


Fig. 9.7

Step 1 : Here we use formula $S = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$ ... (1)

Step 2 :

$$y = c\cosh\frac{x}{c}$$

Differentiating w.r.t. x

$$\therefore \quad \frac{dy}{dx} = c \cdot \sinh\frac{x}{c} \cdot \frac{1}{c}$$

Type 2

Example 9.11 : *Evaluate $\int xy\,ds$ along the arc of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the first quadrant.* [Dec.-2005]

Solution :

Step 1 : Parametric equations of ellipse are x = a cos θ, y = b sin θ.

Along the arc in the positive quadrant θ varies from 0 to θ.

$$\int xy\,ds = \int xy\frac{ds}{d\theta}\,d\theta = \int xy\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

Step 2 : Substituting $x = a\cos\theta$, $y = b\sin\theta$, $\frac{dx}{d\theta} = -a\sin\theta$, $\frac{dy}{d\theta} = b\cos\theta$ and as we have to find the length in the first quadrant integrating between the limits $\theta = 0$ to $\theta = \frac{\pi}{2}$.

$$I = \int_0^{\pi/2} xy\,ds = \int_0^{\pi/2} a\cos\theta\cdot b\sin\theta\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\,d\theta$$

$$= \frac{ab}{2}\int_0^{\pi/2}\sqrt{a^2\sin^2\theta+b^2\left(1-\sin^2\theta\right)}\,[2\sin\theta\cos\theta]\,d\theta$$

$$= \frac{ab}{2}\int_2^{\pi/2}\sqrt{b^2+\left(a^2-b^2\right)\sin^2\theta}\,d\sin^2\theta$$

Step 3 : Integrating w.r.t. $\sin^2\theta$

$$I = \frac{ab}{2}\left[\frac{[b^2+(a^2-b^2)\sin^2\theta]^{3/2}}{\frac{3}{2}\cdot(a^2-b^2)}\right]_0^{\pi/2}$$

$$= \frac{ab}{3(a^2-b^2)}\left[\left(a^2\right)^{3/2}-\left(b^2\right)^{3/2}\right]$$

$$= \frac{ab}{3(a^2-b^2)}[a^3-b^3] = \frac{ab}{3(a^2-b^2)}(a-b)\left(a^2+ab+b^2\right)$$

$$I = \frac{1}{3}\left[\frac{ab\left(a^2+ab+b^2\right)}{(a+b)}\right] \qquad \text{... Ans}$$

Step 3 : Equation (3) and (4) are parametric equations of the curve. To get arc length expression we use formula.

$$S = \int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

From equations (3) and (4), by differentiating,

$$\frac{dx}{d\theta} = f''(\theta)\sin\theta + f'(\theta)\cos\theta + f'''(\theta)\cos\theta - f''(\theta)\sin\theta$$

$$\frac{dy}{d\theta} = f''(\theta)\cos\theta - f'(\theta)\sin\theta - f'''(\theta)\sin\theta - f''(\theta)\cos\theta$$

$$\therefore \quad \frac{dx}{d\theta} = \cos\theta\{f'(\theta) + f'''(\theta)\}$$

$$\frac{dy}{d\theta} = -\sin\theta\{f'(\theta) + f'''(\theta)\}$$

Substituting in formula for S.

$$S = \int \sqrt{\{f'(\theta) + f'''(\theta)\}^2 (\cos^2\theta + \sin^2\theta)}\, d\theta + c$$

$$= \int \{f'(\theta) + f'''(\theta)\}\, d\theta + c$$

Integrating we get, $S = f(\theta) + f''(\theta) + c$ which is the arc length expression for the curve.

Example 9.15 : *For the curve* $x = (a + b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right)$, $y = (a + b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right)$. *Show that* $S = \frac{4a}{a}(a + b)\cos\left(\frac{a\theta}{2b}\right)$; *S being measured from the vertex* $\theta = \frac{\pi b}{a}$.

Solution :

Step 1 :

$$x = (a + b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right)$$

$$\frac{dx}{d\theta} = -(a + b)\sin\theta + (a + b)\sin\left(\frac{a+b}{b}\theta\right)$$

$$y = (a + b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right)$$

$$\frac{dy}{d\theta} = (a + b)\cos\theta - (a + b)\cos\left(\frac{a+b}{b}\theta\right)$$

Step 2 : x = a (t – tanh t)

$$\frac{dx}{dt} = a\left(1 - \sec^2 t\right) = a \tanh^2 t$$

$$y = a \operatorname{sech} t$$

$$\frac{dy}{dt} = -a \operatorname{sech} \cdot \tanh t$$

Step 3 :

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = a^2 \tanh^4 t + a^2 \operatorname{sech}^2 t \tanh^2 t$$

$$= a^2 \cdot \tanh^4 + a^2 \operatorname{sech}^2 t \tanh^2 t$$

$$= a^2 \tanh^2 t \cdot (\tanh^2 + \operatorname{sech}^2 t)$$

$$= a^2 \tanh^2 t(1)$$

From (1) $$S = \int_0^t \sqrt{a^2 \tanh^2 t}\, dt = a\,[\log \cosh t]_0^t$$

$$\therefore \quad S = a \log \cosh t$$

9.2 Astroid

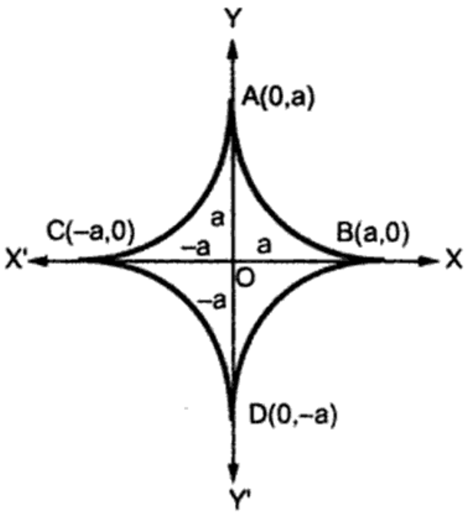


Fig. 9.9

Example 9.17 : *Show that in the Astroid $x^{2/3} + y^{2/3} = a^{2/3}$, $S^3 \propto x^2$, S being measured from the cusp which lies on Y-axis.*

Solution :

Step 1 : A (0, a) is a cuspon Y-axis. P (x, y) is any point on the curve where arc AP = S.

Step 5 : Thus, total length of the curve (1)

$$S = 4\int_0^{\pi/2} \sqrt{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]}\, d\theta$$

$$= 4\int_0^{\pi/2} 3\cos\theta \sin\theta \sqrt{(a^2\cos^2\theta + b^2\sin^2\theta)}\, d\theta$$

Put $a^2\cos^2\theta + b^2\sin^2\theta = t$

$$\Rightarrow (-2a^2\cos\theta\sin\theta + 2b^2\sin\theta\cos\theta)\, d\theta = dt$$

$$\cos\theta \sin\theta\, d\theta = \frac{dt}{2(b^2 - a^2)}$$

$$\therefore \quad 12\int_{a^2}^{b^2} t^{1/2} \frac{dt}{2(b^2 - a^2)} = \frac{6}{(b^2 - a^2)}\left[\frac{t^{3/2}}{3/2}\right]_{a^2}^{b^2}$$

$$= \frac{6}{(b^2 - a^2)}(b^3 - a^3)\cdot\left(\frac{2}{3}\right)$$

$$= 4\frac{(b^2 + ab + a^2)(b - a)}{(b + a)(b - a)}$$

$$= 4\frac{b^2 + ab + a^2}{b + a}$$

9.3 Cycloid

Example 9.20 : *Find the arc of length of cycloid $x = a(\theta + \sin\theta)$; $y = a(1 - \cos\theta)$ from one cusp to another cusp. If s is the length of the arc from origin to point P (x, y). Show that $S^2 = 8ay$.* **(Dec.-2004)**

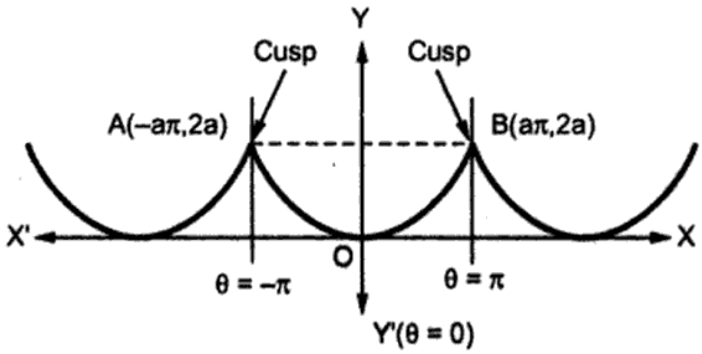


Fig. 9.11

Solution : Step 1 :

Part I

As we are given parametric equations we use formula $S = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

Step 2 : $x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$

$\frac{dx}{d\theta} = a(1 + \cos\theta),\ \frac{dy}{d\theta} = a \sin\theta$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2(1+\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2[1 + 2\cos\theta + \cos^2\theta + \sin^2\theta]$$

$$= 2a^2(1+\cos\theta) = 2a^2\left(2\cos^2\frac{\theta}{2}\right) = 4a^2\cos^2\frac{\theta}{2}$$

Length of arc AB = 2 length OB.

$$S = 2\int_0^{\pi} 2a\cos\frac{\theta}{2}\cdot d\theta = 4a\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 8a$$

which is the required length from one cusp to another cusp.

Step 3 :

Part II

$$S = \int_0^{\theta}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= \int_0^{\theta} 2a\cos\frac{\theta}{2}\,d\theta = 2a\left[2\sin\frac{\theta}{2}\right]_0^{\theta}$$

$$S = 4a\sin\frac{\theta}{2}$$

$$S^2 = 16a^2\sin^2\frac{\theta}{2} = 16a^2\left(\frac{1-\cos\theta}{2}\right)$$

$$8a\big(a(1-\cos\theta)\big) = 8ay$$

Example 9.21 : *Find the length of arc of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ between $x = 0$ to $x = a$.*

Solution :

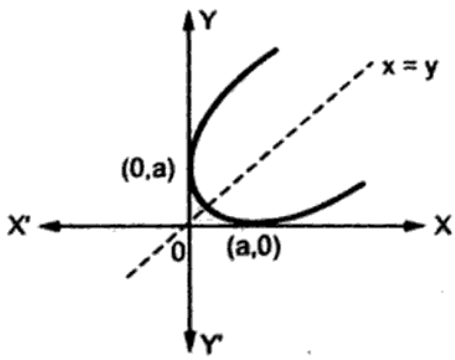


Fig. 9.12

Step 1 : The parametric equations of the curve are $x = a\cos^4\theta$, $y = a\sin^4\theta$.

Step 2 : Arc length is given by,

$$S = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Step 3 : $x = a\cos^4\theta$, $y = a\sin^4\theta$

$$\therefore \quad \frac{dx}{d\theta} = 4a\cos^3\theta\,(-\sin\theta)$$

$$\frac{dy}{d\theta} = 4a\sin^3\theta\cos\theta$$

Step 4 :

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 16a^2\sin^2\theta\cos^2\theta[\cos^4\theta + \sin^4\theta]$$

Step 5 :

$$S = \int_0^{\pi/2} 4a\sin\theta\cos\theta[\sin^4\theta + (1-\sin^2\theta)^2]^{1/2}\, d\theta$$

Put $\sin^2\theta = t$

θ	0	$\pi/2$
t	0	1

$$S = 2a\int_0^{\pi/2}\sqrt{t^2 + (1-t)^2}\,dt$$

$$= 2a\int_0^1 \sqrt{2t^2 - 2t + 1}\,dt$$

$$= 2\sqrt{2}\,a\int_0^1 \sqrt{t^2 - t + \frac{1}{2}}\,dt$$

$$= 2\sqrt{2}\,a\int_0^1 \sqrt{\left(t-\frac{1}{2}\right)^2 + \frac{1}{4}}\,dt$$

Put $t - \frac{1}{2} = u$, $dt = du$

t	0	1
u	$-1/2$	$1/2$

$$= 2\sqrt{2}\,a \int_{-1/2}^{1/2} \sqrt{u^2 + \frac{1}{4}} \cdot du$$

$$= 4\sqrt{2}\,a \int_{0}^{1/2} \sqrt{u^2 + \frac{1}{4}}\, du \qquad [\because \sqrt{u^2 + 1/4} \text{ is an even function of u}$$

$$\because \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \text{ for even function}]$$

$$= 4\sqrt{2}\,a \left[\frac{u}{2}\sqrt{u^2 + \frac{1}{4}} + \frac{1/4}{2}\log\left(u + \sqrt{u^2 + \frac{1}{4}}\right)\right]_0^{1/2}$$

$$= 4\sqrt{2}\,a \left\{\left[\frac{1}{4}\sqrt{\frac{2}{4}} + \frac{1}{8}\log\left(\frac{1}{2} + \sqrt{\frac{2}{4}}\right)\right] - \left[0 + \frac{1}{8}\log\sqrt{\frac{1}{4}}\right]\right\}$$

$$= 4\sqrt{2}\,a \left\{\frac{1}{4\sqrt{2}} + \frac{1}{8}\log\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) - \frac{1}{8}\log\frac{1}{2}\right\}$$

$$= 4\sqrt{2}\,a \left\{\frac{1}{4\sqrt{2}} + \frac{1}{8}\left[\log\frac{\sqrt{2}+2}{2\sqrt{2}}\cdot 2\right]\right\}$$

$$= a\left\{1 + \frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+2}{\sqrt{2}}\right\}$$

$$= a\left\{1 + \frac{1}{\sqrt{2}}\log\left(1 + \sqrt{2}\right)\right\}$$

Exercise 9.2

1) *Find length of arc of the cycloid $x = a(\theta - \sin\theta)$; $y = a(1 - \cos\theta)$ between two consecutive cusps.*

(May-2000) [Ans. : 8a]

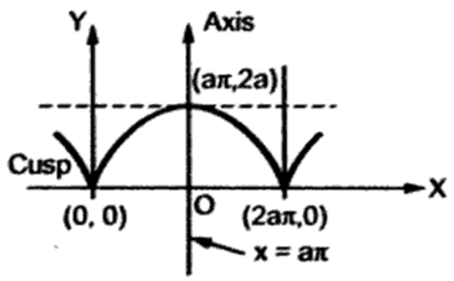


Fig. 9.13

2) *Find centroid of arc of cycloid $x = a(\theta + \sin\theta)$; $y = a(1 - \cos\theta)$ measured from cusp to cusp.*

[Ans. : $0, \frac{2a}{3}$]

3) *If the density at any point of the curve* $x = a(\theta - \sin\theta)$; $y = a(1 - \cos\theta)$ *varies as its distance from the X-axis. Find the distance of its centroid of arc from the X-axis.* **[Ans. :** $\bar{y} = \frac{6a}{5}$**]**

4) *Find the centroid of the arc of the curve* $x = a(\theta + \sin\theta)$; $y = a(1 - \cos\theta)$ *in the positive quadrant.* **(May-98) [Ans. :** $a\left(\pi - \frac{4}{3}\right), \frac{2a}{3}$**]**

5) *Find centroid of arc from cusp to cusp of the cycloid* $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. **[Ans. :** $\left(a\pi, \frac{4a}{3}\right)$**]**

6) *Find length of arc of curve* $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$ *from* $t = 0$ *to* $t = \pi$ **[Ans. :** $2\pi^2 a$**]**

7) *Find length of arc of curve* $x = e^{\theta}\cos\theta$, $y = e^{\theta}\sin\theta$ *from* $\theta = 0$ *to* $\theta = \frac{\pi}{2}$ **[Ans. :** $\sqrt{2}\left(e^{\pi/2} - 1\right)$**]**

8) *Show that the length of the arc of the tractrix* $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $y = a\sin t$ *from* $t = \pi/2$ *to any point t is "a log sin t".*

9) *Prove that the length of the arc of the curve* $x = a\sin 2\theta(1 + \cos 2\theta)$, $y = a\cos 2\theta(1 - \cos 2\theta)$ *measured from origin to any point (x, y) is* $\frac{4}{3}a\sin 3\theta$.

10) *Find centroid of arc bounded by* $\sqrt{x} + \sqrt{y} = \sqrt{a}$ *and the axes in the first quadrant.* **[Ans. :** $\left(\frac{a}{5}, \frac{a}{5}\right)$**]**

[Hint : $x = a\cos^4\theta$, $y = a\sin^4\theta$*]*

11) *Find centroid of arc in positive quadrant of the curve* $x^{2/3} + y^{2/3} = a^{2/3}$ **[Ans. :** $\left(\frac{2a}{5}, \frac{2a}{5}\right)$**]**

12) *Find the length of the arc of the curve* $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ *from* $\theta = 0$ *to* $\theta = 2\pi$.

13) *Show that the length of the arc of the curve* $x = 1 - \cos\theta + \frac{3}{5}$, $y = \frac{4}{5}\sin\theta$ *between* $\theta = 0$ *to* $\theta = \pi$ *is* '$\pi + \frac{6}{5}$'

14) *Find the length of the curve* $x = e^{\theta}\left(\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right)$, $y = e^{\theta}\left(\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right)$ *between* $\theta = 0$ *to* $\theta = \pi$. **[Ans. :** $\frac{5}{2}\left(e^{\pi} - 1\right)$**]**

15) *Show that the length of arc of the curve* $x = \log(\sec\theta + \tan\theta) - \sin\theta$; $y = \cos\theta$ *from* $\theta = 0$ *to* $\theta = t$ *is log sec t.*

16) *Find the distance travelled by a particle P (x, y) whose position at time 't' is given by* $x = \frac{1}{2}t^2$, $y = \frac{1}{3}(2t + 1)^{3/2}$ *from* $t = 0$ *to* $t = 4$. **[Ans. : 12]**

Step 4 : When $\theta = 0$, $t = 0$, $\theta = \frac{3\pi}{2}$, $t = \frac{\pi}{2}$

$$S = a \int_0^{\pi/2} 2 \sin^2 t \cdot 3 \; dt$$

Using reduction formula. we get

$$= 6a \left[\frac{1}{2} \cdot \frac{\pi}{2} \right] = \frac{3\pi a}{2}$$

Example 9.23 : *Find arc length of the curve $r = a e^{m\theta}$ intercepted between radii vectors r_1 and r_2.* **[Dec.-2005]**

Solution :

Step 1 : In this problem it is convenient to use formula.

$$S = \int_{r_1}^{r_2} \sqrt{1 + r^2 \left(\frac{d\theta}{dr} \right)^2} \; dr$$

Step 2 : Differentiating equation of given curve w.r.t. θ, we get

$$\frac{dr}{d\theta} = a \, m e^{m\theta} = mr$$

$$r^2 \left(\frac{d\theta}{dr} \right)^2 = r^2 \cdot \frac{1}{m^2 r^2} = \frac{1}{m^2}$$

$$\therefore \quad S = \int_{r_1}^{r_2} \sqrt{1 + \frac{1}{m^2}} \; dr = \frac{\sqrt{1 + m^2}}{m} [r]_{r_1}^{r_2} = \frac{\sqrt{1 + m^2}}{m} [r_2 - r_1]$$

which is the required arc length.

Example 9.24 : *Prove that length of spiral $r = a e^{\theta \cot \alpha}$ described as r increases from r_1 to r_2 is given by $(r_2 - r_1) \sec \alpha$.*

Solution :

Step 1 : Since limits are interms of r, we use formula,

$$\text{i.e.} \quad S = \int_{r_1}^{r_2} \sqrt{1 + r^2 \left(\frac{d\theta}{dr} \right)^2} \; dr \quad \ldots (1)$$

Step 2 :

$$\because \quad r = a \, e^{\theta \cot \alpha}$$

$$\therefore \quad \frac{dr}{d\theta} = a \cot\alpha \; e^{\theta \cot \alpha} = r \cot\alpha$$

$$\therefore \quad \frac{d\theta}{dr} = \frac{1}{r\cot\alpha}$$

$$\therefore \quad \frac{d\theta}{dr} = \frac{1}{r\cot\alpha} = \frac{\tan\alpha}{r} \qquad \text{From (1)}$$

$$S = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{\tan^2\alpha}{r^2}\right)}\, dr$$

$$= \sec\alpha \int_{r_1}^{r_2} dr = \sec\alpha \cdot (r_2 - r_1)$$

Example 9.25 : *Find the length of parabola* $\frac{2a}{r} = 1 + \cos\theta$ *from vertex to any point* $P(r, \theta)$.

Solution :

Step 1 : The equation $\frac{2a}{r} = 1 + \cos\theta$ represents a parabola with focus at pole 0. V is vertex.

Let, arc VP = S

Then
$$S = \int_0^\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad \ldots (1)$$

Step 2 :

$$\because \quad \frac{2a}{r} = 1 + \cos\theta$$

$$\therefore \quad r = \frac{2a}{1+\cos\theta} = \frac{2a}{2\cos^2\theta/2} = a\sec^2\frac{\theta}{2}$$

$$\therefore \quad \frac{dr}{d\theta} = 2a\sec^2\frac{\theta}{2}\cdot\tan\frac{\theta}{2}\cdot\frac{1}{2}$$

Step 3 : From (1)

$$\therefore \quad S = \int_0^\theta \sqrt{a^2\sec^4\frac{\theta}{2} + a^2\sec^4\frac{\theta}{2}\cdot\tan^2\frac{\theta}{2}}\, d\theta$$

$$S = \int_0^\theta \sqrt{a^2\sec^4\frac{\theta}{2}\left(1+\tan^2\frac{\theta}{2}\right)}\, d\theta$$

Let $\tan\frac{\theta}{2} = t \quad \therefore \quad \frac{1}{2}\sec^2\frac{\theta}{2}\, d\theta = dt$

$$\therefore \quad S = a\int_0^t \sqrt{1+t^2}\; 2\, dt$$

$$= 2a\left[\frac{t\sqrt{t^2+1}}{2}+\frac{1}{2}\log\left(t+\sqrt{t^2+1}\right)\right]_0^{\tan\frac{\theta}{2}}$$

$$= a\left[\tan\frac{\theta}{2}\sec\frac{\theta}{2}+\log\left(\tan\frac{\theta}{2}+\sec\frac{\theta}{2}\right)\right] \qquad \left\{\because t=\tan\frac{\theta}{2}\right\}$$

9.4 Cardiod

Example 9.26 : *Find perimeter of cardiod $r = a(1 + \cos\theta)$ and show that a line $\theta = \frac{\pi}{3}$ divides upper half of the cardiod.*

Solution : Step 1 :

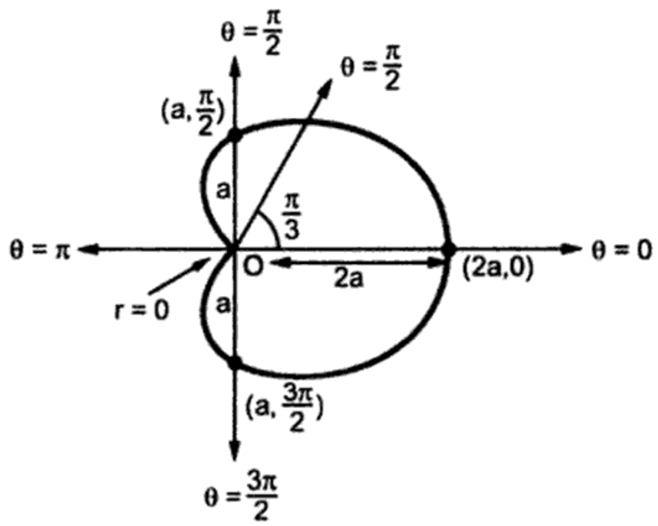


Fig. 9.15

Step 2 : Curve is symmetrical about initial line OX. For upper half of the arc θ varies from $\theta = 0$,to $\theta = \pi$.

Since the curve is given in Polar form we use formula with appropriate limits to find arc length,

$$S = \int_0^{\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta \qquad \text{... (1)}$$

Step 3 : Given

$$r = a(1+\cos\theta)$$

$$\therefore \quad \frac{dr}{d\theta} = -a\sin\theta$$

$$\therefore \quad r^2+\left(\frac{dr}{d\theta}\right)^2 = a^2(1+2\cos\theta+\cos^2\theta+\sin^2\theta)$$

$$= 2a^2(1+\cos\theta)$$

$$S = \int_{x_1}^{x_2} \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\, dx \qquad \text{... (1)}$$

which gives the length 'S' of the arc 'AB' of the given curve. Proceeding in similar way we get the following table.

Formulae of rectification

Equation	ds	S
$y = f(x)$	$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$	$\int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
$x = f(y)$	$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dy$	$\int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dy$
$x = f_1(t)$, $y = f_2(t)$	$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$	$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
$r = f(\theta)$	$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$	$\int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$
$\theta = f(r)$	$\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$	$\int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{dr}{d\theta}\right)^2}\, dr$

1) ARC. $S = \int ds$

2) C.G. of ARC

$$\therefore \qquad \bar{x} = \frac{\int x\rho ds}{\int \rho\, ds}$$

$$\bar{y} = \frac{\int y\rho ds}{\int \rho\, ds}$$

3) M.I. of ARC – M.I. $= \int P^2 \rho\, ds$

University Questions

May - 2003

1. *Find the length of the arc of the curve* $\left(\frac{x}{a}\right)^{2/3} = \left(\frac{y}{b}\right)^{2/3} = 1$ *in the positive quadrant only.*

[5 Marks]

Dec. - 2003

1. *Find the length of the loop of the curve* $3ay^2 = x(x-a)^2 (a > 0)$ **[6 Marks]**

May - 2004

1. *Find the length of the cardiod* $r = a(1+\cos\theta)$ *which lies outside the circle* $r + a\cos\theta = 0$ **[5 Marks]**

Dec. - 2004

1. *Find the arc length of the cycloid* $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ *from one cusp to another cusp.* **[5 Marks]**

May - 2005

1. *Find the total arc length of the curve* $x^{2/3} + y^{2/3} = a^{2/3}$. **[5 Marks]**

Dec. - 2005

1. *Evaluate* $\int xy\, ds$ *along the arc of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *in the first quadrant.* **[5 Marks]**
2. *Find the length of arc of curve* $r = a e^{m\theta}$ *intercepted between radii vectors* r_1 *and* r_2. **[4 Marks]**

May - 2006

1. *Find the whole length of loop of the curve* $3y^2 = x(x-1)^2$, **[5 Marks]**

Dec. - 2006

1. *Find the length of the upper arc of one loop of the curve* $r^2 = a^2 \cos 2\theta$ **[5 Marks]**

May - 2007

1. *Show that the whole length of the loop of the curve*
   $9y^2 = (x+7)(x+4)^2$ *is* $4\sqrt{3}$. **[5 Marks]**

Dec. - 2007

1. *Find the length of the loop of the curve*
   $x = t$, $y = t(1-t^2/3)$ **[5 Marks]**

May - 2008

1. *Evaluate* $\int xyds$ *along the arc of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *in the first quadrant.* **[5 Marks]**

Dec. - 2008

1. *Find the length of the upper arc of one loop of the curve* $r^2 = a^2 \cos 2\theta$ **[5 Marks]**

❑❑❑

10

Planes and Lines

10.1 Introduction

We know that to describe a point in two dimensions we use ordered pair (x, y) or (r, θ) where (x, y) are called Cartesian co-ordinates of a point and (r, θ) are called polar co-ordinates of a point.

In this chapter we study the description of a point in space. There are three simple methods of describing a point in space.

1) The Cartesian co-ordinate system (x, y, z)

2) Spherical polar co-ordinate system (r, θ, ϕ)

3) Cylindrical polar co-ordinate system. (ρ, ϕ, θ).

10.2 The Cartesian Co-ordinate System

In three dimensional geometry we have three mutually perpendicular lines X'OX, Y'OY, Z'OZ intersecting at O. These lines are known as co-ordinate axes and

OX indicates positive direction of X-axis.

OY indicates positive direction of Y-axis.

OZ indicates positive direction of Z-axis.

OX' indicates negative direction of X-axis.

OY' indicates negative direction of Y-axis.

OZ' indicates negative direction of Z-axis

The plane in which X-axis, Y-axis lies is XY plane.

The plane in which Y-axis, Z-axis lies is YZ plane.

The plane in which Z-axis, X-axis lies is ZX plane.

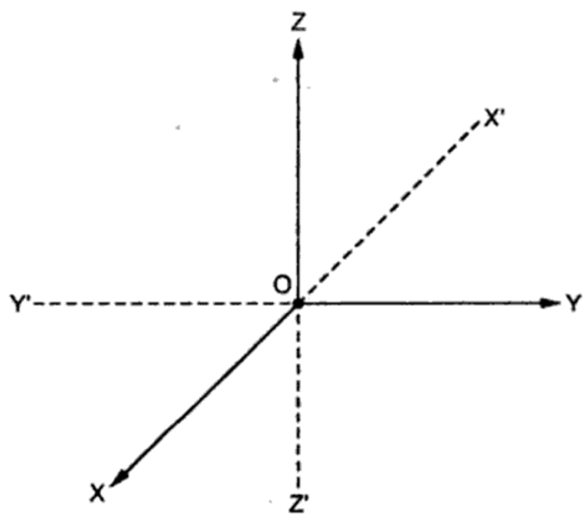


Fig. 10.1

Equation of XY plane or XOY plane is Z = O.

Equation of YZ plane or YOZ plane is X = O.

Equation of ZX plane or ZOX plane is Y = O.

The above three planes are known as co-ordinate planes, these planes divides the entire space in 8 equal parts called as octants. The octant in which X, Y, Z are positive is known as positive octant. Similar to two dimensions the position of a point P in three dimensions is denoted by three real numbers (x, y, z). These are the distances respectively from the origin to the intersection of the perpendicular dropped from point P to X, Y, Z planes respectively.

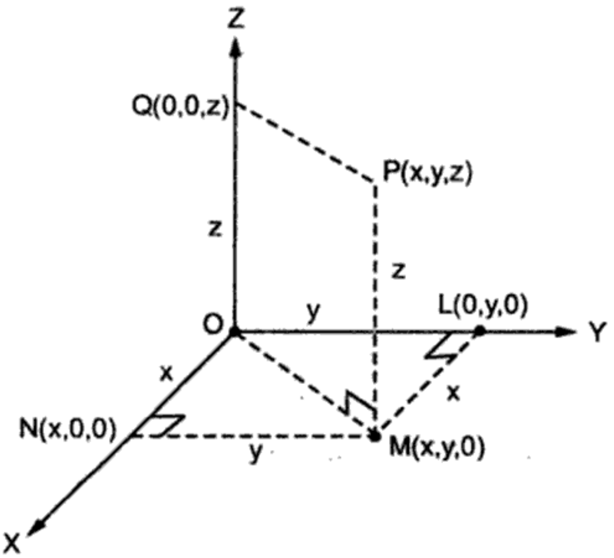


Fig. 10.2

10.3 Spherical Polar Co-ordinate System

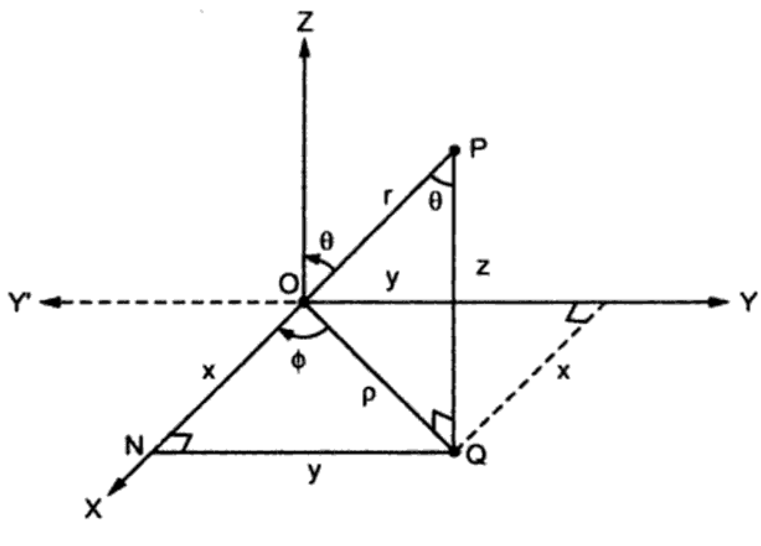


Fig. 10.3

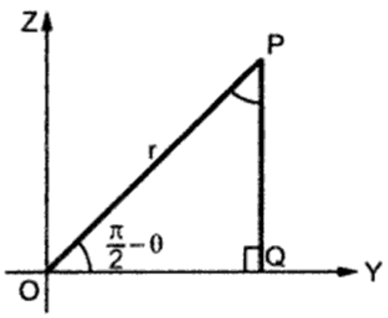


Fig. 10.4

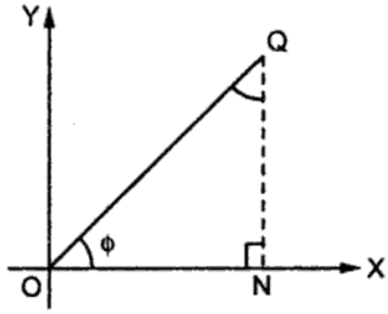


Fig. 10.5

From Fig. 10.3, let OP = r

Let θ be angle made by OP with positive Z-axis.

Let ϕ be angle made by OQ with positive X-axis.

In this system $0 < r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$.

The numbers denoted by (r, θ, ϕ) which can be associated with the point P are called as spherical polar co-ordinates of point P.

10.4 Relation between Cartesian System and Spherical Polar Co-ordinate System

Let (x, y, z) be Cartesian co-ordinates of point P. From Fig. 10.4, OP = Distance of point P from origin.

i.e.

$$\begin{aligned} OP &= \text{Distance between } (0, 0, 0) \text{ and } (x, y, z) \\ &= \sqrt{x^2 + y^2 + z^2} \\ &= r \end{aligned}$$

From Δ OPQ (Fig. 10.4)

$$\left.\begin{aligned} OP &= r \\ PQ &= r \sin\left(\frac{\pi}{2} - \theta\right) \\ \therefore \quad z &= r \cos\theta \\ OQ &= r \cos\left(\frac{\pi}{2} - \theta\right) \\ &= r \sin\theta \end{aligned}\right\} \quad \ldots (1)$$

From Δ ONQ (Fig. 10.5)

$$ON = OQ \cos\phi$$

$$\therefore \qquad x = r \sin\theta \cos\phi \qquad \ldots (2)$$

$$QN = OQ \sin\phi$$

$$y = r \sin\theta \sin\phi \qquad \ldots (3)$$

Thus, from equations (1), (2) and (3)

$$\boxed{\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}}$$

Also from equations (1), (2) and (3), if x, y, z are positive.

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left|\frac{\sqrt{x^2 + y^2}}{z}\right|$$

$$\phi = \tan^{-1}\left|\frac{y}{x}\right|$$

Note :

1) For spherical polar co-ordinates

i) If $z > 0$, then $0 \le \theta \le \pi/2$

ii) If $z < 0$, then $\pi/2 \le \theta \le \pi$

iii) If $x > 0$, $y > 0$ then $0 \le \phi \le \pi/2$

iv) If $x < 0$, $y > 0$ then $\pi/2 \le \phi \le \pi$

v) If $x < 0$, $y < 0$ then $\pi \le \phi \le 3\pi/2$

vi) If $x > 0$, $y < 0$ then $3\pi/2 \le \phi \le 2\pi$

2) If z co-ordinate is + ve then θ is acute i.e $\theta = \tan^{-1}\left|\frac{\sqrt{x^2 + y^2}}{z}\right|$

If z co-ordinate is – ve then θ is obtuse i.e $\theta = \pi - \tan^{-1}\left|\frac{\sqrt{x^2 + y^2}}{z}\right|$

3) The quadrant of the point depends on + ve, and – ve values of x and y, thus ϕ can be calculated using Fig. 10.6 on next page.

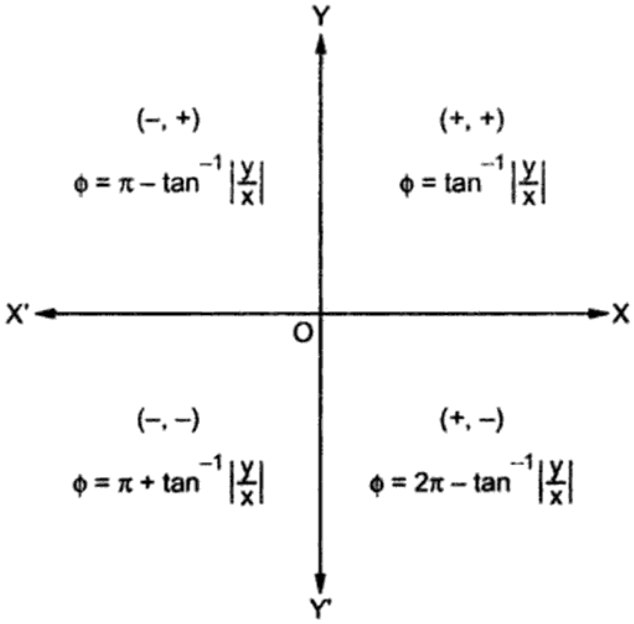


Fig. 10.6

10.5 Cylindrical Polar Co-ordinates

From Fig. 10.3 let $OQ = \rho$

$\therefore \quad PQ = z$

from triangle OQN

$$ON = OQ \cos\phi$$

$$NQ = OQ \sin\phi$$

$\therefore$

$$x = \rho\cos\phi$$

$$y = \rho\sin\phi$$

$$z = z$$

where $\rho = \sqrt{x^2 + y^2}$

$$\phi = \tan^{-1}\left|\frac{y}{x}\right| \qquad \text{(if x, y are + ve)}$$

Note :

1) The quadrant of the point depends on + ve and – ve values of x and y, thus ϕ can be calculated using Fig. 10.6.

10.6 Illustrative Examples

Example 10.1 : *Find the spherical polar and cylindrical co-ordinates of* $(-3, -4, -5)$.

Solution : $r = \sqrt{x^2 + y^2 + z^2}$

Example 10.2 : *Find Cartesian co-ordinates of* $\left(2, \frac{5\pi}{6}, \frac{-3\pi}{4}\right)$.

Solution : Given spherical polar co-ordinates.

$$r = 2, \quad \theta = \frac{5\pi}{6}, \quad \phi = \frac{-3\pi}{4}$$

$$\therefore \quad x = r \sin\theta \cos\phi$$

$$= 2 \sin\left(\frac{5\pi}{6}\right) \cdot \cos\left(\frac{-3\pi}{4}\right)$$

$$= 2 \cdot \left(\frac{1}{2}\right) \cdot \left(-\frac{1}{\sqrt{2}}\right)$$

$$= -\frac{1}{\sqrt{2}}$$

$$y = r \sin\theta \sin\phi$$

$$= 2 \sin\left(\frac{5\pi}{6}\right) . \sin\left(\frac{-3\pi}{4}\right)$$

$$= 2 \cdot \left(\frac{1}{2}\right) \cdot \left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}$$

$$z = r \cos\theta$$

$$= 2 \cos\left(\frac{5\pi}{6}\right)$$

$$= 2\left(-\frac{\sqrt{3}}{2}\right)$$

$$= -\sqrt{3}$$

Thus $$(x, y, z) = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, -\sqrt{3}\right)$$

Exercise 10.1

1) *Find spherical polar co-ordinates of* $[i.e. (x, y, z) \to (r, \theta, \phi)]$

i) $(1, 1, 1)$ **[Ans. :** $\left(\sqrt{3}, 30^\circ, 45^\circ\right)$**]**

ii) $(3, 4, -5)$ **[Ans. :** $\left(5\sqrt{2}, 135^\circ, 53^\circ 8'\right)$**]**

iii) $(-3, 4, -5)$ **[Ans. :** $\left(5\sqrt{2}, 135^\circ, 126.87^\circ\right)$**]**

iv) $(-1, 2, -3)$ **[Ans. :** $\left(\sqrt{14},\ 143.30°,\ 116.56°\right)$**]**

v) $(-2, -1, -3)$ **[Ans. :** $\left(\sqrt{14},\ 143.40°,\ 206.56°\right)$**]**

2) *Find cylindrical polar co-ordinates of* $\left[i.e\ (x, y, z) \to (\rho, \phi, z)\right]$

i) $(1, 1, 1)$ **[Ans. :** $\left(\sqrt{2}, 45°, 1\right)$**]**

ii) $(-3, 4, 12)$ **[Ans. :** $(5, 126.87°, 12)$**]**

iii) $(2, -1, 3)$ **[Ans. :** $\left(\sqrt{5}, -26.56°, 3\right)$**]**

iv) $(-3, -4, -5)$ **[Ans. :** $(5, 233°8', -5)$**]**

v) $(-1, 2, -3)$ **[Ans. :** $\left(\sqrt{5}, 116.56°, -3\right)$**]**

3) *Find Cartesian co-ordinates of*

i) $(2, -\pi/3, \pi/4)$ **[Ans. :** $\left(\frac{1}{\sqrt{2}}, \frac{-\sqrt{3}}{2}, \frac{1}{\sqrt{2}}\right)$**]**

ii) $(2, \pi/3, \pi/4)$ **[Ans. :** $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 1\right)$**]**

10.7 Important Formulae and Definitions

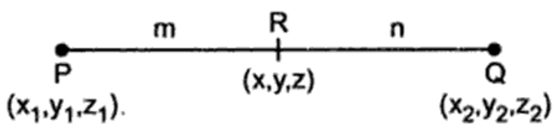
A) Distance formula :

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points then distance between; them is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

B) Section formula :

i) Internal Division :



If the point R divides the line segment PQ internally in the ratio m : n then the co-ordinates of R are given by

$$x = \frac{mx_2 + nx_1}{m + n}, \quad y = \frac{my_2 + ny_1}{m + n}, \quad z = \frac{mz_2 + nz_1}{m + n}$$

ii) External Division : If R divides PQ externally in the ratio m : n then the co-ordinates of R are

D) Direction ratio's of a line :

The numbers a, b, c which are proportional to dc's l, m, n are known as direction ratio's OR dr's of a line. i.e. $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm \frac{1}{\sqrt{a^2+b^2+c^2}}$

i) We can calculate dc's from dr's a, b, c by using

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

ii) Let A (x_1, y_1, z_1), B (x_2, y_2, z_2). The dr's of line AB are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.

iii) If a_1, b_1, c_1 and a_2, b_2, c_2 are dr's of two lines then

a) Angle between them is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}}$$

b) Two lines are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ = Constant.

c) Two lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

E) Projection formula of a line segment :

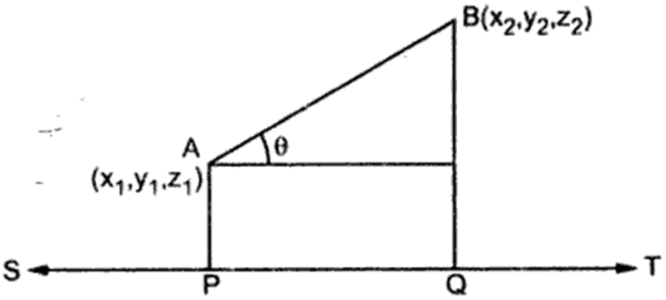


Fig. 10.8

Let l, m, n be dc's of line ST. Let AB be the given line segment its projection on line ST is given by PQ.

where $$PQ = l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

F) Equations of plane :

a) General form :

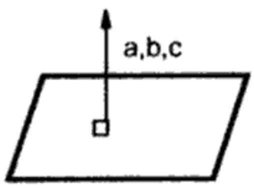


Fig. 10.9

$ax + by + cz + d = 0$ where constants a, b, c are dr's of normal to the plane.

b) Passing through origin :

$$ax + by + cz = 0$$

c) Equation of a plane passing through (x_1, y_1, z_1) and having a, b, c as dr's of normal is given by

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

d) Intercept form : The plane which makes intercepts a, b, c on co-ordinate axes is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

e) Normal form : If l, m, n are dc's of normal to the plane and 'p' is the length of perpendicular from origin to the plane, then its equation is given by

$$lx + my + nz = p$$

f) The equation of plane parallel to the plane $ax + by + cz + d = 0$ is

$ax + by + cz + d_1 = 0$

Note:

i) From any equation of plane the coefficients of x, y, z gives the dr's of normal to the plane.

ii) Two planes are parallel if their normals are parallel.

iii) Two planes are perpendicular if their normals are perpendicular.

iv) Angle between two planes $a_1x+b_1y+c_1z+d_1 = 0$ and $a_2x+b_2y+c_2z+d_2 = 0$ is the angle between their normals having dr's a_1, b_1, c_1 and a_2, b_2, c_2 respectively.

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

G) Length of perpendicular :

a) The length of perpendicular from a point (x_1, y_1, z_1) to the plane

$ax + by + cz + d = 0$ is

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Length of perpendicular from (0, 0, 0) to the plane

$ax + by + cz + d = 0$ is

$$p = \left|\frac{d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

b) Equation of the plane passing through the intersection of two planes

$$a_1x + b_1y + c_1z + d_1 = 0 \text{ and}$$

$$a_2x + b_2y + c_2z + d_2 = 0 \text{ is}$$

$$(a_1x + b_1y + c_1z + d_1) + \lambda\ (a_2x + b_2y + c_2z + d_2) = 0$$

where λ is a parameter.

c) The equation of plane passing through three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

H) Equations of a line :

i) As straight line is the intersection of two planes.

ii) Two point formula : The line joining $(x_1,\ y_1,\ z_1)$ and $(x_2,\ y_2,\ z_2)$ is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

iii) Symmetrical form : The line having dr's a, b, c and passing through $(x_1,\ y_1,\ z_1)$ is given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

I) Coplanirity of two lines :

Two lines $\dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{m_1} = \dfrac{z - z_1}{n_1}$ and

$$\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2} \text{ are}$$

coplanar if $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

It passes through (0, – 2, 4)

$$\Rightarrow \quad 20 - 4v + 8w + d = 0 \quad \text{... (2)}$$

It passes through (3, 1, 4)

$$\Rightarrow \quad 26 + 6u + 2v + 8w + d = 0 \quad \text{... (3)}$$

It passes through (1, 2, 3)

$$\Rightarrow \quad 14 + 2u + 4v + 6w + d = 0 \quad \text{... (4)}$$

It passes through (4, – 4, 2)

$$\Rightarrow \quad 36 + 8u - 8v + 4w + d = 0 \quad \text{... (5)}$$

Solving simultaneously equations (2), (3), (4), (5)

we get $\quad u = -2, v = 1, w = -1, d = -8$

Substituting in equation (1)

$$x^2 + y^2 + z^2 - 4x + 2y - 2z - 8 = 0$$

is the required equation of sphere.

Exercise 11.1

1) *Find the equation of the sphere whose centre is* $(2, -3, 1)$ *and radius 5.*

[Ans. : $x^2 + y^2 + z^2 - 4x + 6y - 2z - 11 = 0$]

2) *Find the equation of the sphere whose centre is* $(-3, 1, 4)$ *and radius 5.*

[Ans. : $x^2 + y^2 + z^2 + 6x - 2y - 8z + 1 = 0$]

3) *Find the equation of the sphere with centre* $(2, -2, 3)$ *and radius 5.*

[Ans. : $x^2 + y^2 + z^2 - 4x + 4y - 6z - 13 = 0$]

4) *Find the centre and radius of sphere*

i) $x^2 + y^2 + z^2 - 4y - 6z + 5 = 0$ **[Ans. : $(-1, 2, 3), 3$]**

ii) $2x^2 + 2y^2 + 2z^2 - x + 3y - 5z = 10$ **[Ans. : $\left(\frac{1}{4}, \frac{-3}{4}, \frac{5}{4}\right), \frac{\sqrt{115}}{4}$]**

5) *Find the equation of the sphere on the join of* $(2, -3, 1)$ *and* $(1, -2, 1)$ *as diameter.*

[Ans. : $x^2 + y^2 + z^2 - 3x + 5y + 7 = 0$]

6) *Find the equation of the sphere on the join of* $(3, 0, 1)$ *and* $(-9, 4, -9)$ *as diameter.*

[Ans. : $x^2 + y^2 + z^2 + 6x - 4y + 8z - 36 = 0$]

7) *Find the equation of a sphere which passes through origin and makes equal intercepts of unit length on the axes.* **[Ans. : $x^2 + y^2 + z^2 - x - y - z = 0$]**

8) *Find the equation of sphere circumscribing the tetrahedron whose faces are*

i) $x = 0, y = 0, z = 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$ **[Ans. : $x^2 + y^2 + z^2 - ax - by - cz = 0$]**

ii) $x = 0, y = 0, z = 0, ax + by + cz + d = 0.$ **[$x^2 + y^2 + z^2 + d\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$]**

9) *Find the equation of sphere circumscribing the tetrahedron, having faces as* $\frac{x}{a} + \frac{y}{b} = 0, \frac{y}{b} + \frac{z}{c} = 0,$

$\frac{x}{a} + \frac{z}{c} = 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

[Ans. : $\left(\frac{x^2 + y^2 + z^2}{a^2 + b^2 + c^2}\right) - \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$]

Hint : Solving three equations at a time we get four points $(0, 0, 0)$ $(a, b, -c)$ $(a, -b, c)$ $(-a, b, c)$.
Then find the sphere passing through four points.

10) *Find the equation of sphere inscribed in the tetrahedron having faces* $x + y = 0, y + z = 0,$ $x + z = 0$ *and* $x + y + z = 1.$

[Ans. : $x^2 + y^2 + z^2 - 2a(x + y + z) + a^2 = 0$ where $a = \frac{3 + \sqrt{6}}{3}$]

Hint : *Let* (a, b, c) *be the centre, equate the radius. Find* $a = \frac{3 \pm \sqrt{6}}{3}$ *and use centre radius form.*

11) *Find the equation of the sphere touching the co-ordinate planes and the plane* $x + y + z = 8.$
[Ans. : $x^2 + y^2 + z^2 - 2a(x + y + z) + 2a^2 = 0$ where $a = \frac{8}{3 - \sqrt{3}}$]

12) *Find the equation of the sphere which passes through the points* $(1, -4, 3), (1, -5, 2), (1, -3, 0)$ *and whose centre lies on the plane* $x + y + z = 0.$ **[Ans. : $x^2 + y^2 + z^2 - 4x + 7y - 3z + 15 = 0$]**

13) *Find the equation of the sphere which passes through* $(2, 1, 1), (0, 3, 2)$ *and has its centre on the line* $2x + y + 3z = 0 = x + 2y + 2z.$ **[Ans. : $9(x^2 + y^2 + z^2) + 28x + 7y - 21z - 96 = 0$]**

14) *Obtain the equation of the sphere passing through the points* $(3, 0, 2), (-1, 1, 1), (2, -5, 4)$ *and having its centre on the plane* $2x + 3y + 3z = 6.$ **[Ans. : $x^2 + y^2 + z^2 + 4y - 6z = 1$]**

15) *Find the equation of sphere passing through*

i) $(4, -1, 2), (0, -2, 3), (1, 5, -1), (2, 0, 1)$ **[Ans. : $x^2 + y^2 + z^2 - 4x - 14y - 22z + 25 = 0$]**

ii) $(0\,0\,0), (0, -1, 1), (-1, 2, 0), (1, 2, 3)$ **[Ans. : $7(x^2 + y^2 + z^2) + 9x - 13y - 27z = 0$]**

iii) $(1, 0, -1), (2, 1, 0), (1, 1, -1), (1, 1, 1)$ **[Ans. : $x^2 + y^2 + z^2 - 2x - y = 0$]**

16) *Find the equation of sphere which passes through the origin and makes equal intercepts of length two on axes.* **[Ans. : $x^2 + y^2 + z^2 - 2(x + y + z) = 0$]**

11.4 Touching Spheres (Externally and Internally)

(i) Two spheres are said to be touch externally, if the distance between their centres is equal to the sum of their radii i.e. $d = r_1 + r_2$. Refer Fig. 11.3.

$\therefore$ The dr's of QC are 4, 3, 0.

As co-ordinates of Q are (8, 5, 4).

$\therefore$ Equation of QC passing through (8, 5, 4) having dr's 4, 3, 0 is

$$\frac{x-8}{4} = \frac{y-5}{3} = \frac{z-4}{0} = k \text{ (say)}$$

$\therefore$ Co-ordinates of C are (4k + 8, 3k + 5, 4). Using distance formula for QC.

$$QC = \sqrt{(4k+8-8)^2 + (3k+5-5)^2 + (4-4)^2}$$

$$= \sqrt{25\,k^2}$$

As the required sphere touches internally to the given sphere

$$x^2 + y^2 + z^2 = 1 \qquad (\therefore C_1 \text{ is } (0\ 0\ 0) \text{ and radius} = 1)$$

$$\therefore \quad QC = CC_1 + 1$$

$$\therefore \quad 5k = \sqrt{(4k+8)^2 + (3k+5)^2 + (4)^2} + 1$$

$$\therefore \quad (5k-1)^2 = (4k+8)^2 + (3k+5)^2 + 4^2$$

Solving we get

$$k = -1$$

$\therefore$ Co-ordinates of C are (4, 2, 4) and radius = 5.

$\therefore$ The required equation is

$$(x-4)^2 + (y-2)^2 + (z-4)^2 = 0$$

i.e. $x^2 + y^2 + z^2 - 8x - 4y - 8z + 11 = 0$

Example 11.3 : ***Prove that the two spheres $x^2 + y^2 + z^2 - 2x + 4y - 4z = 0$ and $x^2 + y^2 + z^2 + 10x + 2z + 10 = 0$ touch each other and find the co-ordinates of point of contact.***

Solution : From the given equations.

$$C_1 = (1, -2, 2) \text{ and } C_2 = (-5, 0, -1)$$

$$r_1 = 3, \quad r_2 = 4$$

Now $\quad d = C_1C_2 = \sqrt{(-5-1)^2 + (0+2)^2 + (-1-2)^2} = 7$

Thus $\quad d = r_1 + r_2$

∴ The two spheres touch each other externally

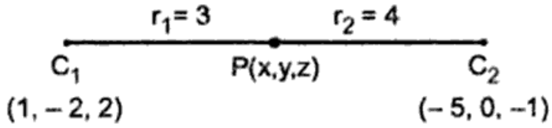


Fig. 11.6

The point of contact P (x, y, z) divides line C_1, C_2 internally in the ratio 3 : 4.

$$\left[\text{i.e. } x = \frac{mx_2 + nx_1}{m+n},\ y = \frac{my_2 + ny_1}{m+n},\ z = \frac{mz_2 + nz_1}{m+n}\right]$$

$$\therefore \quad x = \frac{3(-5) + 4(1)}{3+4} = \frac{-11}{7}$$

$$y = \frac{3(0) + 4(-2)}{3+4} = \frac{-8}{7}$$

$$z = \frac{3(-1) + 4(2)}{3+4} = \frac{5}{7}$$

∴ The point of contact is $\left(\frac{-11}{7}, \frac{-8}{7}, \frac{5}{7}\right)$

Example 11.4 : ***Find the equation of sphere passing through (1 0 0), (0, 1, 0), (0, 0, 1) and having least possible radius.***

Solution : Let $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

be the equation of sphere.

This passes through (1, 0, 0)

$$\Rightarrow \quad 1 + 2u + d = 0$$

i.e. $$u = \frac{(d+1)}{-2}$$

Similarly, $$v = \frac{1+d}{-2},\ w = \frac{1+d}{-2}$$

We know that

$$r^2 = u^2 + v^2 + w^2 - d$$

$$\therefore \quad r^2 = \frac{3(d+1)^2}{4} - d$$

$$= \frac{3(d^2 + 2d + 1) - 4d}{4}$$

$$r^2 = \frac{3d^2 + 2d + 3}{4} = f(d) \text{ say}$$

For least possible radius $f'(d) = 0$

i.e. $6d + 2 = 0$

$\therefore \quad d = -\frac{1}{3}$

$\therefore \quad 2u = 2v = 2w = -\frac{2}{3}$

Hence the required equation of sphere is

$$x^2 + y^2 + z^2 - \frac{2}{3}(x + y + z) - \frac{1}{3} = 0$$

Example 11.5 : ***A sphere with its centre in positive quadrant passes through origin and cuts the co-ordinate planes XOY, YOZ, ZOX in circles of radii $\sqrt{10}$, $\sqrt{2}$, $\sqrt{10}$ respectively. Find the equation of the sphere.***

Solution : Let $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ... (1)

be the general equation of sphere.

As this sphere passes through origin

$\therefore \quad d = 0$

Also the equation of XOY plane is $z = 0$

Substituting we get

$x^2 + y^2 + 2ux + 2vy = 0$

Radius of this circle is $\sqrt{10}$ (given)

$\therefore \quad \sqrt{u^2 + v^2} = \sqrt{10}$

i.e. $u^2 + v^2 = 10$

Similarly $v^2 + w^2 = 2$ and $u^2 + w^2 = 10$

Solving we get $u^2 = 9$, $v^2 = 1$, $w^2 = 1$

$\therefore \quad u = \pm 3, \; v = \pm 1, \; w = \pm 1$

As the centre of the sphere $(-u, -v, -w)$ lies in positive octant

$\therefore \quad u = -3, \; v = -1, \; w = -1$

Hence $x^2 + y^2 + z^2 - 6x - 2y - 2z = 0$

is the required sphere.

As x-axis touches the sphere the two roots of this equation must be equal i.e. discriminant = 0

$$\therefore \quad 4u^2 - 4d = 0$$

$$\therefore \quad u^2 = d$$

Similarly $\quad v^2 = d, \; w^2 = d$

Substituting in equation (2) we get

$$d = 8$$

$$\therefore \quad u^2 = v^2 = w^2 = 8$$

$$\therefore \quad u = v = w = \pm 2\sqrt{2}$$

As the centre is in positive octant

$$\therefore \quad (-u, -v, -w) = \left(2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2}\right)$$

$\therefore$ The equation of sphere is

$$x^2 + y^2 + z^2 - 4\sqrt{2}(x + y + z) + 8 = 0$$

Example 11.8 : ***Find the equation of sphere which touches co-ordinate planes, whose centre is in the positive octant and has radius 11.***

Solution : As the sphere touches the co-ordinate planes and its radius is 4 $\therefore$ its centre will be (4, 4, 4)

$\therefore$ required equation

$$(x - 4)^2 + (y - 4)^2 + (z - 4)^2 = 16.$$

Exercise 11.2

1) *Find the equation of the sphere which has its centre in positive quadrant of XOY plane and which cuts the plaens $x = 0$, $y = 0$, $z = 0$ in circles of radii 3, 4, 5 respectively.*

[Ans. : $x^2 + y^2 + z^2 - 8x - 6y = 0$]

Hint : Refer solved Example No. 11.4

2) *A sphere with its centre in positive octant passes through the origin and cuts the co-ordinate planes in circles of radii $a\sqrt{2}$, $b\sqrt{2}$, $c\sqrt{2}$. Find its equation.*

[Ans. : $\frac{1}{2}(x^2 + y^2 + z^2) - \left(\sqrt{b^2 + c^2 - a^2}\right) x - \left(\sqrt{c^2 + a^2 - b^2}\right) y - \left(\sqrt{a^2 + b^2 - c^2}\right) z = 0$]

Example 11.10 : *A plane passes through a fixed point (a, b, c) and meets the co-ordinate axes in A, B, C. Show that the locus of the centre of the sphere OABC is* $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$.

[May-2003, May-2005]

Solution : Let $A(x_1, 0, 0)$, $B(0, y_1, 0)$, $C(0, 0, z_1)$ be the co-ordinates

∴ Equation of sphere OABC is

$$x^2 + y^2 + z^2 - xx_1 - yy_1 - zz_1 = 0 \qquad \dots (1)$$

whose centre is $\left(\frac{x_1}{2}, \frac{y_1}{2}, \frac{z_1}{2}\right)$

Now equation of plane passing through point A, B, C in intercept form is

$$\therefore \quad \frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$$

This plane passes through (a, b, c)

$$\therefore \quad \frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1} = 1 \qquad \dots (2)$$

Let $\bar{x}, \bar{y}, \bar{z}$ be the point on the locus i.e. centre of the sphere.

$$\therefore \quad \bar{x} = \frac{x_1}{2}, \ \bar{y} = \frac{y_1}{2}, \ \bar{z} = \frac{z_1}{2}$$

$$\therefore \quad x_1 = 2\bar{x}, \ y_1 = 2\bar{y}, \ z_1 = 2\bar{z}$$

Substituting in equation (2)

$$\frac{a}{2\bar{x}} + \frac{b}{2\bar{y}} + \frac{c}{2\bar{z}} = 1$$

Replacing $\bar{x}\ \bar{y}\ \bar{z}$ by x y z.

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$$

Example 11.11 : *A sphere of constant radius r passes through the origin and cuts the axes in A, B, C. Prove that the foot of the perpendicular from origin to the plane ABC is given by* $(x^2 + y^2 + z^2)^2 \left[\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right] = 4r^2$

Solution : Let $A(x_1, 0, 0)$, $B(0, y_1, 0)$, $C(0, 0, z_1)$ be the co-ordinates of A, B, C.

∴ Equation of plane passing through A, B, C in intercept form is

$$\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1 \qquad \dots (1)$$

and equation of sphere OABC is

$$x^2 + y^2 + z^2 - xx_1 - yy_1 - zz_1 = 0 \quad \text{... (2)}$$

whose radius is constant

$$\therefore \quad \sqrt{\frac{x_1^2}{4} + \frac{y_1^2}{4} + \frac{z_1^2}{4}} = r$$

$$\text{i.e.} \quad x_1^2 + y_1^2 + z_1^2 = 4r^2 \quad \text{... (3)}$$

Let $P(\bar{x}, \bar{y}, \bar{z})$ be the foot of the perpendicular from origin to the plane ABC

$\therefore$ dr's of OP (Normal to the plane ABC)

are $\bar{x} - 0, \bar{y} - 0, \bar{z} - 0$ i.e. $\bar{x}, \bar{y}, \bar{z}$

We know that $P(\bar{x}, \bar{y}, \bar{z})$ is one point on the plane

$\therefore$ Equation of plane ABC is

$$\bar{x}(x - \bar{x}) + \bar{y}(y - \bar{y}) + \bar{z}(z - \bar{z}) = 0$$

$$x\bar{x} + y\bar{y} + z\bar{z} = \bar{x}^2 + \bar{y}^2 + \bar{z}^2$$

$$= H \text{ (say)}$$

$$\therefore \quad \frac{x}{(H/\bar{x})} + \frac{y}{(H/\bar{y})} + \frac{z}{(H/\bar{z})} = 1 \quad \text{... (4)}$$

Comparing with equation (1)

$$x_1 = \frac{H}{\bar{x}}, \; y_1 = \frac{H}{\bar{y}}, \; z_1 = \frac{H}{\bar{z}}$$

Substituting in equation (3)

$$H^2\left[\frac{1}{(\bar{x})^2} + \frac{1}{(\bar{y})^2} + \frac{1}{(\bar{z})^2}\right] = 4r^2$$

Substituting H and replacing $\bar{x}, \bar{y}, \bar{z}$ by x y z we get

$$(x^2 + y^2 + z^2)^2\left[\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right] = 4r^2$$

Example 11.12 : *A plane passes through a fixed point (a, b, c). Show that the locus of the foot of the perpendicular to it from the origin is the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$*

Solution : Let $P(\bar{x}, \bar{y}, \bar{z})$ be the foot of the perpendicular

$\therefore$ dr's of OP (i..e dr's of normal to the plane ABC are $(\bar{x} - 0, \bar{y} - 0, \bar{z} - 0)$ and (a, b, c) is one point on the plane ABC.

$\therefore$ its equation is given by,

$$\bar{x}(x - a) + \bar{y}(y - b) + \bar{z}(z - c) = 0 \qquad \text{... (1)}$$

As $P(\bar{x}\ \bar{y}\ \bar{z})$ is one point on the plane it satisfies equation (1)

$$\therefore \quad \bar{x}(\bar{x} - a) + \bar{y}(\bar{y} - b) + \bar{z}(\bar{z} - c) = 0$$

Replacing $\bar{x}$, $\bar{y}$, $\bar{z}$ by x, y, z we get

$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

Example 11.13 : *A point moves so that the sum of the squares of its distances from the six faces of a cube is constant. Show that its locus is a sphere.*

Solution : Take the centre of a cube as (0, 0, 0). Let $P(\bar{x}, \bar{y}, \bar{z})$ be any point on the locus. Distances of this point from the 6 faces are

$$a - \bar{x},\ a + \bar{x},\ a - \bar{y},\ a + \bar{y},\ a - \bar{z},\ a + \bar{z}.$$

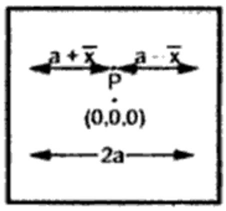


Fig. 11.7

Given

$$(a - \bar{x})^2 + (a + \bar{x})^2 + (a - \bar{y})^2 + (a + \bar{y})^2 + (a - \bar{z})^2 + (a + \bar{z})^2 = K$$

Replacing $\bar{x}$, $\bar{y}$, $\bar{z}$ by x y z we get

$$x^2 + y^2 + z^2 = \left(\frac{K}{2} - 3a^2\right) \text{ which is the sphere.}$$

Example 11.14 : *A is the point $A(1, 3, 4)$ and B is $(1, -2, -1)$. A point P moves so that 3PA = 2PB. Prove that the locus of P is the sphere $x^2 + y^2 + z^2 - 2x - 14y - 16z + 42 = 0$.*

Solution : Let $P(\bar{x}, \bar{y}, \bar{z})$ be the point on the locus

Given $\qquad 3PA = 2PB$

$$\therefore \quad 9(PA)^2 = 4(PB)^2$$

$$\therefore \quad 9[(\bar{x} - 1)^2 + (\bar{y} - 3)^2 + (\bar{z} - 4)^2] = 4[(\bar{x} - 1)^2 + (\bar{y} + 2)^2 + (\bar{z} + 1)^2]$$

i.e. $5\bar{x}^2 + 5\bar{y}^2 + 5\bar{z}^2 - 10\bar{x} - 70\bar{y} - 80\bar{z} + 210 = 0$

Replacing $\bar{x}$ $\bar{y}$ $\bar{z}$ by x y z

we get $x^2 + y^2 + z^2 - 2x - 14y - 16z + 42 = 0$.

Exercise 11.3

1) *A plane passes through the point (a, a, a) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere is $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{2}{a}$.*

   Hint : *Refer solved Example no. 11.10.*

2) *A sphere of constant radius 2R passes through the origin and meets the co-ordinates axes is A, B, C. Find the locus of centroid of.*

   i) *Tetrahedron OABC.* **[Ans. : $x^2 + y^2 + z^2 = R^2$]**

   ii) *Triangle ABC.* **[Ans. : $9(x^2 + y^2 + z^2) = 16R^2$]**

   Hint : *Refer solved Example no. 11.9*

3) *If 'O' the origin be the centre of the sphere of radius 'a' and A, B be two points on the line with 'O' such that $OA \cdot OB = a^2$. If P be a variable point on the sphere. Show that $PA \cdot PB$ = constant.*

4) *A plane passes through a point (2, 2, 2) and cuts the axes in A, B, C. Show that the locus of the centre of the centre of the sphere OABC is $x^{-1} + y^{-1} + z^{-1} = 1$.* **[May-2002]**

5) *A plane passes through (2, 3, 4). Find the locus of the foot of the perpendicualr from the origin.* **[Ans. : $x^2 + y^2 + z^2 - 2x - 3y - 4z = 0$]**

6) *Prove that the locus of a point which moves such that the sum of the squares of its distances from the planes. $x + y + z = 0$, $x - z = 0$, $x - 2y + z = 0$, is 9 is a sphere of radius 3.*

7) *A sphere of radius 3 passes through origin and meets the co-ordinate axes in A, B, C. Find the locus of the centroid of Δ ABC.* **[Dec.-2004] [Ans. : $x^2 + y^2 + z^2 = 11.4$]**

11.7 Tangent Plane

The equation of tangent plane to the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$

at $(x_1\ y_1\ z_1)$ is given by

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0.$$

From Fig. 11.8 dr's of CP are $x_1 + u$, $y_1 + v$, $z_1 + w$.

$\therefore$ Equation of tangent plane is

$$x(x_1 + u) + y(y_1 + v) + z(z_1 + w) = K \text{ (say)} \quad \dots (1)$$

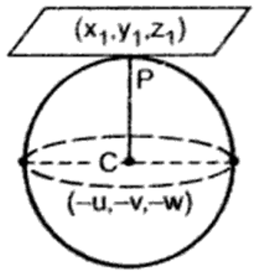


Fig. 11.8

As (x_1, y_1, z_1) lies on equation (1)

$$\therefore x_1(x_1 + u) + y_1(y_1 + v) + z_1(z_1 + w) = K$$

$$\therefore \quad K = x_1^2 + y_1^2 + z_1^2 + x_1 u + y_1 v + z_1 w$$

Substituting in equation (1)

$$x(x_1 + u) + y(y_1 + v) + z(z_1 + w) = x_1^2 + y_1^2 + z_1^2 + x_1 u + y_1 v + z_1 w$$

i.e. $\quad xx_1 + yy_1 + zz_1 + ux + vy + wz = x_1^2 + y_1^2 + z_1^2 + ux_1 + vy_1 + wz_1 \quad \dots (2)$

As (x_1, y_1, z_1) lies on the sphere

$$\therefore x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0.$$

i.e. $\quad x_1^2 + y_1^2 + z_1^2 + ux_1 + vy_1 + wz_1 = -ux_1 - vy_1 - wz_1 - d$

$\therefore$ From equation (2)

$$xx_1 + yy_1 + zz_1 + ux + vy + wz = -ux_1 - vy_1 - wz_1 - d$$

$$\boxed{\therefore \quad xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0}$$

is the required equation of tangent plane to the given sphere at point (x_1, y_1, z_1).

Note :

i) To find the equation of tangent plane to the sphere at (x_1, y_1, z_1)

Replace x^2 by xx_1

Replace y^2 by yy_1

Replace z^2 by zz_1

Replace $2x$ by $x + x_1$

Replace $2y$ by $y + y_1$

Replace $2z$ by $z + z_1$ in the equation of sphere.

ii) **Tangent plane property :** If a plane touches a sphere, then length of perpendicular from the centre of the sphere to the plane must be equal to **radius** of the sphere.

$\therefore$ required planes are

$$2x - y + 2z - 20 = 0$$

or $$2x - y + 2z - 2 = 0$$

Example 11.17 : *Show that the plane $4x - 3y + 6z - 35 = 0$ is tangential to the sphere. $x^2 + y^2 + z^2 - y - 2z - 14 = 0$ and find the point of contact.* **[Dec.-95]**

Solution :

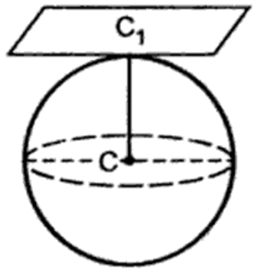


Fig. 11.9

From $$x^2 + y^2 + z^2 - y - 2z - 14 = 0$$

Co-ordinates of centre C are $\left(0, \frac{1}{2}, 1\right)$

radius $$CC_1 = \sqrt{0 + \frac{1}{4} + 1 + 14}$$

$$= \frac{\sqrt{61}}{2}$$

Now perpendicular distance from the centre $\left(0, \frac{1}{2}, 1\right)$ to the plane

$$4x - 3y + 6z - 35 = 0$$

is $$p = \left| \frac{0 - 3\left(\frac{1}{2}\right) + 6(1) - 35}{\sqrt{0 + \frac{1}{4} + 1}} \right| = r$$

$$= \frac{\sqrt{61}}{2}$$

Which is equal to the radius. Thus the given plane is tangential to the sphere.

To find the point of contact (C_1) write the equation of normal line CC_1. From equation of plane the coefficients of (x, y, z) gives dr's of normal.

$\therefore$ (4, – 3, 6) are dr's of normal Co-ordinates of C are $\left(0, \frac{1}{2}, 1\right)$.

$$\therefore \quad \frac{x - 0}{4} = \frac{y - 1/2}{-3} = \frac{z - 1}{6} = t \text{ (Say)}$$

$$\therefore \quad x = 4t, \; y = -3t + \frac{1}{2}, \; z = 6t + 1$$

Substituting in $$4x - 3y + 6z - 35 = 0$$

We get $4(4t) - 3\left(-3t + \frac{1}{2}\right) + 6(6t + 1) - 35 = 0$

$$\Rightarrow \qquad t = \frac{1}{2}$$

Hence substituting in (x, y, z) we get (2, – 1, 4) as the point of contact.

Example 11.18 : *Prove that the line $2(x+1) = 2 - y = z + 3$ touches the sphere $9(x^2 + y^2 + z^2) = 5$ and find the point of contact.*

Solution : Given line is

$$\frac{x+1}{1} = \frac{y-2}{-2} = \frac{z+3}{2} = t \text{ (Say)}$$

$$\therefore \qquad x = t - 1, \; y = -2t + 2, \; z = 2t - 3.$$

Substituting in $\quad 9(x^2 + y^2 + z^2) = 5$ we get

$$9[(t-1)^2 + (-2t+2)^2 + (2t-3)^2] = 5$$

$$\Rightarrow \qquad 81t^2 - 198t + 121 = 0$$

$$\Rightarrow \qquad (9t - 11)^2 = 0$$

$$t = \frac{11}{9} \text{ twice.}$$

$\therefore$ the values of t are real and equal

$\therefore$ the given line intersects the sphere into two co-incident points and hence touches the sphere.

The point of contact is $\left(\frac{2}{9}, \frac{-4}{9}, \frac{-5}{9}\right)$

Example 11.19 : *Find the equation of tangent planes at the points where the line $\frac{x+3}{4} = \frac{y+4}{3} = \frac{z-8}{-5}$ intersects the sphere $x^2 + y^2 + z^2 + 2x - 10y = 23$ ... (1)*

Solution : Let $\frac{x+3}{4} = \frac{y+4}{3} = \frac{z-8}{-5} = t$ (Say)

$$x = 4t - 3, \; y = 3t - 4, \; z = -5t + 8.$$

Substituting in equation (1) we get

$$50t^2 - 150t + 100 = 0$$

$$\Rightarrow \qquad t^2 - 3t + 2 = 0$$

$$\Rightarrow \qquad t = 1, \; t = 2 \text{ Substitute in } (x, y, z)$$

$\therefore$ (1, – 1, 3) and (5, 2, – 2) are the points.

Equation of tangent plane at $(x_1 \ y_1 \ z_1)$ is

$$xx_1 + yy_1 + zz_1 + (x + x_1) - 5(y + y_1) - 23 = 0$$

$\therefore$ at (1, – 1, 3) the tangent plane is

$$x - y + 3z + (x + 1) - 5(y - 1) - 23 = 0$$

$$\Rightarrow \quad 2x - 6y + 3z - 17 = 0 \quad \ldots (2)$$

Also at (5, 2, – 2) the tangent plane is

$$5x + 2y - 2z + (x + 5) - 5(y + 2) - 23 = 0$$

$$\text{i.e.} \quad 6x - 3y - 2z - 28 = 0 \quad \ldots (3)$$

Equation (2) and (3) are the required tangent planes.

Example 11.20 : *Find the equations of tangent planes to the sphere $x^2 + y^2 + z^2 = 9$ which pass through the line $x + y = 6$, $x - 2z = 3$.* **[Dec.-94]**

Note : If $u_1 = 0$ and $u_2 = 0$ are two planes then equation of plane which passes through the line of intersection of given two planes is given by $u_1 + ku_2 = 0$

Solution : The required plane is

$$(x + y - 6) + k(x - 2z - 3) = 0$$

$$\text{i.e.} \quad (1 + k)x + y - 2kz - (6 + 3k) = 0 \quad \ldots (1)$$

The centre of the given sphere is (0, 0, 0) and radius is 3.

The length of perpendicular from (0, 0, 0) to

$$(1 + k)x - y - 2kz - (6 + 3k) = 0 \text{ must be } 3.$$

$$\therefore \quad 3 = \left| \frac{0 - 0 - 0 - (6 + 3k)}{\sqrt{(1 + k)^2 + (1) + (2k)^2}} \right|$$

$$\Rightarrow \quad k = 1, \ k = -\frac{1}{2}$$

Substituting in equation (1) we get

$$2x + y - 2z - 9 = 0$$

$$\text{and} \quad x + 2y + 2z - 9 = 0.$$

Example 11.21 : *Find the equations of tangent planes to the sphere* $x^2 + y^2 + z^2 + 6x - 2z + 1$ *which pass through the line* $3(16 - x) = 3z = 2y + 30$.

[Dec.-98, Dec.-2001]

Solution : From given line

$$48 - 3x = 3z \text{ and } 3z = 2y + 30$$

i.e. $$x + z = 16 \text{ and } 2y - 3z + 30 = 0$$

$\therefore$ The required plane is

$$(x + z - 16) + k(2y - 3z + 30) = 0$$

i.e. $$x + 2ky + (1 - 3k)z - 16 + 30k = 0 \quad \text{... (1)}$$

The centre of the given sphere is (– 3, 0, 1) and the radius is 3.

The length of perpendicular from (– 3, 0, 1) to

$x + 2ky + (1 - 3k)z - 16 + 30k = 0$ must be 3.

$$\therefore \quad 3 = \left|\frac{-3 + 2k(0) + (1 - 3k) - 16 + 30k}{\sqrt{1 + (2k)^2 + (1 - 3k)^2}}\right|$$

$$\Rightarrow \quad 3 = \left|\frac{-18 + 27k}{\sqrt{2 - 6k + 13k^2}}\right|$$

$$(-18 + 27k)^2 = 9(13k^2 - 6k + 2)$$

$$81(9k^2 - 12k + 4) = 9(13k^2 - 6k + 2)$$

$$68k^2 - 102k + 34 = 0$$

$$\therefore \quad 2k^2 - 3k + 1 = 0$$

$$(2k - 1)(k - 1) = 0$$

$$\Rightarrow \quad k = 1, \; k = \frac{1}{2}$$

13) *Find the equation of tangent plane to the sphere* $x^2 + y^2 + z^2 = a^2$ *at* (r, θ, ϕ).

[Ans. : $x \; r\sin\theta\cos\phi + y\; r\sin\theta\sin\phi + z\; r\cos\theta = a^2$]

Hint : $x_1 = r\sin\theta\cos\phi$, $y_1 = r\sin\theta\sin\phi$, $z_1 = r\cos\theta$.

14) *Show that the condition that the plane* $ax + by + cz = P$ *should touch the sphere* $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ *is*

$$(au + bv + cw + p)^2 = (a^2 + b^2 + c^2)^2 (u^2 + v^2 + w^2 - d).$$

15) *If any tangent plane to the sphere* $x^2 + y^2 + z^2 = r^2$ *makes intercepts a, b, c on axes then prove that* $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{r^2}$.

11.8 Illustrations on Sphere Touching the Given Plane

Example 11.22 : ***Find the equation of the sphere which passes through the point (1, 0, 0) and touches the plane 2x – y – 2z = 4 at the point (1, 2, –2).***

Solution : Given plane is

$$2x - y - 2z = 4$$

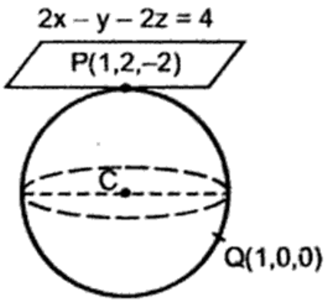


Fig. 11.10

$\therefore$ Coefficients of (x, y, z) i.e. (2, – 1, 2) are the dr's of the normal (i.e. CP).

Co-ordinates of P are (1, 2, – 2).

Thus equation of CP is

$$\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z+2}{-2} = t \text{ (Say)}$$

$$\therefore \quad x = 2t + 1, \; y = -t + 2, \; z = -2t - 2.$$

From Fig. 11.10

$$CP^2 = CQ^2$$

$$(2t + 1 - 1)^2 + (-t + 2 - 2)^2 + (-2t - 2 + 2)^2 = (2t + 1 - 1)^2 + (-t + 2)^2 + (-2t - 2)^2$$

$$\Rightarrow \quad 4t^2 + t^2 + 4t^2 = 4t^2 + t^2 - 4t + 4 + 4t^2 + 8t + 4$$

$$\Rightarrow \quad t = -2$$

$\therefore$ Co-ordinates of 'C' are (– 3, 4, 2)

Also
$$CQ^2 = (-3-1)^2 + (4-0)^2 + (2-0)^2$$
$$= 16 + 16 + 4$$
$$= 36$$

$\therefore$ We use centre radius form

$$(x+3)^2 + (y-4)^2 + (z-2)^2 = 36$$

$$\Rightarrow \quad x^2 + y^2 + z^2 + 6x - 8y - 4z - 7 = 0$$

Example 11.23 : *Find the equations of the spheres passing through the points $(4, 1, 0)$, $(2, -3, 4)$, $(1, 0, 0)$ and touching the plane $2x + 2y - z = 11$.*

Solution : Let (a, b, c) be the centre.

$\therefore$ Length of perpendicular from (a, b, c) to the plane $2x + 2y - z = 11$ will be radius r.

$$r = \left| \frac{2a + 2b - c - 11}{\sqrt{4+4+1}} \right|$$

$$\therefore \qquad (2a + 2b - c - 11)^2 = 9r^2 \qquad \text{... (1)}$$

From the three points.

$$r^2 = (a-4)^2 + (b-1)^2 + c^2 \qquad \text{... (2)}$$

$$= (a-2)^2 + (b+3)^2 + (c-4)^2 \qquad \text{... (3)}$$

$$= (a-1)^2 + b^2 + c^2 \qquad \text{... (4)}$$

(2) = (4) gives $3a + b = 8 \Rightarrow b = 8 - 3a$

(3) = (4) gives $a - 3b + 4c = 14$

$$\therefore \qquad 4c = 14 - a + 3b$$

$$4c = 14 - a + 24 - 9a$$

$$c = \frac{38 - 10a}{4}$$

$$c = \frac{19 - 5a}{2}$$

Also from equation (1)

$$(2a + 2b - c - 11)^2 = 9(a - 1)^2 + b^2 + c^2)$$

Substituting b and c.

$$\left(2a + 16 - 6a - \frac{19}{2} + \frac{5a}{2} - 1\right)^2 = 9\left[(a-1)^2 + (8-3a)^2 + \left(\frac{19-5a}{2}\right)^2\right]$$

Simplifying we get quadratic in a

Example 11.25 : *Find the equation of the sphere, which has its centre at* $(2, 3, -1)$ *and touches the line* $\frac{x+1}{-5} = \frac{y-8}{3} = \frac{z-4}{4}$.

Solution :

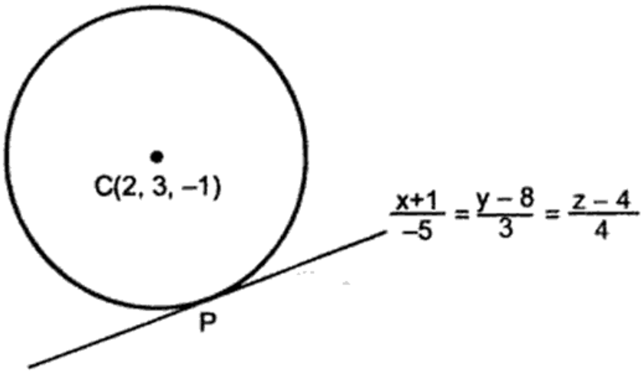


Fig. 11.11

Let P the point of contact of the line with the required sphere

Let $$\frac{x+1}{-5} = \frac{y-8}{3} = \frac{z-4}{4} = k$$

$(-5k - 1, 3k + 8, 4k + 4)$ for some value of k, are the co-ordinates of P.

Direction ratios of CP are $-5k - 3, 3k + 5, 4k + 5$ ... (1)

Direction ratios of tangent line are $-5, 3, 4$. ... (2)

CP is perpendicular to the tangent line

$$\therefore \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$\therefore$ From equation (1), $2 - 5(-5k - 3) + 3(3k + 5) + 4(4k + 5) = 0$

$$25k + 15 + 9k + 15 + 16k + 20 = 0$$

$$50k + 50 = 0, \quad k = -1$$

Substituting we get P as (4, 5, 0)

$$\text{Radius}^2 = CP^2 = (4-2)^2 + (5-3)^2 + (0+1)^2$$

$$= 4 + 4 + 1$$

$$= 9$$

$\therefore$ Equation of required sphere is

$$(x-2)^2 + (y-3)^2 + (z+1)^2 = 9$$

Exercise 11.5

1) *Find the equation of the sphere tangential to the plane $x - 2y - 2z = 7$ at $(3, -1, -1)$ and passing through the point $(1, 1, -3)$.* **[Ans. : $x^2 + y^2 + z^2 - 10y - 10z - 31 = 0$]**

2) *Find the equation of sphere with $x = y = z$ as diameter and touching the planes $x + y + z = 0$ and $x + y + z = 8$.* **[Ans. : $3(x^2 + y^2 + z^2) + 8(x + y + z) = 0$]**

   ***Hint** : Let $(9, 9, 9)$ be the centre.*

3) *Find the equation of the sphere touching three co-ordinate planes and the plane $2x + y + 2z = 6$.* **[Ans. : $x^2 + y^2 + z^2 - 6x - 6y - 6z + 18 = 0$]**

4) *Find the equations of the two spheres in positive octant which.*

   i) *Touches three co-ordinate plane $x + y + z = 1$.* **[Ans. : radius a, centre (a, a, a) where $a = \frac{3 \pm \sqrt{3}}{6}$]**

   ii) *Touches three co-ordinate planes and the plane $x + 2y + 2z = 8$.* **[Ans. : $x^2 + y^2 + z^2 - 2x - 2y - 2z + 2 = 0$ and $x^2 + y^2 + z^2 - 8x - 8y - 8z + 32 = 0$]**

11.9 The Section of a Sphere by a Plane

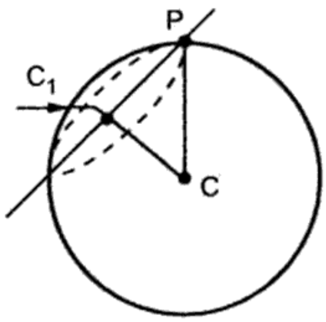


Fig. 11.12

Note :

1) The section of a sphere by a plane gives a circle.
2) The curve of intersecion of two spheres is also a circle.
3) The section of a sphere by a plane through the centre of the sphere is called **great circle**. Its centre and radius are same as that of the given sphere.
4) If $S_1 = 0$ and $S_2 = 0$ are two spheres then $S_1 - S_2 = 0$ represents a plane in which the circle lies.

   It is known as the **radical plane.**

   Radical plane of two spheres is the locus of the point from which the square of the length of tangents of two given spheres are equal.
5) If $S = 0$ and $U = 0$ are the equations of sphere and plane respectively, then $S = 0$, $U = 0$ together represents a circle.
6) If $S_1 = 0$ and $S_2 = 0$ are two spheres then $S_1 = 0$, $S_2 = 0$ together represents a circle.

7) **Sphere through the circle :**

If S = 0 and U = 0 represents a circle then $S + \lambda U = 0$ represents family of spheres passing through the circle.

If $S_1 = 0$ and $S_2 = 0$ represents a circle then $S_1 + \lambda S_2 = 0$ reperesents family of spheres passing through the circle.

11.10 Illustrations on Centre and Radius of Circle : S = 0, U = 0

Example 11.26 : *Find the centre and radius of the circle* $x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0, x + 2y + 2z + 7 = 0$. *Also find the orthogonal projection of the area of circle in XOY plane.*

Solution :

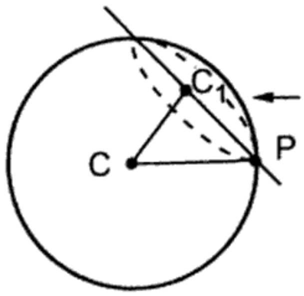


Fig. 11.13

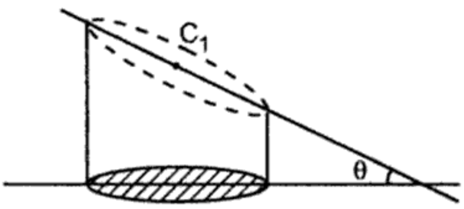


Fig. 11.14

From the equation of the given sphere co-ordinates of C are (– 1, 1, 2) and radius

$$CP = \sqrt{1 + 1 + 4 + 19} = 5$$

CC_1 = Perpendicular distance from C (– 1, 1, 2) to the plane x + 2y + 2z + 7 = 0

$$= \left| \frac{-1 + 2 + 4 + 7}{\sqrt{1 + 4 + 4}} \right| = 4$$

Now from Δ CPC_1

$$CP^2 = CC_1^2 + C_1P^2$$

$$\therefore \quad C_1P^2 = CP^2 - CC_1^2$$

$$= 25 - 16 = 9$$

$\therefore$ Radius of circle = 3

To find co-ordinates of C_1, find the equation of CC_1.

From equation of plane $x + 2y + 2z + 7 = 0$ the coefficients of x, y, z represents dr's of normal $\therefore$ (1, 2, 2,) are dr's of CC_1 and C (– 1, 1, 2) is one point on CC_1 thus

$$\frac{x+1}{1} = \frac{y-1}{2} = \frac{z-2}{2} = t$$

is the equation of CC_1

$$\therefore \quad x = t - 1$$

$$y = 2t + 1$$

$$z = 2t + 2$$

is any point on CC_1

As C_1 lies on $x + 2y + 2z + 7 = 0$

$$\therefore \quad (t-1) + 2(2t+1) + 2(2t+2) + 7 = 0$$

$$\Rightarrow \quad t = -\frac{4}{3}$$

Substituting in x, y, z we get $\left(-\frac{7}{3}, \frac{-5}{3}, -\frac{2}{3}\right)$ as the centre of the circle.

To find the orthogonal projection. Consider dr's of normal to the plane i.e. dr's of CC_1 i.e. (1, 2, 2) and consider dr's of normal to XOY plane i.e. dr's of z-axis i.e. (0, 0, 1)

$$\therefore \quad \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{0+0+2}{\sqrt{1+4+4}\sqrt{0+0+1}} = \frac{2}{3}$$

The orthogonal projection of area of the circle on XOY plane is Area $\cos\theta$.

(see Fig. 11.14)

$$= \left(\pi R^2\right) \cdot \cos\theta$$

$$= \pi \cdot 9 \cdot \frac{2}{3} = 6\pi$$

$$CM^2 = (1+2)^2 + (2-3)^2 + (2+1)^2$$

$$= 9 + 1 + 9 = 19$$

From Δ CMA

$$MA^2 = (CA)^2 - (CM)^2$$

$$= 24 - 19 = 5$$

$\therefore$ radius $= AM = \sqrt{5}$

Exercise 11.6 : Problems on Circle

1) *Find the centre and radius of the circle of intersection of the sphere* $x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$ *by the plane* $x + 2y + 2z = 15$. **(Dec.- 99) [Ans. : (1, 3, 4), $\sqrt{7}$]**

2) *Find the centre and radius of the circle* $x^2 + y^2 + z^2 - 2x + 4y + 2z - 6 = 0$, $x + 2y + 2z - 4 = 0$. *Find the orthogonal projection of the area of circle in YOZ plane.* **(Dec.- 2003)**

[Ans. : (2, 0, 1), $\sqrt{3}$, π]

3) *Determine the centre and radius of the circle* $x^2 + y^2 + z^2 = 16$, $2x + y + 2z = 9$.

[Ans. : (2, 1, 2), $\sqrt{7}$]

4) *Find the centre and radius of*

i) $x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$, $x + 2y + 2z = 15$. **[Ans. : (1, 3, 4), 3]**

ii) $x^2 + y^2 + z^2 - 2x + 4y + 6z - 2 = 0$, $x + 2y + 2z = 20$. **[Ans. : (2, 4, 5), $\sqrt{7}$]**

iii) $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ **[Ans. : $\left(\frac{10}{29}, \frac{15}{29}, \frac{20}{29}\right) \sqrt{\frac{236}{29}}$]**

iv) $x^2 + y^2 + z^2 = 16$, $2x + y + 2z = 9$. **[Ans. : (2, 1, 2) $\sqrt{7}$]**

v) $x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0$, $x + 2y + 2z + 7 = 0$. *Also find orthogonal projection on XOY plane.* **[Ans. : $\left(\frac{-7}{3}, \frac{-5}{3}, \frac{-2}{3}\right)$, 3, Projection 6π]**

5) *Find the centre and radius of the circle passing through the points* (1, 0, 0), (0, 2, 0), (0, 0, 3).

[Ans. : $\left(\frac{13}{98}, \frac{40}{49}, \frac{123}{98}\right)$ Rad $= \frac{5\sqrt{26}}{14}$]

6) *Prove that the radius of the circle* $x^2 + y^2 + z^2 + x + y + z = 4$, $x + y + z = 0$ *is 2.*

7) *Obtain the equation of circle lying on the sphere*

i) $x^2 + y^2 + z^2 + 4x - 6y + 2z - 10 = 0$ *and with centre* (1, 2, 2). **[Ans. : S = 0 and U = 0 together represents a circle where plane is $3x - y + 3z - 7 = 0$]**

ii) $x^2 + y^2 + z^2 - 2x + 4y + 6z + 3 = 0$ *with centre* (2, 3, −4). **[Ans. : S = 0, U = 0 U is $x + 5y - 7z - 45 = 0$]**

iii) $x^2 + y^2 + z^2 - 2x + 4y - 6z - 43 = 0$ *with centre* (2, 1, −4). **[Ans. : S = 0 and $x + 3y - 7z - 33 = 0$]**

Example 11.30 : *Find the equation of sphere which passes through* $(3, 1, 2)$ *and meets XOY plane in a circle of radius 3 units with the centre at* $(1, -2, 0)$.

Solution :

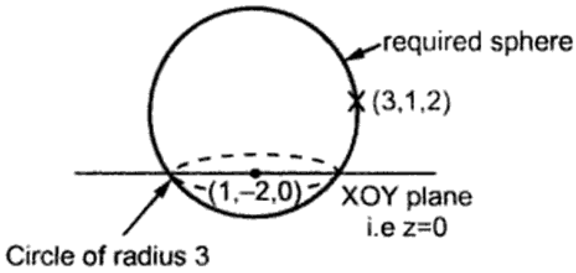


Fig. 11.17

From Fig. 11.17 equation of circle in XOY plane whose centre is $(1, -2, 0)$ and radius 3 is

$$(x-1)^2+(y+2)^2 = 9$$

Thus equation of sphere whose intersection with z = 0 of the above circle is

$$(x-1)^2+(y+2)^2+z^2 = 9 \qquad \text{(Note this step)}$$

$$\therefore \qquad x^2+y^2+z^2-2x+4y-4 = 0$$

Thus the required sphere is

$$S+\lambda U = 0$$

$$x^2+y^2+z^2-2x+4y-4+\lambda z = 0 \qquad \dots (1)$$

which passes through (3, 1, 2)

$$\therefore \qquad 9+1+4-6+4-4+2\lambda = 0$$

$$\Rightarrow \qquad \lambda = -4$$

$\therefore$ Substituting λ in equation (1)

$$x^2+y^2+z^2-2x+4y-4z-4 = 0$$

is the required equation of sphere.

Example 11.31 : *Find the sphere through the circle* $x^2+y^2+z^2 = 4$, $z = 0$, *meeting the plane* $x+2y+2z = 0$ *in a circle of radius 3.* **[Dec.-92, May-95, May-96, May-2001, Dec.-2002, May-2004]**

Solution :

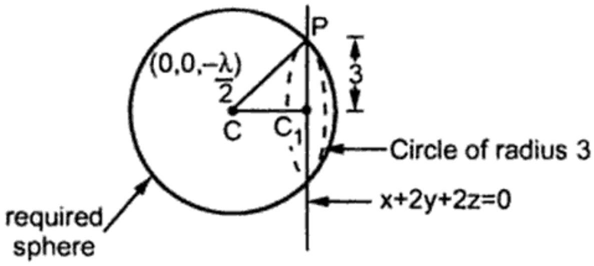


Fig. 11.18

Required sphere is $S + \lambda U = 0$

i.e. $x^2 + y^2 + z^2 - 4 + \lambda z = 0$

centre C is $\left(0, 0, \frac{-\lambda}{2}\right)$ and radius R is $\sqrt{\frac{\lambda^2}{4} + 4}$

$$CC_1 = \text{Perpendicular distance from } \left(0, 0, \frac{-\lambda}{2}\right) \text{ on}$$

$$x + 2y + 2z = 0$$

$$= \left|\frac{0 + 0 - \lambda}{\sqrt{1 + 4 + 4}}\right| = \left|-\frac{\lambda}{3}\right|$$

From $\Delta\, CC_1P$

$$CP^2 = CC_1^2 + C_1P^2$$

$$\frac{\lambda^2}{4} + 4 = \frac{\lambda^2}{9} + 9$$

$$\Rightarrow \qquad \lambda = \pm 6$$

$\therefore$ Substituting in equation (1) we get the required sphere

$$x^2 + y^2 + z^2 \pm 6z - 4 = 0$$

Example 11.32 : *Find the equation of the sphere passing through the circle* $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ *and point* $(1, 2, 3)$.

Solution : Equation of the circle is given by

$$S = x^2 + y^2 + z^2 - 9 = 0$$

and

$$U = 2x + 3y + 4z - 5 = 0$$

Equation of the sphere passing through this circle is

$$S + \lambda U = 0$$

$\Rightarrow \ \lambda = 2$

Substituting in equation (1)

$\therefore$ The required sphere is

$$x^2 + y^2 + z^2 + x + 13y + 2z - 15 = 0$$

Example 11.34 : *Find the equation of the sphere which passes through the circle $x^2 + y^2 + z^2 - 2x + 2z - 2 = 0$, $y = 0$ and touches the plane $y - z - 7 = 0$.*

[Dec.-2002]

Solution : The sphere through the circle $x^2 + y^2 + z^2 - 2x + 2z - 2 = 0$, $y = 0$ is $S + \lambda U = 0$.

i.e. $x^2 + y^2 + z^2 - 2x + 2z - 2 + \lambda y = 0$ ... (1)

Centre C is $\left(1, \frac{-\lambda}{2}, -1\right)$

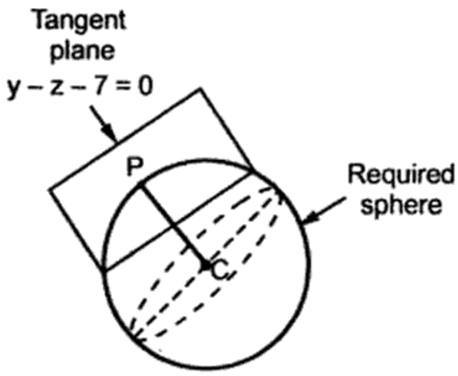


Fig. 11.20

Centre C of the required sphere is $C\left(1, \frac{-\lambda}{2}, -1\right)$ and radius

$$CP = \sqrt{1 + \frac{\lambda^2}{4} + 1 + 2} = \sqrt{4 + \frac{\lambda^2}{4}}$$

Also CP = Length of perpendicular from C on the plane $y - z - 7 = 0$

$$= \left|\frac{-\frac{\lambda}{2} + 1 - 7}{\sqrt{2}}\right|$$

or $$\sqrt{4 + \frac{\lambda^2}{4}} = \left|\frac{-\left(\frac{\lambda}{2} + 6\right)}{\sqrt{2}}\right|$$

On squaring both sides, we get

$$\frac{\lambda^2}{4} + 4 = \left(\frac{\lambda}{2} + 6\right)^2 \frac{1}{2}$$

or
$$\frac{\lambda^2}{4} + 4 = \left(\frac{\lambda^2}{4} + 6\lambda + 36\right)\frac{1}{2}$$

$$\Rightarrow \frac{\lambda^2}{2} - \frac{\lambda^2}{4} + 8 - 6\lambda - 36 = 0$$

$$\Rightarrow \quad \frac{\lambda^2}{4} - 6\lambda - 28 = 0$$

i.e. $$\lambda^2 - 24\lambda - 28 \times 4 = 0$$

$$(\lambda + 4)(\lambda - 28) = 0$$

$$\Rightarrow \lambda = -4, 28$$

Substituting λ in equation (1), we get

The spheres are

$$x^2 + y^2 + z^2 - 2x - 4y + 2z - 2 = 0$$

$$x^2 + y^2 + z^2 - 2x + 28y + 2z - 2 = 0$$

Example 11.35 : *Find the equation of the sphere through the circle $x^2 + y^2 + z^2 + 2x + 3y + 6 = 0$, $x - 2y + 4z - 9 = 0$ and the centre of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$.*

Solution : The equation of a sphere passing through the given circle is

$$(x^2 + y^2 + z^2 + 2x + 3y + 6) + \lambda (x - 2y + 4z - 9) = 0$$

This passes through centre of the second sphere i.e. the point (1, – 2, 3)

Substituting it in equation (1) we get $\lambda = -2$.

Hence the equation of the sphere

$$x^2 + y^2 + z^2 + 7y - 8z + 24 = 0$$

Exercise 11.8

Problems on spheres passing through circle with one more condition.

1) *Find the equation of sphere which passes through (4, 6, 3) and meets XOY plane in a circle of radius 5 units having centre at (1, 2, 0).* **[May-2005]**

Hint : Refer solved Example No. 11.30. **[Ans. : $x^2 + y^2 + z^2 - 2x - 4y - 3z - 20 = 0$]**

2) *Find the equation of the sphere through the circle $x^2 + y^2 = 9$, $z = 0$ and the point (α, β, γ).*

[Ans. : $\gamma (x^2 + y^2 + z^2 - 9) = (\alpha^2 + \beta^2 + \gamma^2 - 9) z$]

3) *Find the equation of the sphere through circle* $x^2 + y^2 + z^2 + 5x - 4y + 6z - 11 = 0$, $x^2 + y^2 + z^2 - 3x + 8y - 10z - 3 = 0$ *and the point* $(1, -2, 3)$

[Ans. : $2(x^2 + y^2 + z^2) + 7y - 8z - 12 = 0$]

4) *Find the equation of the sphere which touches the sphere* $4(x^2 + y^2 + z^2) + 10x - 25y - 2z = 0$ *at the point* $(1, 2, -2)$ *and passses through* $(-1, 0, 0)$. **[May-1991]**

[Ans. : $x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$]

5) *Find the equation of the sphere passing through the circle* $x^2 + y^2 + z^2 = 5$, $x + 2y + 3z = 3$ *and touch the plane* $4x + 3y = 15$. **[Ans. : $x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0$**

$x^2 + y^2 + z^2 - 4x - 8y - 12z - 13 = 0$]

6) *Find the equations of the sphere which pass through the circle* $x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0$, $2x + y + z - 4 = 0$ *and touch the plane* $3x + 4y - 14 = 0$.

[Ans. : $x^2 + y^2 + z^2 - 2x + y + z - 3 = 0$, $x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0$]

11.13 Orthogonal Spheres

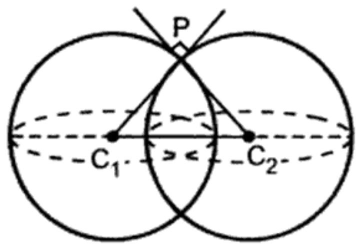


Fig. 11.21

Two spheres are said to be Orthogonal if the tangent planes to the two spheres at the points of intersection are at right angles. i.e. the normals to the two tangent planes at the point of intersection through C_1 and C_2 are perpendicular. In $\Delta\, C_1PC_2$, we have,

i.e. $$(C_1C_2)^2 = (C_1P)^2 + (C_2P)^2 \quad \text{... (1)}$$

If $x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$ and

$x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$

then $$(C_1C_2)^2 = (u_1 - u_2)^2 + (v_1 - v_2)^2 + (w_1 - w_2)^2$$

$$(C_1P)^2 = u_1^2 + v_1^2 + w_1^2 - d_1$$

$$(C_2P)^2 = u_2^2 + v_2^2 + w_2^2 - d_2$$

Substituting in equation (1)

$$(u_1 - u_2)^2 + (v_1 - v_2)^2 + (w_1 - w_2)^2 = u_1^2 + v_1^2 + w_1^2 - d_1 + u_2^2 + v_2^2 + w_2^2 - d_2$$

We get the condition of orthogonality on simplification,

$$\boxed{2(u_1u_2 + v_1v_2 + w_1w_2) = d_1 + d_2}$$

11.14 Great Circle

The section of a sphere by a plane passing through the centre of the sphere is called the great circle i.e. circle of maximum radius obtained from the given sphere. In this case radius of sphere will be equal to the radius of circle.

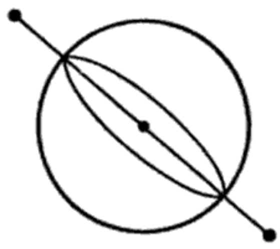


Fig. 11.22

11.15 Tangent Line to a Sphere

A line is said to be a tangent to a sphere if it cuts the sphere at only one point or two coincident points.

Length of the tangent line

Let A (x_1, y_1, z_1) be any point outside the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

Then the length of the tangent line from A to the sphere is given by

$$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d$$

Students can easily derive it since

$$AT^2 = CA^2 - CT^2$$

$$CT = \text{Radius}$$

$$= \sqrt{u^2 + v^2 + w^2 - d}$$

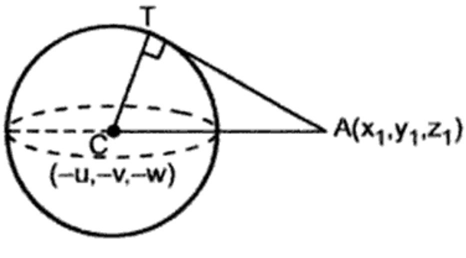


Fig. 11.23

11.16 Radical Plane

Radical plane of given spheres is the locus of the point from which the length of tangent to given spheres are equal.

Let S_1 and S_2 be two spheres whose equations are

$$S_1 = x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$$

and $$S_2 = x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

Let $P(x_1, y_1, z_1)$ be a point the locus.

$$PT_1^2 = PT_2^2$$

Where PT_1 and PT_2 are the two tangents.

Or $x_1^2 + y_1^2 + z_1^2 + 2u_1x_1 + 2v_1y_1 + 2w_1z_1 + d_1$

$x_1^2 + y_1^2 + z_1^2 + 2u_2x_1 + 2v_2y_1 + 2w_2z_1 + d_2$

Dropping the suffix we get the locus as

$$2(u_1 - u_2)x + 2(v_1 - v_2)y + 2(w_1 - w_2)z + d_1 - d_2 = 0$$

Or equation of the radical plane is

$$S_1 - S_2 = 0$$

11.17 Tangent to a Circle at a Point (x_1, y_1, z_1)

Let S = 0 and P = 0 be the given equation of the circle.

The tangent line to the circle (P = 0 and S = 0) is given by the line of intersection of the tangent plane of the sphere at (x_1, y_1, z_1) and the plane P = 0.

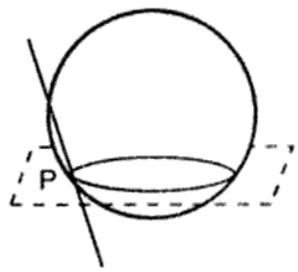


Fig. 11.24

Let $S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$P = lx + my + nz - p = 0$

be the circle. Then equation of the tangent line to the circle is

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$$

and $lx + my + nz - p = 0$

11.18 Illustrations on Orthogonal Spheres and Great Circle

Example 11.36 : *Find the equation of the sphere which touches plane $3x + 2y - z + 2 = 0$ at the point $(1, -2, 1)$ and also cuts the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$ Orthogonally.* **[Dec.-1999]**

Solution : $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$

Centre C_2 s (2, – 3, 0) and

$$\text{Radius} = C_2A = \sqrt{4 + 9 - 4} = 3$$

C_1M is perpendicular to $3x + 2y - z + 2 = 0$ where C_1 is the centre of required sphere and M is (1, – 2, 1) direction ratios of C_1M are 3, 2, – 1.

Equation of C_1M is $\frac{x-1}{3} = \frac{y+2}{2} = \frac{z-1}{-1} = k$

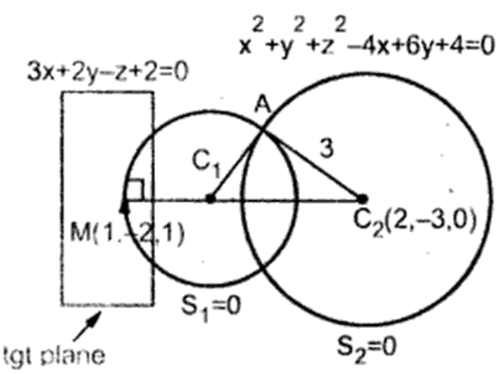


Fig. 11.25

Solving we get k = – 1

Substituting k in equation (1) we get

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$$

Example 11.40 : *Obtain the equation of the sphere through the circle $x^2 + y^2 + z^2 - x + 9y - 5z - 5 = 0$, $x^2 + y^2 + z^2 + 10y - 4z - 8 = 0$ as a great circle.*

Solution : Let $S_1 = x^2 + y^2 + z^2 - x + 9y - 5z - 5 = 0$

and $S_2 = x^2 + y^2 + z^2 + 10y - 4z - 8 = 0$

The plane on which the circle lies is given by $S_1 - S_2 = 0$.

$$\begin{aligned} P &= x^2 + y^2 + z^2 - x + 9y - 5z - 5 \\ &\quad -\left(x^2 + y^2 + z^2 + 10y - 4z - 8\right) = 0 \\ &= -x - y - z + 3 = 0 \end{aligned}$$

Or $P = x + y + z - 3 = 0$

Equation of the sphere through the circle is

$$S_1 + \lambda P = 0$$

Or $x^2 + y^2 + z^2 - x + 9y - 5z - 5 + \lambda(x + y + z - 3) = 0$

Or $x^2 + y^2 + z^2 + x(\lambda - 1) + y(\lambda + 9) + z(\lambda - 5) - 5 - 3\lambda = 0$ ... (1)

Centre of the required sphere is

$$\left(-\frac{\lambda - 1}{2}, -\frac{\lambda + 9}{2}, -\frac{\lambda - 5}{2}\right)$$

Or $\frac{1 - \lambda}{2}, \frac{-\lambda - 9}{2}, \frac{5 - \lambda}{2}$

Since the cross-section circle is a great circle is the centre lies on the plane $x + y + z = 3$ itself

Or $\frac{1 - \lambda}{2} + \frac{-\lambda - 9}{2} + \frac{5 - \lambda}{2} - 3 = 0$

$1 - \lambda - \lambda - 9 + 5\lambda - 6 = 0$

$-3\lambda = 9$

$\lambda = -3$

Hence the required equation of the sphere becomes

$x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0.$

Example 11.41 : *A sphere S has points (1, −2, 3), (4, 0, 6) as opposite ends of a diameter. Find the equations of the sphere having the intersection of S with plane $x + y - 2z + 6 = 0$ as great circle.* [May-2000]

Solution : The equation of sphere S with points (1, − 2, 3), (4, 0, 6) as diameter is

$$(x-1)(x-4)+(y+2)y+(z-3)(z-6) = 0$$

i.e. $x^2 + y^2 + z^2 - 5x + 2y - 9z + 22 = 0$

Equation of the sphere passing through the circle of intersection of

$$x^2 + y^2 + z^2 - 5x + 2y - 9z + 22 = 0$$

$$x + y - 2z + 6 = 0$$

$$x^2 + y^2 + z^2 - 5x + 2y - 9z + 22 + k(x + y - 2z + 6) = 0$$

$$x^2 + y^2 + z^2 + x(k-5) + y(k+2) - z(9+2k) + (22+6k) = 0 \qquad \text{... (1)}$$

As the given circle of intersection is a great circle of this sphere (1)

∴ The centre of sphere (1) lies in the plane

$$x + y - 2z + 6 = 0$$

Centre C is $\left(\frac{k-5}{-2}, \frac{k+2}{-2}, \frac{-(9+2k)}{-2}\right)$

C lies on $x + y - 2z + 6 = 0$

$$\frac{-(k-5)}{2} - \frac{k+2}{2} - \frac{2(9+2k)}{2} + 6 = 0$$

$$-k + 5 - k - 2 - 18 - 4k + 12 = 0$$

$$-6k - 2 = 0$$

$$k = -\frac{1}{2}$$

$$x^2 + y^2 + z^2 - \frac{11}{2}x + \frac{3}{2}y - 8z + 19 = 0$$

$$2(x^2 + y^2 + z^2) - 11x + 3y - 16z + 38 = 0$$

Example 11.42 : *The plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x + z - 2 = 0$ in a circle of radius unity. Find the equation of the sphere which has this circle for one of its great circles.* [Dec.-2000]

Solution : The given circle of intersection is

$$S = x^2 + y^2 + z^2 - x + z - 2 = 0$$

5) *Prove that the equation of the sphere which cuts orthogonally each of the spheres* $x^2 + y^2 + z^2 = a^2 + b^2 + c^2$, $x^2 + y^2 + z^2 + 2ax = a^2$, $x^2 + y^2 + z^2 + 2by = b^2$, $x^2 + y^2 + z^2 + 2cz = c^2$ *is* $x^2+y^2+z^2+\left(\frac{b^2+c^2}{a}\right)x+\left(\frac{c^2+a^2}{b}\right)y+\left(\frac{a^2+b^2}{c}\right)z+\left(a^2+b^2+c^2\right)=0.$

6) *Find the equation of the sphere for which the circle* $x^2 + y^2 + z^2 + 7y - 22 = 0$, $2x + 3y + 4z = 8$ *is a great circle.* **[Ans. : $x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$]**

7) *Prove that every sphere through the circle* $x^2 + y^2 - 2ax + r^2 = 0$, $z = 0$ *cuts orthogonally every sphere through the circle* $x^2 + z^2 = r^2$, $y = 0$.

8) *A sphere S has* $(1, -2, 3)$ *and* $(4, 0, 6)$ *as opposite ends of a diameter. Find the equation of the sphere having the intersection of S with the plane* $x + y - 2z + 6 = 0$ *as its great circle.*

[Ans. : $2(x^2 + y^2 + z^2) - 11x + 3y - 16z + 38 = 0$]

Miscellaneous Problems

Example 11.43 : ***Find the equation of lines passing through the point (1, 1, 1) tangent to the sphere $x^2 + y^2 + z^2 = 2$ and parallel to the plane $4x + 3y - z = 0$.***

Solution : Let P be the point (1, 1, 1). We have to find the equations of the tangent lines from P to the sphere.

Let the tangent lines meet the sphere at A and B.

We have to find the equations of PA and PB.

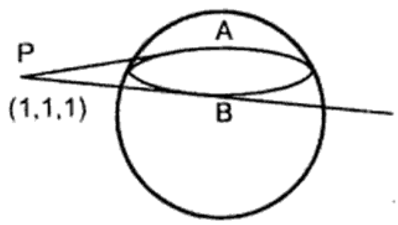


Fig. 11.26

Let the point of contact be (x_1, y_1, z_1) (either A or B) (x_1, y_1, z_1) lies in the sphere.

$$x_1^2 + y_1^2 + z_1^2 = 2$$

Also since the line is parallel to $4x - 3y - z = 0$.

PA is perpendicular to normal to the plane.

The dr's of PA is $x_1 - 1, y_1 - 1, z_1 - 1$.

The dr's of normal is 4, 3, – 1.

$$\therefore \quad 4(x_1 - 1) + 3(y_1 - 1) - 1(z_1 - 1) = 0$$

Or $$4x_1 + 3y_1 - z_1 = 6 \qquad \text{... (1)}$$

Equation of the tangent plane to the sphere

$$x^2 + y^2 + z^2 = 2 \text{ at } (x_1, y_1, z_1) \text{ is}$$

$$xx_1 + yy_1 + zz_1 = 2$$

Now line PA lies in this plane.

$\therefore$ the point P given by (1, 1, 1) satisfy this

OA is rotated about the diameter PQ through an angle θ to OB.

Let DR of PQ be (a, b, c)

DR of OA = 4, – 3, 2

DR of OB = 5, 0, – 2

PQ is perpendicular to OA and OB

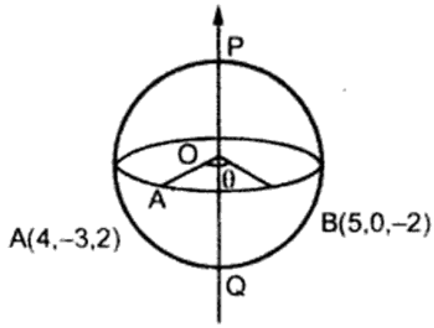


Fig. 11.27

$$\therefore \quad 4a - 3b + 2c = 0$$

$$5a + 0b - 2c = 0$$

$$\frac{a}{6} = \frac{b}{18} = \frac{c}{15}$$

∴ DR of diameter PQ are 2, 6, 5.

Equation of diameter PQ.

$$\frac{x-0}{2} = \frac{y-0}{6} = \frac{z-0}{5}$$

Angle θ between OA and OB is given by

$$\cos\theta = \frac{20}{\sqrt{29}} - \frac{4}{\sqrt{29}} = \frac{16}{29}$$

$$\theta = \cos^{-1}\left(\frac{16}{29}\right)$$

Example 11.45 : *Find the equation of tangent line in symmetrical form to the circle.* $x^2 + y^2 + z^2 + 2x - 4y + 8z - 9 = 0, x + y + z = 3$ *at the point* $(1, 1, 1)$.

[May-1995]

Solution : Line joining centre of the circle and the point (1, 1, 1) is perpendicular to the tangent line to the circle. Similarly, normal to the tangent plane at (1, 1, 1) is also perpendicular to the tangent line. Centre of given sphere is (– 1, 2, – 4).

Equation of line normal to the plane x + y + z = 3, passing through (– 1, 2, – 4) is given by

$$\frac{x+1}{1} = \frac{y-2}{1} = \frac{z+4}{1} = k$$

Any point on this line is (k – 1, k + 2, k – 4), putting for x, y, z in the plane x + y + z = 3, we get

$$k - 1 + k + 2 + k - 4 = 3$$

$$\therefore \quad 3k = 6 \text{ or } k = 2$$

∴ Centre of circle P is (2 – 1, 2 + 2, 2 – 4) or (1, 4, – 2).

If Q is the point (1, 1, 1) then the dr's of PQ are (1 – 1, 1 – 4, 1 + 2) i.e. (0, – 3, 3).

$$\therefore \qquad r_2 = 3 \qquad \ldots (1)$$

For the first sphere centre C_1 is (1, 2, 3) and radius

$$r_1^2 = 1 + 4 + 9 - 10 = 4$$

$$\therefore \qquad r_1 = 2 \qquad \ldots (2)$$

The distance d between the centres is given by

$$d^2 = (3-1)^2 + (1-2)^2 + (-1-3)^2 = 21 \qquad \ldots (3)$$

If P is any one of the point of intersection of the spheres then by cosine formula for ΔPC_1C_2, we get

$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2\, r_1 r_2}$$

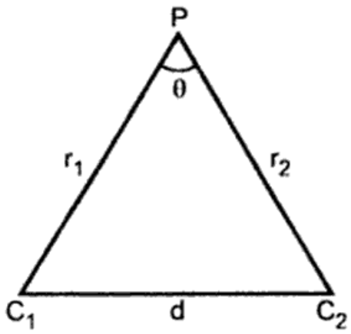


Fig. 11.28

Hence from equation (1), (2) and (3) angle of intersection for the given sphere is

$$\cos\theta = \frac{4+9-21}{2(2)(3)} = -\frac{8}{12} = -\frac{2}{3}$$

$$\therefore \qquad \theta = \cos^{-1}\left(-\frac{2}{3}\right)$$

Exercise 11.10

1) *Find the equation of the diameter of the sphere $x^2 + y^2 + z^2 = 17$. Such that the rotation about it will transfer the point (2, 3, 2) to the point (4, 0, 1) along a great circle of the sphere. Find the angle through which the sphere must be rotated.* **[Ans. : $\theta = \cos^{-1}\frac{10}{17}$]**

2) *Prove that the plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x + z - 2 = 0$ in a circle of radius unity and find the equation of the sphere which has this circle for one of its great circle.* **[Ans. : $x^2 + y^2 + z^2 - 2x - 2y + 2z = -2$]**

University Questions

May - 2003

1. *A sphere S has points (1, – 2, 3), (4, 0, 6) as opposite ends of a diameter. Find the equation of the sphere having the intersection of S with the plane $x + y - 2z + 6 = 0$ as its great circle.* **[5 Marks]**

2. *Find the equation of the sphere which touches the sphere $4(x^2 + y^2 + z^2) + 10x - 25y - 2z = 0$ at the point (1, 2, – 2) and passes through the point (– 1, 0, 0).* **[5 Marks]**

lie on the same sphere. Find its equation. **[5 Marks]**

2. *Find the equation of the sphere which touches the co-ordinate axes, whose centre is in the positive octant and has radius 4.* **[6 Marks]**

May - 2007

1. *A sphere of constant radius r passes through the origin and meets the co-ordinate axis in A, B, C. Show that the locus of centroid of the triangle ABC is a sphere* $9(x^2+y^2+z^2) = 4r^2$. **[6 Marks]**
2. *Find the equation of the sphere through the circle* $x^2+y^2+z^2 = 4,\ z = 0$, *meeting the plane* $x+2y+2z = 0$ *in a circle of radius 3.* **[6 Marks]**

Dec. - 2007

1. *Find the equation of the sphere which touches the sphere* $x^2+y^2+z^2-x+3y+2z-3 = 0$ *at the point (1, 1, –1) and passes through the point (0, 0, 3).* **[6 Marks]**
2. *Determine the centre and radius of the circle*

   $x^2+y^2+z^2+2x-2y-4z-19 = 0;$

   $x+2y+2z+7 = 0.$

   Also find the orthogonal projection of the area of the circle in xoy plane. **[6 Marks]**

May - 2008

1. *A sphere S has points (1, – 2, 3) and (4, 0, 6) as opposite ends of a diameter. Find the equation of sphere having the intersection of S with the plane* $x+y-2z+6 = 0$ *as its great circle.* **[6 Marks]**
2. *Find the equation of a sphere passing through the point (3, 1, 2) and which meets XOY plane in a circle with centre as (1,– 2, 0) and radius 3 units.* **[6 Marks]**

Dec. - 2008

1. *Find the equation of sphere which touches the sphere* $x^2+y^2+z^2-x+3y+2z-3 = 0$ *at (1, 1, – 1) and passes through (0, 0, 3).* **[6 Marks]**
2. *Find the center and radius of the circle* $x^2+y^2+z^2-2x+4y+2z-6 = 0, x+2y+2z-4 = 0.$
   Also find the orthogonal projection of the area of the circle in yz-plane. **[6 Marks]**

❑❑❑

60

12

The Cone

12.1 Definition

A cone is a surface generated by a line which passes through a fixed point and satisfies one more geometrical condition like passes through a given curve of surface. The fixed point is called vertex of a cone and the given curve is called guiding curve of cone.

Any line through the vertex and guiding curve is called a generator.

12.2 Cone with Vertex at the Origin

The equation of a cone whose vertex is at origin is homogeneous equation of second degree in x, y, z and conversely.

Proof : Let the equation of the cone whose vertex is origin be

$$\phi(x, y, z) = 0 \qquad \text{... (1)}$$

Let $P(x_1, y_1, z_1)$ be any point on generator.

$\therefore$ P satisfies equation (1)

$$\therefore \quad \phi(x_1, y_1, z_1) = 0 \qquad \text{... (2)}$$

As P lies on cone, the equation of OP is

$$\frac{x-0}{x_1-0} = \frac{y-0}{y_1-0} = \frac{z-0}{z_1-0} = t \text{ say}$$

$$\therefore \quad x = x_1 t,\ y = y_1 t,\ z = z_1 t$$

are the co-ordinates of any point on the generator, it also satisfies equation (1)

$$\therefore \quad \phi(x_1 t, y_1 t, z_1 t) = 0 \qquad \text{... (3)}$$

From (2) and (3) $\phi(x, y, z) = 0$ is a homogeneous equation.

Conversely : Consider the general second degree homogeneous equation in x, y, z.

i.e. $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ ... (4)

Let $P(x_1, y_1, z_1)$ be any point satisfying (4).

$$\therefore ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 = 0 \quad \text{... (5)}$$

From above any point lying on the line OP is in the form

$$x = x_1t, \; y = y_1t, \; z = z_1t$$

Substituting in (4)

$$a(x_1t)^2 + b(y_1t)^2 + c(z_1t)^2 + 2f(y_1t)(z_1t) + 2g(z_1t)(x_1t) + 2h(x_1t)(y_1t) = 0$$

i.e. $t^2\left[ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1\right] = 0$

i.e from (5) any point on OP satisfies equation (4).

Therefore line OP lies entirely on the locus of (4). Thus the surface is generated by lines passing through the origin and hence, it is a cone with vertex at origin.

12.3 Type 1 : Problems on Cone with Vertex Origin

Procedure for problem solving :

Necessary data :

1) One second degree equation.

2) One first degree equation.

We know that every 2nd degree homogeneous equation represents a cone with vertex origin.

$\therefore$ Consider the given second degree equation and using the given 1st degree equation make it homogeneous.

Example 12.1 : ***Find the equation of a cone whose vertex is at origin and guiding curve is*** $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \; y = b$

Solution : $y = b$ can be expressed as $\frac{y}{b} = 1$

Now consider $$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$$

Now making this equation homogeneous as

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = (1)^2$$

i.e. $$\frac{x^2}{a^2} + \frac{z^2}{c^2} = \left(\frac{y}{b}\right)^2$$

Hence the required equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$$

Example 12.2 : *A variable plane is parallel to the given plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ and meets co-ordinate axes in A, B, C. Find the equation of cone whose vertex is the origin and guiding curve is the circle ABC.*

Solution : The equation of plane ABC is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = k$$

i.e $$\frac{x}{ak}+\frac{y}{bk}+\frac{z}{ck} = 1 \quad \text{... (1)}$$

Also the equation of sphere OABC is

$$x^2+y^2+z^2-kax-kby-kbz = 0 \quad \text{... (2)}$$

(i.e. intercept form of plane and sphere).

Now we have one 1st degree equation and one 2nd degree equation.

Thus from (2)

$$x^2+y^2+z^2-k(ax+by+cz)\cdot(1) = 0$$

$$\therefore x^2+y^2+z^2-k\cdot(ax+by+cz)\left(\frac{x}{ak}+\frac{y}{bk}+\frac{z}{ck}\right)=0.$$

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{a}{c}+\frac{c}{a}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right) = 0$$

is the required equation of cone.

Example 12.3 : *Find the equation of cone with vertex origin and base is a circle $x=a$, $y^2+z^2=b^2$ and show that the section of the cone by a plane parallel to XOY plane is a hyperbola.* **(May-1985, Dec.-1987)**

Solution : Consider $y^2+z^2=b^2$

$$y^2+z^2 = b^2(1)^2$$

$$\therefore \quad y^2+z^2 = b^2\left(\frac{x}{a}\right)^2$$

$$\therefore \quad a^2(y^2+z^2) = b^2x^2$$

is the required equation of cone. If this cone is cut by $z = c$. (Plane || to XOY)

then $a^2y^2+a^2c^2 = b^2x^2$

$$\therefore \quad b^2x^2-a^2y^2 = a^2c^2$$

Which represents a hyperbola.

Exercise 12.1 : Problems on Type 1

1) *Find the equation of the cone with origin and following guiding curve.*

i) $ax^2 + by^2 + cz^2 = 1,\ lx + my + nz = p$ [Ans. : $p^2(ax^2 + by^2 + cz^2) = (lx + my + nz)^2$]

ii) $x^2 + y^2 + z^2 - x - 1 = 0,\ x^2 + y^2 + z^2 + y - 2 = 0$ [Ans. : $z^2 - 3xy - x^2 = 0$]

Hint : One 1^{st} degree equation is not given $\therefore$ for finding 1^{st} degree equation consider $s_1 - s_2 = 0$ represents the plane of circle as intersection of two spheres.

iii) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,\ z = c$ [Ans. : $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0$]

iv) $x^2 - 2y^2 + 3z^2 - 4x + 9 = 0,\ x + y - z = 4$

[Ans. : $9x^2 + 41y^2 - 39z^2 + 34xy - 34zx - 18yz = 0$]

v) $x^2 + y^2 + z^2 + x - 2y + 3z = 4,\ x^2 + y^2 + z^2 + 2x - 3y + 4z = 5$

[Ans. : $2x^2 + y^2 - 5xy + 4xz - 3yz = 0$]

vi) $ax^2 + by^2 + cz^2 = 1,\ lx^2 + my^2 = 2z$ [Ans. : $4z^2(ax^2 + by^2 + cz^2) = (lx^2 + my^2)^2$]

vii) $x = a,\ y^2 + z^2 = b^2$ [Ans. : $a^2(y^2 + z^2) = b^2x^2$]

viii) $x^2 + y^2 + z^2 = 9,\ x + y + z = 1$ [Ans. : $4(x^2 + y^2 + z^2) + 9(xy + yz + zx) = 0$]

ix) $x^2 + y^2 + z^2 + 2x - y + 3z = 1,\ x^2 + y^2 + z^2 + x + 2z = 5$

[Ans. : $3x^2 + 11y^2 + 3z^2 + 14xy + 18yz - 30xz = 0$]

x) $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{1} = 1,\ x + y + z = 1$ [Ans. : $27x^2 + 32y^2 + 72(xy + yz + zx) = 0$]

2) *The plane $x + y + z = 1$ meets the co-ordinate axes in A, B, C. Prove that the equation of the cone generated by lines through O, the origin to meet ABC is $xy + yz + zx = 0$.*

12.4 The Direction Cosines or Direction Ratio's of a Generator of a Cone whose Vertex is the Origin Satisfy the Equation of the Cone

Proof : Let $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ ... (1)

be the general equation of cone with vertex origin.

Let l, m, n be direction cosines (or direction ratio's) of any generator of cone.

$\therefore$ The equation of generator will be

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n} = k \text{ (say)}$$

$\therefore x = lk,\ y = mk,\ z = nk$ are the co-ordinates of any point on a generator substituting x, y, z in (1)

$\therefore a(lk)^2 + b(mk)^2 + c(nk)^2 + 2f(mk)(nk) + 2g(nk)(lk) + 2h(lk)(mk) = 0$

$\therefore al^2 + bm^2 + cn^2 + 2fmn + 2gln + 2hlm = 0$

Which shows that l, m, n satisfies equation (1) conversely, if $al^2 + bm^2 + cn^2 + 2fmn + 2gln + 2hlm = 0$ then we can say that the line with direction cosines l, m, n is a generator of the cone.

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

12.5 Quadratic Cone through the Axes

The general equation of a cone of second degree passing through the three co-ordinate axes is

$$fyz + gzx + hxy = 0.$$

Proof :

let $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ ... (1)

be the equation of cone.

Since x-axis is the generator of the cone dc's of x-axis i.e. (1, 0, 0) must satisfy equation (1).

$\therefore$ Substituting (1, 0, 0) in equation (1) we get a = 0.

Similarly we get b = 0, c = 0, hence from (1) the required equation is

$$fyz + gzx + hxy = 0 \quad \text{... (2)}$$

12.6 Type 2 : Problems on Quadratic Cones through Axes

Example 12.4 : *Find the equation of the cone which passes through the three co-ordinate axes as well as the two lines* $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$, $\frac{x}{3} = \frac{y}{-1} = \frac{z}{1}$

Solution : The equation of cone through three co-ordinate axes is

$$fyz + gzx + hxy = 0 \quad \text{... (1)}$$

The direction ratio's of given two lines are (1, – 2, 3) and (3, – 1, 1) they satisfies equation (1).

$$\therefore \quad -6f + 3g - 2h = 0$$

and $\quad -f + 3g - 3h = 0.$

Using cramers rule we get

$$\frac{f}{\begin{vmatrix} 3 & -2 \\ 3 & -3 \end{vmatrix}} = \frac{-g}{\begin{vmatrix} -6 & -2 \\ -1 & -3 \end{vmatrix}} = \frac{h}{\begin{vmatrix} -6 & 3 \\ -1 & 3 \end{vmatrix}}$$

$$\frac{f}{-9+6} = \frac{-g}{18-2} = \frac{h}{-18+3}$$

$$\frac{f}{-3} = \frac{-g}{16} = \frac{h}{-15}$$

Substituting f, g, h in (1) we get,

$$-3yz - 16zx - 15xy = 0$$

i.e $$3yz + 16zx + 15xy = 0.$$

which is the required equation.

Example 12.5 : ***Find the equation of the cone with vertex origin and direction cosines satisfying the relation $2l^2 + 4m^2 + 6n^2 = 0$.***

Solution : Equation of any line through (0, 0, 0) is

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n} \quad \text{... (1)}$$

given that dc's satisfy

$$2l^2 + 4m^2 + 6n^2 = 0 \quad \text{... (2)}$$

Eliminating l, m, n from (1) and (2) we get,

$$2x^2 + 4y^2 + 6z^2 = 0.$$

which is the required equation of cone.

Exercise 12.2 : Problems on Type 2

1) *Find the equation of cone whose vertex is origin which passes through three co-ordinate axes and as well as.*

   i) $\frac{x}{1} = \frac{y}{2} = \frac{z}{-2}, \frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ **[Ans. : 3yz − 5zx + xy = 0]**

   ii) $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}, \frac{x}{1} = \frac{y}{-1} = \frac{z}{-1}, \frac{x}{5} = \frac{y}{4} = \frac{z}{1}$ **[Ans. : 5yz + 8zx − 3xy = 0]**

   iii) $\frac{x}{2} = \frac{y}{1} = \frac{z}{-1}, \frac{x}{1} = \frac{y}{3} = \frac{z}{5}, \frac{x}{8} = \frac{y}{-11} = \frac{z}{5}$ **[May-2006] [Ans. : 16yz − 33zx − 25xy = 0]**

2) *Find the equation of the cone generated by the line* $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ *where* $l^2 - m^2 + 3n^2 = 0$.

   [Ans. : $x^2 - y^2 + 3z^2 = 0$]

3) *Prove that the line* $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ *where* $2l^2 + 3m^2 - 5n^2 = 0$ *is a generator of the cone* $2x^2 + 3y^2 - 5z^2 = 0$.

4) *Show that the lines drawn through the point* (α, β, γ) *whose direction cosines satisfy* $al^2 + bm^2 + cn^2 = 0$ *generates the cone* $a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 = 0$.

12.7 Type 3 : Problems on Cone with given Vertex and given Guiding Curve

Example 12.6 : *Find the equation of the cone whose vertex is the point (0, 0, 1) and whose directrix is ellipse* $\frac{x^2}{25}+\frac{y^2}{9}=1,\ z=3$. *Obtain the section of the derived cone by plane* $y = 0$. **[Dec.-2000]**

Solution : Step 1 : Let l, m, n be direction ratio's of the generator. As (0, 0, 1) is the vertex.

∴ The equation of generator is

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-1}{n} \qquad \text{... (1)}$$

Step 2 : Put z = 3

$$\therefore \qquad \frac{x}{l} = \frac{y}{m} = \frac{2}{n}$$

Step 3 : Find x, y

$$\therefore \qquad x = \frac{2l}{n},\quad y = \frac{2m}{n}$$

Step 4 : Substituting x, y in the given equation.

i.e. $\frac{x^2}{25}+\frac{y^2}{9} = 1$ we get

$$\frac{4}{25}\left(\frac{l}{n}\right)^2 + \frac{4}{9}\left(\frac{m}{n}\right)^2 = 1$$

Step 5 : Using equation (1) eliminate l, m, n

$$\frac{4}{25}\left(\frac{x}{z-1}\right)^2 + \frac{4}{9}\left(\frac{y}{z-1}\right)^2 = 1$$

i.e. $\quad 36x^2 + 100y^2 = (z-1)^2 \cdot 225$

Which is the required equation of cone. To find the section of this cone put y = 0

$$\therefore \qquad 36x^2 = 225(z-1)^2$$

$$6x = \pm\, 15(z-1)$$

$$2x = 15\,(z-1)$$

$$\therefore \qquad 2x - 5z + 5 = 0 \text{ or } 2x + 5z - 5 = 0$$

4) *Find the equation of the cone whose vertex is* $(1, 1, 0)$ *and base is* $x^2 + z^2 = 4,\ y = 0.$

[Ans. : $x^2 - 3y^2 + z^2 - 2xy + 8y - 4 = 0$]

5) *Find the equation of the curve whose vertex is* (α, β, γ) *and base is*

i) $ax^2 + by^2 = 1,\ z = 0$ **[Ans. : $a(\alpha z - \gamma x)^2 + b(\beta z - \gamma y)^2 = (z - \gamma)^2$]**

ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ z = 0$ **[Ans. : $\left(\frac{\alpha z - \gamma x}{a}\right)^2 + \left(\frac{\beta z - \gamma y}{b}\right)^2 = (z - \gamma)^2$]**

6) *Vertex is* $(1, 2, 3)$ *guiding curve is* $x^2 - 2y^2 + z^2 = 4,\ x - y + z = 3$ **[Nov.-1988]**

[Ans. : $x^2 + 4y^2 + 5z^2 - 10yz + 4xz - 6xy - 2x + 20y - 14z + 2 = 0$]

7) *Vertex is* $(5, 4, 3)$ *with* $3x^2 + 2y^2 = 6$ *and* $y + z = 0$ *as base.*

[Ans. : $147x^2 + 87y^2 + 101z^2 - 210xy + 90yz - 210xz + 84y + 84z - 294 = 0$]

8) *Vertex is* $(3, 4, 5)$ *and with* $3y^2 + 4z^2 = 16,\ z + 2x = 0$ *as base.*

[Ans. : $48x^2 + 33y^2 + 16z^2 - 48xy - 24yz - 32zx - 64x + 32z - 176 = 0$]

12.8 Type 4 : Right Circular Cone

A right circular cone is the surface generated by a straight line which passes through a fixed point and makes constant angle with fixed line through the fixed point.

The section of a right circualr cone by a plane perpendicular to axis is a circle.

The fixed point is called the vertex and the fixed line as the axis and the constant angle is the semi vertical angle.

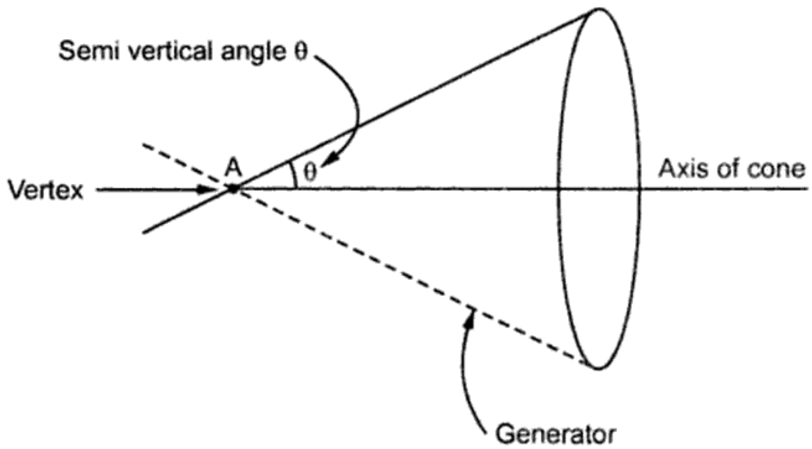


Fig. 12.1

Procedure for solving the problems on Right circular cone :

Necessary data :

i) Co-ordinates of vertex A (a, b, c) (say)

ii) dr's of axis l, m, n (say)

iii) Semi vertical angle θ (say)

If the above three things are not given then 1st find all the three things and then proceed as under.

Step 1 : Let P (x, y, z) be any point on generator.

Step 2 : Consider A (a, b, c) i.e. vertex.

$\therefore$ dr's of line AP are x – a, y – b, z – c

Step 3 : Consider dr's of axis l, m, n.

Step 4 : Use the formula for $\cos \theta$.

$$\cos \theta = \frac{l(x-a)+m(y-b)+n(z-c)}{\sqrt{l^2+m^2+n^2}\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}$$

Step 5 : Simplifying we get the required equation of cone.

Example 12.10 : *Find the equation of the right circular cone whose vertex is* $(1, -1, 2)$ *and axis is the line* $\frac{x-1}{2} = \frac{y+1}{1} = \frac{z-2}{-2}$ *and semivertical angle 45°.*

[May-2001, Dec-2005]

Solution : Step 1 : Let P (x, y, z) be any point on generator.

Step 2 : Given vertex A (1, – 1, 2)

$\therefore$ drs of AP are x – 1, y + 1, z – 2

Step 3 : dr's of axis 2, 1, – 2.

Step 4 : Use formula for $\cos \theta$.

$$\cos 45^\circ = \frac{2(x-1)+1(y+1)-2(z-2)}{\sqrt{4+1+4}\sqrt{(x-1)^2+(y+1)^2+(z-2)^2}}$$

Step 5 : Squaring both sides we get

$$9\left[(x-1)^2+(y+1)^2+(z-2)^2\right] = x\left[2x+y-2z+3\right]^2$$

$$5x^2+8y^2+13z^2-4xy+4yz+8xz-30x+12y-24z+45 = 0$$

Example 12.11 : *Find the equation of right circular cone which passes through the point* $(1, 1, 2)$ *and has its axis as the line 6x = – 3y = 4z and vertex origin.*

[May-2004]

Solution : The axis of the cone is

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{3}$$

$\therefore$ dr's of axis are 2, – 4, 3.

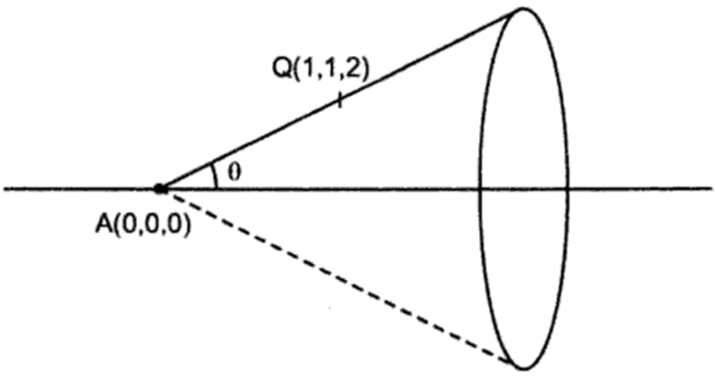


Fig. 12.2

The direction ratio's of line joining A (0, 0, 0) and given point Q (1, 1, 2) are 1, 1, 2.

∴ The semi vertical angle of the cone is the angle between the axis and line AQ.

∴ Using dr's of axis 2, – 4, 3 and dr's of AQ are 1, 1, 2.

$$\therefore \qquad \cos\theta = \frac{2(1) - 4(1) + 3(2)}{\sqrt{4+16+9}\,\sqrt{1+1+4}}$$

$$\cos\theta = \frac{4}{\sqrt{174}}$$

Now we know that

i) Co-ordinates of vertex A (0, 0, 0)

ii) dr's of axis 2, – 4, 3

iii) Semi vertical angle.

$$\theta = \cos^{-1}\frac{1}{\sqrt{174}}$$

Now we can follow our standard procedure.

Step 1 : Let P (x, y, z) be any point on generator.

Step 2 : Given vertex is A (0, 0, 0)

∴ dr's of AP are x – 0, y – 0, z – 0

Step 3 : dr's of axis are (2, – 4, 3).

Step 4 : Use formula for cos θ.

$$\cos\theta = \frac{2x - 4y + 3z}{\sqrt{4+16+9}\,\sqrt{x^2+y^2+z^2}}$$

$$\frac{4}{\sqrt{174}} = \frac{2x - 4y + 3z}{\sqrt{29}\,\sqrt{x^2+y^2+z^2}}$$

Step 5 : Squaring both sides we get

$$3\left(2x - 4y + 3z\right)^2 = 8\left(x^2+y^2+z^2\right)$$

$$\cos\theta = \frac{1}{\sqrt{3}}$$

$$\therefore \qquad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \text{ is the semi vertical angle.}$$

Now we have

i) Vertex (0, 0, 0)

ii) dr's of axis (– 1, 1, 1)

iii) Semi vertical angle $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\therefore$ We can follow our standard procedure for finding right circular cone.

Step 1 : Let P (x, y, z) be any point on the generator.

Step 2 : As the vertex A is (0, 0, 0). $\therefore$ dr's of AP are (x – 0, y – 0, z – 0).

Step 3 : dr's of axis are – 1, 1, 1.

Step 4 : Use formula for cos θ.

$$\cos\theta = \frac{-x+y+z}{\sqrt{x^2+y^2+z^2}\,\sqrt{1+1+1}}$$

$$\frac{1}{\sqrt{3}} = \frac{(-x+y+z)}{\sqrt{x^2+y^2+z^2}\left(\sqrt{3}\right)}$$

Step 5 : Squaring both sides.

$$x^2+y^2+z^2 = (-x+y+z)^2$$

i.e. $\qquad xy - yz + zx = 0$

is the required equation of right circular cone.

Exercise 12.4 : Problems on Type 4

1) *Find the equation of the cone with vertex (1, 2, – 3) semi vertical angle* $\cos^{-1}\left(1/\sqrt{3}\right)$ *and the line* $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z+1}{-1}$ *as axis of the cone.* **[May-98]**

[Ans. : $x^2 - 2y^2 + z^2 + 4yz + 2xz - 4xy + 12x - 24y - 4z - 36 = 0$]

2) *Find the equation of the right circular cone whose vertex is at the origin whose axis is the line* $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ *and which has a semi vertical angle of 30°.* **[Dec.-2001]**

[Ans. : $19x^2 + 13y^2 + 3z^2 - 8xy - 24yz - 12zx = 0$]

Exercise 12.5 : Problems on Type 5

1) *Prove that the equation* $x^2 - 2y^2 + 3z^2 - 4xy + 5yz - 6zx + 8x - 19y - 2z - 20 = 0$ *represents a cone. Find its vertex.* **[Ans. : (1, −2, 3)]**

2) *Prove that the equation* $x^2 + 2y^2 + z^2 - 4yz - 6zx - 2x + 8y - 2z + 9 = 0$ *represents a cone with vertex* $(1, -2, 0)$.

3) *Prove that the equation* $4x^2 - y^2 + 2z^2 - 3yz + 2xy + 12x - 21y + 6z + 4 = 0$ *represents a cone with vertex* $(-1, -2, -3)$.

4) *Prove that the equation* $2y^2 - 8yz - 4zx - 8xy + 6x - 4y - 2z + 5 = 0$ *represents a cone with vertex* $\left(-\frac{7}{6}, \frac{1}{3}, \frac{5}{6}\right)$.

5) *Prove that* $7x^2 + 2y^2 + 2z^2 - 10zx + 10xy + 26x - 2y + 2z - 17 = 0$ *represents a cone with vertex* $(1, -2, 2)$.

12.10 Type 6 : Enveloping Cone

Definition :

The locus of tangent lines from a given point to a sphere is called the enveloping cone from the point to the sphere.

a) The equation of enveloping cone from the point x_1, y_1, z_1 to the sphere $x^2 + y^2 + z^2 = a^2$ is given by $SS_1 = T^2$ where

$$S \equiv x^2 + y^2 + z^2 - a^2$$

$$S_1 \equiv x_1^2 + y_1^2 + z_1^2 - a^2$$

$$T \equiv xx_1 + yy_1 + zz_1 - a^2$$

b) The equation of enveloping cone from (x_1, y_1, z_1) to $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is also $SS_1 = T^2$ where

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d$$

$$S_1 \equiv x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d$$

$$T \equiv xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d$$

Proof of equation of enveloping cone :

Consider the sphere $x^2 + y^2 + z^2 = a^2$

Let P (x_1, y_1, z_1) be the point from which tangent lines are drawn to the sphere and Q (x, y, z) be any other point on the tangent line.

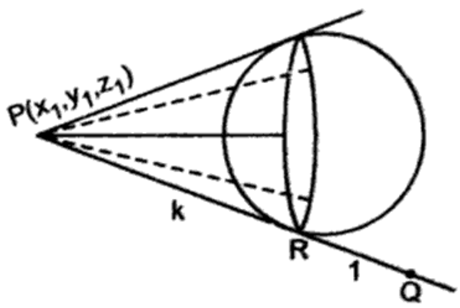


Fig. 12.5

Then any point R will divide PQ in the ratio k : 1 and co-ordinates of

$$R = \frac{kx + x_1}{k+1}, \frac{ky + y_1}{k+1}, \frac{kz + z_1}{k+1}$$

R satisfies the equation of sphere.

$$\therefore \left(\frac{kx + x_1}{k+1}\right)^2 + \left(\frac{ky + y_1}{k+1}\right)^2 + \left(\frac{kz + z_1}{k+1}\right)^2 = a^2$$

Will have two equal roots as the line PQ is a tangent line. The equation written as a quadratic in k becomes,

$$k^2\left(x^2 + y^2 + z^2 - a^2\right) + 2k\left(xx_1 + yy_1 + zz_1 - a^2\right)$$

$$+ x_1^2 + y_1^2 + z_1^2 - a^2 = 0$$

Since the two values of k should be equal, discriminant

$$b^2 - 4ac = 0$$

$$\therefore \quad b^2 = 4ac$$

Substituting a, b, c we get,

$$\left(xx_1 + yy_1 + zz_1 - a^2\right)^2 = \left(x^2 + y^2 + z^2 - a^2\right)\left(x_1^2 + y_1^2 + z_1^2 - a^2\right)$$

i.e. $\quad T^2 = SS_1$

which is the required equation.

Example 12.14 : *Find the equation of the cone with vertex at (1, 1, 1) and generators touching the sphere $x^2 + y^2 + z^2 - 2x + 4z = 1$.*

Solution : The point (x_1, y_1, z_1) is (1, 1, 1)

Equation of the enveloping cone is

$$SS_1 = T^2$$

$$S_1 = x_1^2 + y_1^2 + z_1^2 - 2x_1 + 4z_1 - 1$$

$$= 1 + 1 + 1 - 2 + 4 - 1 = 4$$

$$S = x^2 + y^2 + z^2 - 2x + 4z - 1$$

$$T = xx_1 + yy_1 + zz_1 - (x + x_1) + 2(z + z_1) - 1$$

$$= x + y + z - x - 1 + 2z + 2 - 1$$

$$= y + 3z$$

$$SS_1 = T^2$$

$$\therefore \quad 4\left(x^2 + y^2 + z^2 - 2x + 4z - 1\right) = (y + 3z)^2$$

$$4x^2 + 3y^2 - 5z^2 - 6yz - 8x + 16z - 4 = 0$$

Exercise 12.6 : Problems on Type 6

1) *Find the equation of the cone of tangents drawn from the point (2, 0, 0) to the sphere* $x^2 + y^2 + z^2 = 1$. **[Ans. : $x^2 - 3y^2 - 3z^2 - 4x + 4 = 0$]**
2) *Find the enveloping cone to the sphere* $x^2 + y^2 + z^2 + 2x - 2y = 2$ *with the vertex (1, 1, 1).* **[Ans. : $3x^2 - y^2 + 4xz - 10x + 2y - 4z + 6 = 0$]**
3) *Find the equation of the cone with vertex (0, 0, 0) and generators touching the sphere* $x^2 + y^2 + z^2 - 2x + 4z = 1$. **[Ans. : $2x^2 + y^2 + 5z^2 - 4xz = 0$]**
4) *Show that the plane z = 0 cuts the enveloping cone of the sphere* $x^2 + y^2 + z^2 = 1$ *with vertex (2, 4, 1) in a rectangular hyperbola.* **[Ans. : $10\left(x^2 + y^2 - 11\right) = (2x + 4y - 11)^2$]**

12.11 Type 7 : Angle between the Lines in which a Plane through the Vertex Cuts a Cone

Consider a cone with vertex at the origin. Any plane through the vertex origin cuts the cone, in two lines. The angle between the two lines is found as in the following example.

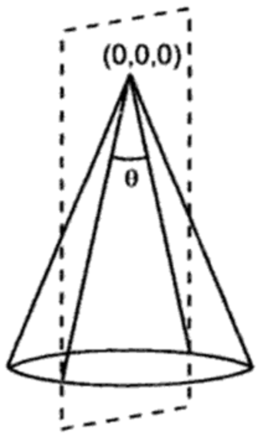


Fig. 12.6

Example 12.15 : ***Find the angle between the lines in which the plane*** $x + 3y - 2z = 0$ ***meets the cone*** $x^2 + 9y - 4z^2 = 0$.

Solution : Let the equation of the line in which the plane cuts the cone be

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n}$$

Then since the line lies on the plane x + 3y − 2z = 0 it will perpendicular to the normal dr's are 1, 3, − 2.

$$\therefore \quad l + 3m - 2n = 0 \qquad \ldots (1)$$

Also since the dr's of any generator of a cone with vertex at the origin satisfies the equation of the cone itself we have,

$$l^2 + 9m^2 - 4n^2 = 0 \qquad \ldots (2)$$

From (1) we have $2n = l + 3m$

Substituting in (2) we have

$$l^2 + 9m^2 - (l + 3m)^2 = 0$$

$\rightarrow -6ml = 0$

i.e. $l = 0$ or $m = 0$

If $l = 0$, then $3m = 2n$

$\therefore$ The dr's of the line can be (0, 2, 3).

If $m = 0$ then $l = 2n$ then dr's of the lines are (2, 0, 1)

$\therefore$ The two lines are

$\frac{x}{0} = \frac{y}{2} = \frac{z}{3}$ and $\frac{x}{2} = \frac{y}{0} = \frac{z}{1}$

If 'θ' is the angle between the lines then

$$\cos\theta = \frac{3}{\sqrt{13}\sqrt{5}} = \frac{3}{\sqrt{65}}$$

$$\therefore \qquad \theta = \cos^{-1}\frac{3}{\sqrt{65}}$$

Example 12.16 : *Find the angle between the lines of section of the plane $3x + y + 5z = 0$ and cone $6yz - 2xz + 5xy = 0$.*

Solution : Let $\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n}$ be the line of section of the given cone by the given plane.

So that the direction ratios (l, m, n) of generator satisfies the two equations and we get,

$$6mn - 2ln + 5lm = 0$$

$$\text{and} \qquad 3l + m + 5n = 0 \qquad \ldots (1)$$

$$\rightarrow \qquad m = -(3l + 5n)$$

Substituting we get,

$$-6n(3l + 5n) - 2ln - 5l(3l + 5n) = 0$$

$$\therefore \qquad 15l^2 + 45ln + 30n^2 = 0$$

$$\text{i.e.} \qquad (l + n)(l + 2n) = 0$$

$\therefore \quad l = -n \qquad$ from (i) $3l + m + 5n = 0 \Rightarrow \frac{l}{1} = \frac{m}{2} = \frac{n}{-1}$

$l = -2n \qquad$ from (i) $3l + m + 5n = 0 \Rightarrow \frac{l}{-2} = \frac{m}{1} = \frac{n}{1}$

Thus (1, 2, – 1) and (– 2, 1, 1) are dr's of two lines.

$$\therefore \quad \cos \theta = \frac{-2+2-1}{\sqrt{1+4+1}\sqrt{4+1+1}} = \frac{-1}{6}$$

$$\theta = \cos^{-1}\left(\frac{-1}{6}\right)$$

Example 12.17 : ***Prove that the plane $ax + by + cz = 0$ cuts the cone $xy + yz + zx = 0$ in perpendicular lines if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$.***

Solution : Let $\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n}$ be the line of section.

$\therefore l$, m, n satisfies equation of cone and plane.

$$\therefore \quad lm + mn + ln = 0$$

$$\text{and} \quad al + bm + cn = 0 \qquad \dots (1)$$

$$\Rightarrow \quad n = \frac{al + bm}{-c}$$

Substituting we get,

$$lm - \frac{m}{c}(al + bm) - \frac{l}{c}(al + bm) = 0$$

$$\therefore \quad al^2 + lm(a + b - c) + bm^2 = 0$$

$$\therefore \quad a\left(\frac{l}{m}\right)^2 + (a + b - c)\left(\frac{l}{m}\right) + b = 0$$

which is a quadratic in $\frac{l}{m}$.

Let $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$ be the two roots of this equation.

$\therefore$ Product of the roots $= \frac{b}{a}$

$$\text{i.e.} \quad \frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{b}{a}$$

$$\therefore \qquad \frac{l_1 l_2}{1/a} = \frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c} \text{ by symmetry}$$

Now two lines will be at right angle

if $$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

i.e. $$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$$

Example 12.18 : *If $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is one of the three mutually perpendicular generators of cone $5yz - 8zx - 3xy = 0$. Find equations of other two lines.*

Solution : The equation of plane passing through origin having the above line as normal is $x + 2y + 3z = 0$.

Let $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ be one of the perpendicular lines.

$\therefore l$, m, n satisfies equation of cone and plane.

$$\therefore \; 5mn - 8nl - 3lm = 0$$

and $$l + 2m + 3n = 0$$

$$\Rightarrow \qquad n = \left(\frac{l + 2m}{-3}\right)$$

Substituting we get

$$5m\left(\frac{l + 2m}{-3}\right) - 8\left(\frac{l + 2m}{-3}\right)l - 3lm = 0$$

i.e. $$8l^2 + 2lm - 10m^2 = 0$$

i.e. $$4\left(\frac{l}{m}\right)^2 + \left(\frac{l}{m}\right) - 5 = 0$$

$$\therefore \qquad \frac{l}{m} = \frac{-5}{4}$$

$$\frac{l}{m} = 1$$

Let (l_1, m_1, n_1) and (l_2, m_2, n_2) be the dr's of two lines.

Firstly $$\frac{l_1}{m_1} = \frac{-5}{4} \text{ or } \left(\frac{l_1}{-5} = \frac{m_1}{4}\right)$$

i.e. in $$l + 2m + 3n = 0 \qquad \text{... (1)}$$

$$-\frac{5m_1}{4} + 2m_1 + n_1 = 0$$

13 Cylinder

13.1 Definition

A cylinder is a surface generated by straight line which is parallel to a fixed line and satisfies one more geometrical condition like intersecting a fixed curve called the guiding curve or directrix or touches a given surface.

The fixed straight line is called the axis and the given curve or surface is called as the guiding curve of the cylinder. The moving straight line is called the generator of the cylinder.

13.2 Equation of Cylinder

To find the equation to the cylinder whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ and intersects the curve $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ and $z = 0$.

Let $P(x_1, y_1, z_1)$ be any point on the cylinder.

Then the equation of the generator through P parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

The line meets the plane $z = 0$ at

$$x = x_1 - l\frac{z_1}{n}$$

$$y = y_1 - \frac{mz_1}{n}$$

If this point also lies on the given curve

$$a\left(x_1 - \frac{l z_1}{n}\right)^2 + \left(y_1 - \frac{mz_1}{n}\right)^2 + 2h\left(x_1 - \frac{lz_1}{n}\right)\left(y_1 - \frac{mz_1}{n}\right)$$

$$+ 2g\left(x_1 - \frac{lz_1}{n}\right) + 2f\left(y_1 - \frac{mz_1}{n}\right) + 1 = 0$$

Dropping the suffix, the locus of the point $P(x_1, y_1, z_1)$ becomes after simplification.

$$a(nx - lz)^2 + b(ny - mz)^2 + 2h(nx - lz)(ny - mz) + 2ng(nx - lz)$$

$$+ 2nf(ny - mz) + cn^2 = 0 \quad \text{... (1)}$$

which is the required equation of the cylinder.

Note : If the generators are parallel to the z-axis, then $l = 0$, $m = 0$, $n = 1$ and the above equation (i) becomes

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

which is free from z.

Hence to find the equation of a cylinder whose guiding curve is $S = 0$, $P = 0$ and parallel to the co-ordinate axes say for example z axis we eliminate z from the two the equations.

Similarly if the generators are parallel to the x axis we eliminate the variable x using the two equations of the guiding curve.

13.3 Type 1 Illustrations

Example 13.1 : *Find the equations of the cylinder whose generators are parallel to i) OX axis ii) OY axis and iii) OZ axis and which passes through the curve of intersection of the surfaces represented by $x^2 + y^2 + 2z^2 = 12$ and $x - y + z = 1$.*

Solution : 1) Generators parallel to x axis

If the generators are parallel to x axis we have to eliminate x from the two equations

$$x^2 + y^2 + 2z^2 = 12 \quad \text{... (1)}$$

and $$x - y + z = 1 \quad \text{... (2)}$$

we have $x = 1 + y - z$

Substituting in the equation (1) we get

$$(1 + y - z)^2 + y^2 + 2z^2 = 12$$

$$2y^2 + 3z^2 + 2y - 2yz - 2z - 11 = 0$$

is the equation of the cylinder whose generators are parallel to x axis.

2) Generators parallel to y axis

We eliminate y from the given equations

$$y = x + z - 1$$

Substituting in [Using equation (2)] equation (1) we get

$$x^2 + (x + z - 1)^2 + 2z^2 = 12$$

$$2x^2 + 3z^2 + 2xy - 2z - 2z - 11 = 0$$

is the cylinder whose generators are parallel to y axis.

3) Similarly eliminating z from the two equations

$$x - y + z = 1$$

and $x^2 + y^2 + 2z^2 = 12$

We get $\quad x^2 + y^2 + 2(1 - x + y)^2 = 12$

$$3x^2 + 3y^2 - 4x - 4y - 10 = 0$$

is the equation of the cylinder whose generators are parallel to the z axis.

Example 13.2 : ***Find the equation of the cylinder whose guiding curve is $ax^2 + by^2 = 2z$ and $lx + my + nz = p$ and generators parallel to the z axis.***

[May-2005]

Solution : Since the generators are parallel to the z axis we have to eliminate z from the above two equations of the guiding curve

$$ax^2 + by^2 = 2z$$

and $\quad lx + my + nz = p$

i.e. $\quad lx + my + n\dfrac{(ax^2 + by^2)}{2} = p$

or the required equation is

$$n(ax^2 + by^2) + 2lx + 2my - 2p = 0$$

Exercise 13.1 : Problems on Type 1

1) *Find the equation of cylinder whose generators are parallel to x axis whose guiding curve is $ax^2 + by^2 + cz^2 = 1$, $lx + my + nz = p$.* **[Ans. : $(am^2 + bl^2)y^2 + (an^2 + cl^2)z^2 + 2amnyz - 2ampy - 2anpz + ap^2 - l^2 = 0$]**

2) *Find the equation of cylinder whose generators are parallel to z axis and which passes through the curve of intersection of $x^2 + y^2 + z^2 = 1$, $x + y + z = 1$.* **[Ans. : $x^2 + y^2 + xy - x - y = 0$]**

3) *Find the equation of cylinder whose generators are parallel to x axis and which passes through the curve of intersection of $x^2 + y^2 + z^2 = 1$, $2x + 3y + 5z = 4$.* **[Ans. : $(4 - 3y - 5z)^2 + 4y^2 + 4z^2 = 4$]**

4) *Find the equation of cylinder whose generators are parallel to z axis and which passes through the curve of intersection of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $lx + my + nz = p$.* **[Ans. : $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{1}{n^2c^2}(p - lx - my)^2 = 1$]**

> 5) *Show that the cylinder with generators parallel to z axis and the guiding curve $3x^2 + 2y^2 + z^2 = 18$, $y - z = 3$ is a right circular cylinder. Find the area of the section of the cylinder by XOY plane.* **[Ans. : $x^2 + y^2 - 2y - 3 = 0$, Area of section = 4π]**

13.4 Type 2 Problems on Cylinder with given Guiding Curve

For this type we follow the following standard procedure.

Example 13.3 : ***Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{-1} = \frac{y}{2} = \frac{z}{3}$ whose guiding curve is $x^2 + y^2 = 9$, $z = 1$.***

Solution :

Step 1 : Let $P(x_1, y_1, z_1)$ be any point on the cylinder as $-1, 2, 3$ are dr's of the generator. Thus the equation of the generator is

$$\frac{x - x_1}{-1} = \frac{y - y_1}{2} = \frac{z - z_1}{3} \qquad \text{... (1)}$$

Step 2 : This generator meets the plane z = 1.

$$\therefore \quad \frac{x - x_1}{-1} = \frac{y - y_1}{2} = \frac{1 - z_1}{3}$$

Step 3 : Find x and y in terms of x_1, y_1, z_1.

$$x = x_1 - \frac{(1 - z_1)}{3}, \quad y = y_1 + \frac{2}{3}(1 - z_1)$$

$$x = \frac{(3x_1 + z_1 - 1)}{3}, y = \frac{(3y_1 - 2z_1 + 2)}{3}$$

Step 4 : This point lies on the conic $x^2 + y^2 = 9$.

$\therefore$ Substituting x and y.

$$\frac{(3x_1 + z_1 - 1)^2}{9} + \frac{(3y_1 - 2z_1 + 2)^2}{9} = 9$$

Step 5 : Simplifying and replacing (x_1, y_1, z_1) by (x, y, z)

$9x^2 + 9y^2 + 5z^2 + 6xz - 12yz - 6x + 12y - 10z - 76 = 0.$

is the required equation of cylinder.

Example 13.4 : ***Find the equation of the cylinder whose generators are parallel to $\frac{x}{3} = \frac{y}{1} = \frac{z}{\sqrt{6}}$ and whose guiding curve is $x^2 + y^2 = 25$, $z = 0$.*** **[Dec.-2004, May-2006]**

Solution :

Step 1 : Let (x_1, y_1, z_1) be any point on the generator. As 3, 1, $\sqrt{6}$ are the dr's of the generator $\therefore$ equation of the generator is

$$\frac{x - x_1}{3} = \frac{y - y_1}{1} = \frac{z - z_1}{\sqrt{6}} \quad \text{... (1)}$$

Step 2 : This generator meets the plane z = 0.

$$\therefore \quad \frac{x - x_1}{3} = \frac{y - y_1}{1} = \frac{z - 0}{\sqrt{6}}$$

Step 3 : Find x, y in terms of x_1, y_1, z_1.

$$x = x_1 - \frac{3z_1}{\sqrt{6}}$$

$$y = y_1 - \frac{z_1}{\sqrt{6}}$$

Step 4 : This point lies on $x^2 + y^2 = 25$.

$\therefore$ Substituting x and y.

$$\left(x_1 - \frac{3z_1}{\sqrt{6}}\right)^2 + \left(y_1 - \frac{z_1}{\sqrt{6}}\right)^2 = 25$$

$$\left(\sqrt{6}\, x_1 - 3 z_1\right)^2 + \left(\sqrt{6}\, y_1 - z_1\right)^2 = 150$$

Step 5 : Simplifying and replacing (x_1, y_1, z_1) by (x, y, z) we get

$6x^2 + 6y^2 + 10z^2 - 2\sqrt{6}\, yz - 6\sqrt{6}\, xz - 150 = 0$

which is the required locus of (x_1, y_1, z_1) i.e. the equation of cylinder.

Note : For the following problem the given plane is not parallel to the standard planes thus the procedure for solving will change.

Example 13.5 : *Find the cylinder with generator parallel to x = y = z and with guiding curve $x^2 + 2y^2 + 6xy - 2z + 8 = 0$, $x - 2y + 3 = 0$.*

Solution :

Step 1 : Let P (x_1, y_1, z_1) be any point on cylinder. As 1, 1, 1 are the dr's of generator $\therefore$ equation of generator is

$$\frac{x - x_1}{1} = \frac{y - y_1}{1} = \frac{z - z_1}{1} = (k) \text{ say}$$

Step 2 : Find x, y, z in terms of k, $x = x_1 + k$, $y = y_1 + k$, $z = z_1 + k$.

Step 3 : Substitute x, y, z in plane to find k.

i.e. $x - 2y + 3 = 0$

$$\Rightarrow \quad (x_1 + k) - 2(y_1 + k) + 3 = 0$$

$$\Rightarrow \quad k = x_1 - 2y_1 + 3$$

Step 4 : Substitute k in x, y, z

$$x = 2x_1 - 2y_1 + 3$$

$$y = x_1 - y_1 + 3$$

$$z = x_1 - 2y_1 + z_1 + 3$$

Step 5 : Substitute x, y, z in equation of conic

$$x^2 + 2y^2 + 6xy - 2z + 8 = 0$$

$$(2x_1 - 2y_1 + 3)^2 + 2(x_1 - y_1 + 3)^2 + 6(2x_1 - 2y_1 + 3)(x_1 - y_1 + 3)$$

$$-2(x_1 - 2y_1 + z_1 + 3) + 8 = 0$$

Step 6 : Simplifying and replacing (x_1, y_1, z_1) by (x, y, z) we get

$$18x^2 + 18y^2 - 36xy + 76x - 74y - 2z + 83 = 0$$

Exercise 13.2 : Problems on Type 2

1) *Find the equation of cylinder whose generators are parallel to the line* $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ *and guiding curve* $x^2 + 2y^2 = 1, z = 3$. **(May-1998)**
[Ans. : $3(x^2 + 2y^2 + z^2) + 8yz - 2xz + 6x - 18z - 24y + 24 = 0$]

2) *Find the equation of the cylinder whose generators are parallel to* $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ *whose guiding curve is* $x^2 + y^2 = 16, z = 0$. **(May 02) [Ans. : $9x^2 + 9y^2 + 5z^2 - 6xz - 12yz - 14y = 0$]**

3) *Find the equation of the cylinder whose generators are parallel to the line* $6x = -3y = 2z$ *and guiding curve is* $x^2 + \frac{y^2}{4} = \frac{1}{4}, z = 0$. **[Ans. : $36x^2 + 9y^2 + 17z^2 + 6yz - 48zx - 9 = 0$]**

4) *Find the equation of the cylinder whose generators are parallel to* $6x = -3y = 2z$ *and the guiding curve is* $x^2 + y^2 = 1, z = 3$. **[Ans. : $9x^2 + 9y^2 + 5z^2 + 12yz - 6zx + 18x - 36y = 0$]**

5) *Find the equation of the cylinder whose generating lines have direction cosines l, m, n and which pass through* $x^2 + z^2 = a^2$ *in ZOX plane.* **[Ans. : $(mx - ly)^2 + (mz - ny)^2 = a^2m^2$]**

6) *Find the equation of cylinder whose guiding curve is* $x^2 + z^2 - 4x - 2z + 4 = 0, y = 0$ *and whose axis contain the point* $(0, 3, 0)$.
[Ans. : $9x^2 + 5y^2 + 9z^2 + 12xy + 6yz - 36x - 30y - 18z + 36 = 0$]

Hint : *Centre of given circle is C (2, 0, 1). Axis also passes through D (0, 3, 0).*

Thus dr's of axis CD in (2, – 3, 1).

$\therefore$ *Equation of generator* $\frac{x - x_1}{2} = \frac{y - y_1}{-3} = \frac{z - z_1}{1}$ *and then proceed as usual.*

13.5 Right Circular Cylinder

Right circular cylinder is a surface generated by straight line parallel to a fixed line and is at a constant distance from it.

The fixed line is called as the axis and the constant distance is the radius of the cylinder.

The section of a right circular cylinder by any plane perpendicular to axis is a circle.

13.6 Equation of a Right Circular Cylinder

General Form :

To find the equations of the right circular cylinder whose radius is r and axis is the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$.

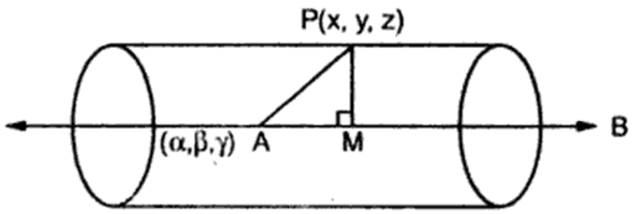


Fig. 13.1

Let A be the point (α, β, γ) and AB the axis whose equations are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

The dr's of the line are l, m, n

$\therefore$ The dc's of the line are

$$\frac{l}{\sqrt{l^2+m^2+n^2}}, \frac{m}{\sqrt{l^2+m^2+n^2}}, \frac{n}{\sqrt{l^2+m^2+n^2}}$$

Let P (x, y, z) be any point on the cylinder and PM perpendicular to AB.

Then PM = Radius = r

$$PA = \sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}$$

$$AM = \text{Projection of AP on line AB}$$

$$AM = \frac{(x-\alpha)\, l}{\sqrt{l^2+m^2+n^2}} + \frac{(y-\beta)\, m}{\sqrt{l^2+m^2+n^2}} + \frac{(z-\gamma)\, n}{\sqrt{l^2+m^2+n^2}}$$

From Δ APM

$$AP^2 = AM^2 + PM^2$$

Since line (1) is a tangent line, the line should touch the sphere in only one point and therefore equation (2) has two equal roots or the discriminant of the equation (2) will be zero.

$$4\left(lx_1 + my_1 + nz_1\right)^2 - 4\left(l^2 + m^2 + n^2\right)\left(x_1^2 + y_1^2 + z_1^2 - a^2\right) = 0$$

$\therefore$ The locus of the point (x, y, z) is

$$\left(lx + my + nz\right)^2 - \left(l^2 + m^2 + n^2\right)\left(x^2 + y^2 + z^2 - a^2\right) = 0$$

Note : Enveloping cylinder of a sphere is always right circular.

Note : For solving the problems on right circular cylinder the following three things are necessary

i) one point on axis

ii) dr's of axis

iii) radius of cylinder

If these three things are known then we can follow the following standard procedure for solving the problems.

13.8 Type 3 Illustrations : Problems on Right Circular Cylinder and Enveloping Cylinder

Example 13.6 : *Find the equation of right circular cylinder of radius 2 and equation of axis is* $\dfrac{x-1}{2} = \dfrac{y-2}{-3} = \dfrac{z-3}{6}$ **[Dec.-99, May-2000, Dec.-2003, May-2004]**

Solution : Given i) One point on axis = (1, 2, 3)

ii) dr's of axis = 2, – 3, 6

iii) Radius of cylinder = 2

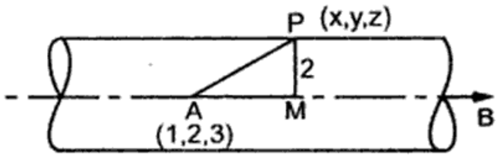


Fig. 13.3

Step 1 : Let P (x, y, z) be any point on cylinder and A (1, 2, 3) be one point on axis and M be the perpendicular from P on axis.

Step 2 :

$$\text{Distance AP} = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$$

Distance PM = radius = 2

and distance AM = Projection of AP on axis AB

Now from dr's (2, – 3, 6) we can find dc's of axis.

$\therefore$ dc's of AB are $\left(\frac{2}{7}, \frac{-3}{7}, \frac{6}{7}\right)$

$$\therefore \quad AM = \frac{2}{7}(x-1) - \frac{3}{7}(y-2) + \frac{6}{7}(z-3)$$

$$AM = \frac{2x - 3y + 6z - 14}{7}$$

Step 3 : From Δ APM

$$AP^2 = AM^2 + MP^2$$

Step 4 : Substituting AP, AM, MP we get

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = \frac{(2x - 3y + 6z - 14)^2}{49} + 4$$

Step 5 : Simplify to get required equation of right circular cylinder

$$45x^2 + 40y^2 + 13z^2 + 12xy + 36yz - 24xz - 42x - 280y - 126z + 294 = 0$$

Example 13.7 : ***Find the equation of right circular cylinder whose guiding curve is*** $x^2 + y^2 + z^2 = 9, x - y + z = 3.$ **[May-2003, May-2005]**

Solution :

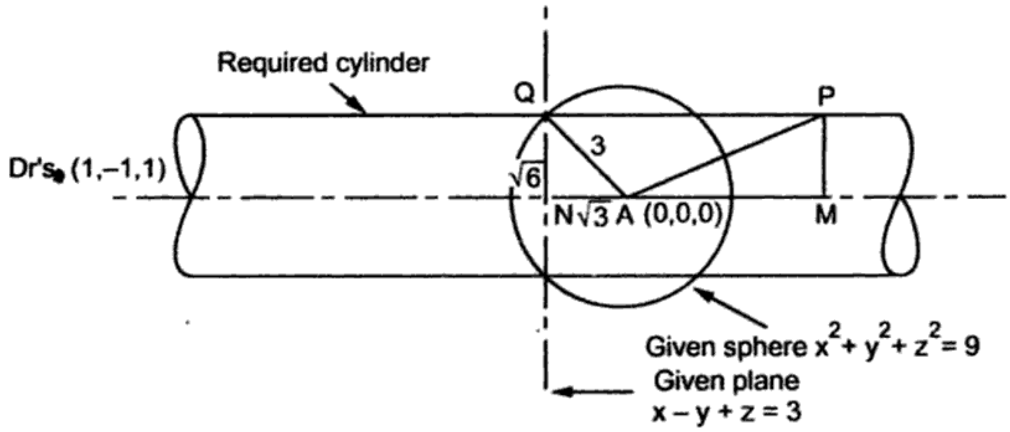


Fig. 13.4

Given sphere $x^2 + y^2 + z^2 = 9$

$\therefore$ radius of sphere = 3

Centre of sphere = (0, 0, 0)

$\therefore$ AQ = 3 and A is (0, 0, 0)

Step 4 : Substituting AP, AM, PM

$$x^2 + y^2 + z^2 = \frac{(x - y + z)^2}{3} + 6$$

Step 5 : Simplify to get the required equation of right circular cylinder

$$3\left(x^2 + y^2 + z^2\right) - (x - y + z)^2 = 18$$

$$x^2 + y^2 + z^2 + xy + yz - zx = 9$$

Example 13.8 : *Find the equation of the right circular cylinder which passes through the section of the sphere $x^2 + y^2 + z^2 = 25$, $x + 2y + 2z = 0$.* **[May-1999]**

Solution :

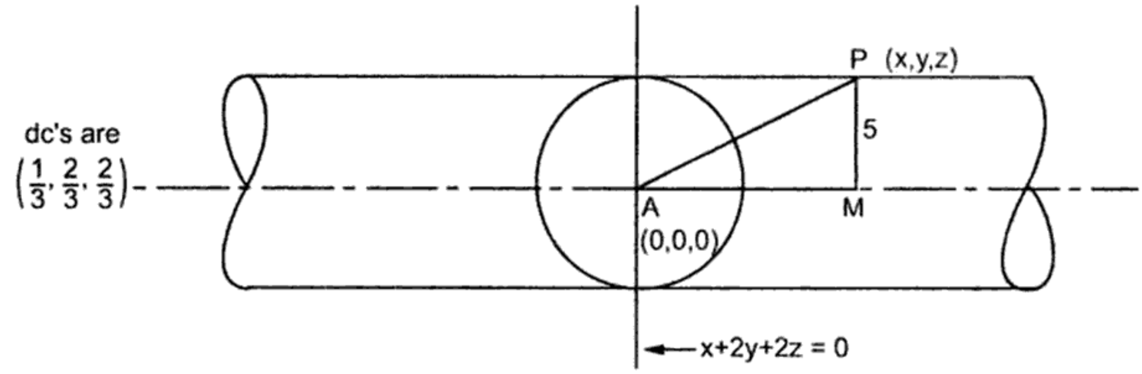


Fig. 13.5

Given sphere $x^2 + y^2 + z^2 = 25$

$\therefore$ radius of sphere = 5 and centre of sphere = (0, 0, 0)

Note that this centre satisfies $x + 2y + 2z = 0$.

Thus radius of the cylinder = radius of sphere = 5

i.e. the required cylinder is the enveloping cylinder of the given sphere

Now we know that

i) (0, 0, 0) one point on axis

ii) (1, 2, 2) dr's of axis ($\because$ the axis is normal to the plane)

iii) radius of cylinder = 5

Thus

Step 1 : Let P (x, y, z) be any point on the cylinder A (0, 0, 0) is one point on axis. Let M be the foot of the perpendicular from point P on the axis.

Step 2 : $$AP = \sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2}$$

$$PM = 5 \text{ (radius of cylinder)}$$

Exercise 13.3 : Problems on Type 3

1) *Find the equation of right circular cylinder of radius 3 whose axis is the line* $\frac{x-1}{2}=\frac{y-3}{2}=\frac{z-5}{-1}$.

[Ans. : $5x^2 + 5y^2 + 8z^2 - 8xy - 4xz - 6x - 42y - 96z + 225 = 0$]

2) *Find the equation of right circular cylinder of radius 2 whose axis passes through A (1, – 2, 4) and has dc's proportional to 2, 3, 6.* **[May-2006]**

[Ans. : $45x^2 + 40y^2 + 13z^2 - 12xy - 36yz - 24zx - 18x - 316y - 152z + 433 = 0$]

3) *The radius of normal section of a right circular cylinder is 2 units, the axis lies along* $\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z-2}{5}$ *Find the equation of right circular cylinder.*

[Ans. : $26x^2 + 29y^2 + 5z^2 + 4xy - 20xz + 10yz + 150y + 50z + 75 = 0$]

4) *Find the equation of the right circular cylinder of radius 2 whose axis is the line* $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$

[Ans. : $5x^2 + 8y^2 + 5z^2 - 4yz - 8zx - 4xy + 22x - 16y - 14z - 10 = 0$]

5) *Obtain the equation of a right circular cylinder of radius 5 and axis is* $\frac{x-2}{2}=\frac{y-3}{1}=\frac{z+1}{1}$.

[Ans. : $10x^2 + 13y^2 + 5z^2 + 6yz - 12xz + 4xy - 64x - 80y + 16z - 154 = 0$]

6) *Find the equation of right circular cylinder whose axis is* $\frac{x-2}{2}=\frac{y-1}{1}=\frac{z}{3}$ *and which passes through the point (0, 0, 3)*

[Ans. : $10x^2 + 13y^2 + 5z^2 - 4xy - 6yz - 12zx - 36x - 18y + 30z - 135 = 0$]

7) *The axis of the right circular cylinder of radius 2 is* $\frac{x-1}{2}=\frac{y}{3}=\frac{z-3}{1}$ *Show that its equation is*

$10x^2 + 5y^2 + 13z^2 - 12xy - 6yz - 4xz - 8x + 30y - 74z + 59 = 0$

8) *Find the equation of the right circular cylinder whose axis is* $x = 2y = -z$ *and radius is 4. Prove that the area of the section of this cylinder by the plane* $z = 0$ *is* 24π.

[Ans. : $5x^2 + 8y^2 + 5z^2 + 4yz + 8zx - 4xy - 144 = 0$]

9) *Find the equation of the right circular cylinder whose radius is 3, whose axis passes through (1, – 1, 2) has dc's proportional to (2, – 1, 3)*

[Ans. : $10x^2 + 13y^2 + 5z^2 + 6yz - 12xz + 4xy + 8x + 10y - 2z - 123 = 0$]

10) *Show that the foot of the perpendicular from the point* (α, β, γ) *on the line* $x = +y = -z$ *is* $\left(\frac{\alpha+\beta-\gamma}{3}, \frac{\alpha+\beta-\gamma}{3}, \frac{\alpha+\beta-\gamma}{-3}\right)$. *Deduce the equation of right circular cylinder of radius having its axis along the given line.*

Hint :

dr's AB are (1, 1, – 1)

From equation $x = y = -z$ *of line AB.*

Let $(k, k, -k)$ *be co-ordinates of M.*

$\therefore$ *dr's of PM are* $(\alpha - k, \beta - k, \gamma + k)$

As AB $\perp$ *PM.*

$\therefore k = \frac{\alpha+\beta-\gamma}{3}$

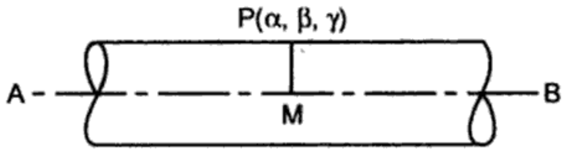


Fig. 13.7

Hence we know co-ordinates of M.

Also $(PM)^2 = a^2$

Now using distance formula find PM and replace α, β, γ *by* x, y, z *to get equation of cylinder.*

$$\left[\text{Ans.}: x^2 + y^2 + z^2 - xy + yz + zx = \frac{3a^2}{2}\right]$$

11) *Find the equation of right circular cylinder of radius 'a' whose axis passes through origin and makes equal angles with co-ordinate axis.*

$$\left[\text{Ans.}: x^2 + y^2 + z^2 - xy - yz - zx = \frac{3a^2}{2}\right]$$

12) *Show that the equation of right circular cylinder described on the circle through three points* $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ *as the guiding curve is* $x^2 + y^2 + z^2 - xy - yz - zx = 1$.

13) *Show that the equation of right circular cylinder described on the circle through three points* $(3, 0, 0)$, $(0, 3, 0)$, $(0, 0, 3)$ *as the guiding curve is* $x^2 + y^2 + z^2 - xy - yz - zx = 9$.

14) *Find the equation of the enveloping cylinder of the sphere* $x^2 + y^2 + z^2 - 2x + 4y = 1$ *having its generators parallel to the line* $x = y = z$.

$$[\text{Ans.}: x^2 + y^2 + z^2 - xy - yz - zx - 4x + 5y - z - 2 = 0]$$

15) *Find the equation of the enveloping cylinder of the sphere* $x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$ *having its generator parallel to the line* $x = -2y = 2z$.

$$[\text{Ans.}: 2x^2 + 5y^2 + 5z^2 + 4xy - 4xz + 2yz + 4x - 14y - 11z - 67 = 0]$$

University Questions

May - 2003

1. *Find the equation of right circular cylinder whose guiding curve is* $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$. **[6 Marks]**

Dec. - 2003

1. *Find the equation of right circular cylinder of radius 2 whose axis passes through (1,2, 3) and has direction cosines proportional to 2, – 3, 6.* **[6 Marks]**

May - 2004

1. *Find the equation of the right circular cylinder described on the circle through (3, 0, 0), (0, 3, 0), (0, 0, 3).* **[5 Marks]**
2. *Find the equation of right circular cylinder of radius 2 whose axis passes through (1,2, 3) and has direction cosines proportional to 2, – 3, 6.* **[5 Marks]**

Dec. - 2004

1. *Find the equation of the right circular cylinder of radius 2 and whose axis lies along the straight line* $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{5}$. **[5 Marks]**
2. *Find the equation of the cylinder whose generators are parallel to the line* $\frac{x-1}{3} = \frac{y-1}{1} = \frac{z+2}{\sqrt{6}}$ *and whose guiding curve is,* $x^2 + y^2 = 25$, $z = 0$. **[5 Marks]**

May - 2005

1. *Find the equation of the right circular cylinder whose guiding curve is* $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$. **[5 Marks]**
2. *Find the equation of the cylinder whose generators are parallel to x-axis and whose guiding curve is* $ax^2 + by^2 + cz^2 = 1$, $lx + my + nz = p$. **[6 Marks]**

Dec. - 2005

1. *Find the equation of the right circular cylinder of radius 2 whose axis the line* $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$ **[5 Marks]**
2. *Find the equation of right circular cylinder of radius 2 whose axis passes through (1, 2, 3) and has direction cosines proportional to 2, – 3, 6.* **[6 Marks]**

May - 2006

1. *Find the equation of the right circular cylinder of radius 2 whose axis passes through the point (1, – 2, 4) and has direction cosines proportional to 2, 3, 6.* **[6 Marks]**
2. *Find the equation of the cylinder whose generators are parallel to the line* $\frac{x}{3} = \frac{y}{1} = \frac{z}{\sqrt{6}}$ *and whose guiding curve is* $x^2 + y^2 = 25$, $z = 0$. **[5 Marks]**

Dec. - 2006

1. *Find the equation of the right circular cylinder whose guiding curve is* $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$. **[6 Marks]**
2. *Find the equation of the cylinder whose generators are parallel to the line* $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$, *and guiding curve is* $x^2 + 2y^2 = 1$, $z = 3$. **[5 Marks]**

May - 2007

1. *Find the equation of right circular cone whose vertex is at the origin, whose axis is the line* $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ *and which has a semivertical angle of 30°.* **[5 Marks]**
2. *Find the cylinder with generator parallel to* $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$ *and with guiding curve* $x^2 + 2y^2 + 6xy - 2z + 8 = 0$, $x - 2y + 3 = 0$. **[5 Marks]**

Dec. - 2007

1. *Find the equation of the right circular cylinder whose axis is* $x = 2y = -z$ *and radius is 4.* **[5 Marks]**
2. *Find the equation of the cylinder whose generators are parallel to the line* $\frac{x}{-1} = \frac{y}{2} = \frac{z}{3}$ *and whose guiding curve is* $x^2 + y^2 = 9$, $z = 1$. **[5 Marks]**

May - 2008

1. *Find the equation of right circular cylinder which passes through the section of the sphere* $x^2+y^2+z^2=25$ *made by plane* $x+2y+2z=0$. **[5 Marks]**

2. *Find the equation of right circular cylinder whose axis is* $\frac{x-2}{2}=\frac{y-1}{1}=\frac{z}{3}$ *and which passes through the point* $(0, 0, 3)$. **[5 Marks]**

Dec. - 2008

1. *Find the equation of right circular cylinder whose guiding curve in* $x^2+y^2+z^2=9, x-y+z=3$. **[6 Marks]**

2. *Find the equation of cylinder with generators parallel to* $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$ *and with guiding curve* $x^2+2y^2+6xy-2z+8=0,\ x-2y+3=0$. **[5 Marks]**

3. *Find the equation of the right circular cylinder whose axis is* $x=2y=-z$ *and radius is 4.*

□□□

14

Multiple Integrals

14.1 Introduction

The students are already familiar with single integrals. In this chapter we shall deal with multiple integrals (mainly double and triple integrals) and its applications to find area Volume, Mean and Root Mean Square Values, Mass, Centre of Gravity and Moment of Inertia. Double and triple integrals also useful in vector integrations and electromagnetic theory.

14.2 Double Integration

(I) Definition :

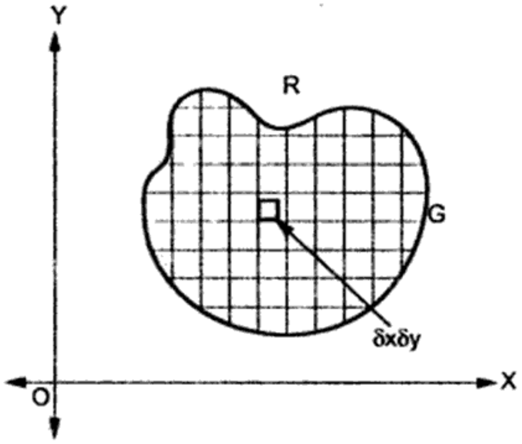


Fig. 14.1

The single integral $\int_a^b f(x)\,dx$ is defined as limit of sum of

$$f(x_1)\delta x_1 + f(x_2)\delta x_2 + \ldots + \delta(x_n)\delta x_n \text{ as } n \to \infty$$

i.e. $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} f(x_r)\cdot\delta x_r$

Similarly If f(x, y) be continuous and single valued function defined over the region R bounded by closed curve C. Divide the region R into subregions R_1, R_2, ... R_n of Areas δA_1, δA_2, ... δA_n.

Let P (x_r, y_r) be any point inside r^{th} subregions of area δA_r. Adding these areas together we get

$$f(x_1, y_1)\delta A_1 + f(x_2, y_2)\delta A_2 + \ldots + f(x_n, y_n)\delta A_n = \sum_{r=1}^{n} f(x_r, y_r)\delta A_r$$

Thus, Double integral of the function f (x, y) over the region R is

$$\boxed{\iint_R f(x, y)\,dx\,dy = \lim_{\substack{n\to\infty \\ \delta A \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r}$$

(II) Evaluation of Double Integral

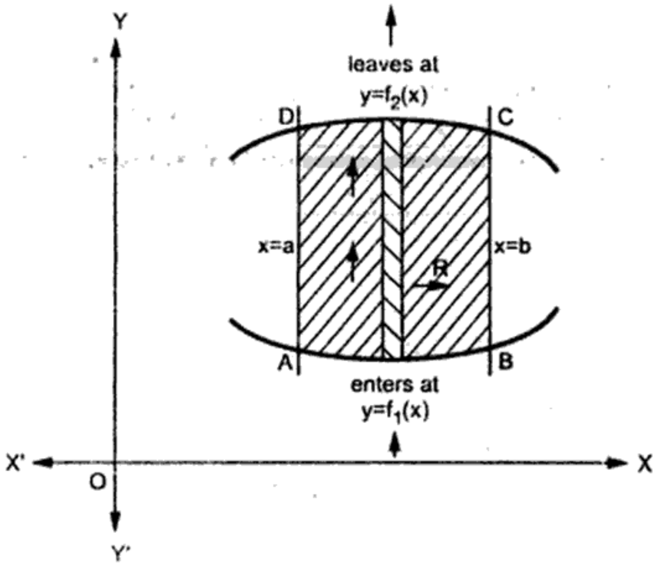


Fig. 14.2

The evaluation of double integral depends upon the nature of the curves bounding the Region R (ABCD). Let R be the region bounded by the curves $y = f_1(x)$, $y = f_2(x)$, $x = a$, $x = b$ Fig. 14.2.

then

$$\iint_R f(x, y)\, dxdy = \int_{x=a}^{x=b} \int_{y=f_1(x)}^{y=f_2(x)} f(x, y)\, dy\, dx$$

$$= \int_{x=a}^{x=b} \left[\int_{y=f_1(x)}^{y=f_2(x)} f(x, y)\, dy \right] dx \qquad \dots (1)$$

R.H.S of (1) indicates that the integral f (x, y) to be integrated w.r.t. y first over the limits $y = f_1(x)$ **to** $y = f_2(x)$.

Keeping x constant and the resulting expression to be integrated w.r.t. x over the limits **x = a to x = b.**

Similarly, if the Double integral is of the form :

$$\iint_R f(x, y)\, dxdy = \int_{y=c}^{y=d} \int_{x=f_1(y)}^{x=f_2(y)} f(x, y)\, dxdy \qquad \dots (2)$$

then the integral f (x, y) to be integrated w.r.t x over the limits $x = f_1(y)$ **to** $x = f_2(y)$ first and the resulting expression w.r.t y over the limits **y = c to y = d.**

Note : (A) From (1) and (2), we observe that the function (Integrand) f (x, y) to be integrated w.r.t. the variable whose limits are variable and the w.r.t the variable whose limits are constant.

(B) If x, y has constant limits i.e. x = a, x = b, y = c, y = d then it is immaterial whether we integrate w.r.t. x first or w.r.t y first i.e.

$$\iint_R f(x, y)\, dxdy = \int_{x=a}^{x=b} \left[\int_{y=c}^{y=d} f(x, y)\, dy \right] dx = \int_{y=c}^{y=d} \left[\int_{x=a}^{x=b} f(x, y)\, dx \right] dy \qquad \dots (3)$$

14.3 Illustrated Examples

Type I Direct Evaluation of Double Integral

Example 14.1 : *Evaluate* $\int_0^1 \int_0^x e^{y/x}\, dy\, dx$

Solution : Let

$$I = \int_{x=0}^{x=1} \int_{y=0}^{y=x} e^{y/x}\, dy\, dx \qquad \text{(w.r.t y first)}$$

$$= \int_0^1 \left(\int_{y=0}^{x} e^{y/x}\, dy \right) dx$$

$$= \int_0^1 \left(\frac{e^{y/x}}{\frac{1}{x}} \right)_0^x dx$$

$$= \int_0^1 \left(x\, e^{y/x} \right)_0^x dx$$

$$= \int_0^1 [(x\, e) - x]\, dx$$

$$= (e - 1)\left(\frac{x^2}{2} \right)_0^1$$

$$\boxed{I = \frac{(e-1)}{2}} \qquad \text{... Ans}$$

Example 14.2 : *Evaluate* $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{1 + x^2 + y^2}$

Solution : Let

$$I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1+x^2}} \frac{dx\, dy}{1 + x^2 + y^2} \qquad \text{(w.r.t y first)}$$

$$= \int_0^1 \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{dy}{(1 + x^2 + y^2)} \right] dx$$

$$= \int_0^1 \left[\int_{y=0}^{a} \frac{dy}{a^2 + y^2} \right] dx \qquad \text{Let } a = \sqrt{1 + x^2} = \text{constant}$$

Example 14.4 : *Evaluate* $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\, dx\, dy$

Solution : Let

$$I = \int_{y=0}^{a} \int_{0}^{\sqrt{a^2-y^2}} \sqrt{a^2 - y^2 - x^2}\, dx\, dy \qquad \text{(w.r.t. x first)}$$

$$= \int_{y=0}^{a} \left[\int_{x=0}^{m} \sqrt{m^2 - x^2}\, dx \right] dy \qquad \text{where } m = \sqrt{a^2 - y^2} = \text{constant}$$

$$= \int_{y=0}^{a} \left[\frac{x}{2}\sqrt{m^2 - x^2} + \frac{m^2}{2} \sin^{-1}\left(\frac{x}{m}\right) \right]_0^m dy$$

$$= \int_0^a \left[\frac{m^2}{2} \frac{\pi}{2} \right] dy \qquad \left(\text{since } m^2 = a^2 - y^2\right)$$

$$= \frac{\pi}{4} \int_0^a \left(a^2 - y^2\right) dy = \frac{\pi}{4} \left(a^2 y - \frac{y^3}{3}\right)_0^a$$

$$\boxed{I = \frac{\pi a^3}{6}} \qquad \text{... Ans}$$

Example 14.5 : *Evaluate* $\int_0^a \int_0^{\cos^{-1}\left(\frac{r}{a}\right)} r \sin\theta\, dr\, d\theta$

Solution : Let

$$I = \int_{r=0}^{a} \int_{\theta=0}^{\cos^{-1}\left(\frac{r}{a}\right)} r \sin\theta\, dr\, d\theta$$

$$= \int_{r=0}^{a} r \left[\int_{\theta=0}^{\cos^{-1}\left(\frac{r}{a}\right)} \sin\theta\, d\theta \right] dr$$

$$= \int_{r=0}^{a} r(-\cos\theta)_0^{\cos^{-1}\left(\frac{r}{a}\right)} dr$$

$$= -\int_{r=0}^{a} r\left(\frac{r}{a} - 1\right) dr$$

$$= \int_0^a \left(r - \frac{r^2}{a}\right) dr$$

$$= \int_0^1 \left(\frac{y^2}{2}\right)_{x^2}^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 [(2-x)^2 - x^4]\, dx$$

$$= \frac{1}{2}\int_0^1 (4 - 4x + x^2 - x^4).dx$$

$$= \frac{1}{2}\left(4x - 4\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^5}{5}\right)_0^1$$

$$= \frac{1}{2}\left[4 - 2 + \frac{1}{3} - \frac{1}{5}\right] - \frac{1}{2}[0]$$

$$\boxed{I = \frac{16}{15}} \qquad \text{... Ans}$$

Example 14.8 : *Evaluate* $\iint_R (x^2 + y^2)dxdy$ *over the area of triangle whose vertices are (0, 1) (1, 1) and (1, 2).*

Solution : Let $$I = \iint_R (x^2 + y^2)\, dxdy \qquad \text{... (1)}$$

Where R is triangle whose vertices are (0, 1), (1, 1) and (1, 2) as shown in the Fig. 14.5.

Equation of line AC is

$$\frac{y-2}{1} = \frac{x-1}{1} \Rightarrow y = x + 1$$

To find limits for x and y, consider vertical strip in the shaded region over which y varies from **y = 1 to y = x + 1,**

Moving the strip from **x = 0 to x = 1,** we get the complete region of integration.

$\therefore$ From (1), we have

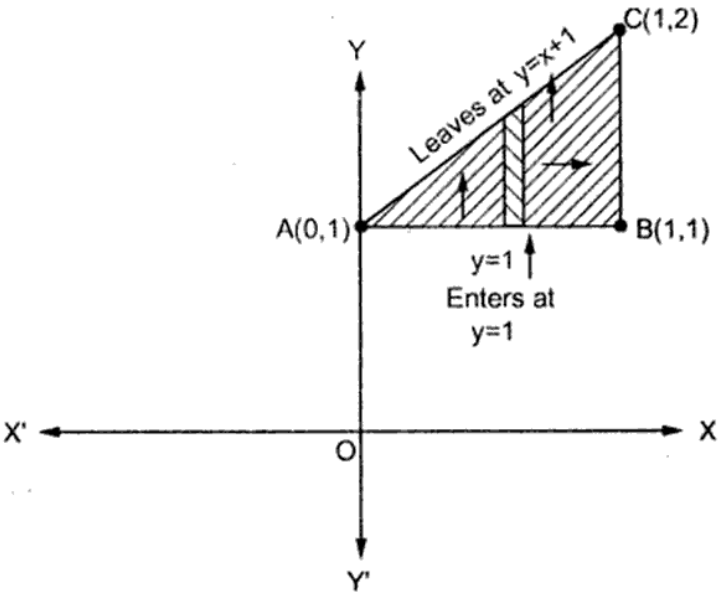


Fig. 14.5

$$I = \int_{x=0}^{1}\int_{y=1}^{x+1} (x^2 + y^2)\, dxdy \qquad \text{(w.r.t. 'y' first keeping 'x' as constant)}$$

7) $\iint \sin(ax + by)\,dxdy$ *over the area of triangle bounded by* $x = 0$, $y = 0$ *and* $ax + by = 1$.

[Ans. : $\frac{1}{\pi\, ab}$]

8) $\iint\limits_A (5 - 2x - y)\,dxdy$ *where A is area bounded by* $y = 0$, $x + 2y = 3$, $x = y^2$.

[Ans. : $\frac{217}{60}$]

9) $\iint e^{3x + 4y}\,dxdy$ *over* $x = 0$, $y = 0$ *and* $x + y = 1$.

[Ans. : $\frac{1}{12}\left(3e^4 - 4e^3 + 1\right)$]

10) $\iint x^2\,dxdy$ *over the area bounded by hyperbola* $xy = 16$ *and the lines* $y = x$, $y = 0$ *and* $x = 8$

[Ans. : 448]

11) $\iint e^{y^2}\,dxdy$ *over the triangle whose vertices are (0, 0) (2, 1) and (0, 1)*

[Ans. : e – 1]

12) *Evaluate* $\iint\limits_R y\,dxdy$ *over the ellipse* $4x^2 + 9y^2 = 36$ *in positive quadrant.*

(Dec.-2000) [Ans. : 3]

Type III Evaluation of Double Integral by Changing the Order of Integration

In many cases it happen that the function f (x, y) in the double integral.

$$\int_{x=a}^{x=b}\left[\int_{y=f_1(x)}^{y=f_2(x)} f(x, y)\,dy\right]dx \qquad \dots (1)$$

is difficult or even impossible to integrate w.r.t. y first, however, easy to integrate w.r.t. x first. In this case it is required to chaʳ ·e or reverse the order of integration. The new limits are obtained by sketching the region of integration and considering reverse strip in the region of integration. i.e. we put the integral (1) in the form :

$$\int_{y=c}^{y=d}\left[\int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x, y)\,dx\right]dy \qquad \dots (2)$$

and evaluate it. The following examples illustrate this point/concept.

14.4 Illustrated Examples

Example 14.11 : ***Evaluate*** $\int_0^1 \int_{4y}^{4} e^{x^2}\,dx\,dy$

Solution : Let

$$I = \int_{y=0}^{1}\left[\int_{x=4y}^{x=4} e^{x^2}\,dx\right]dy \qquad \dots (1)$$

The integrand e^{x^2} is difficult to integrate w.r.t x over the limits x = 4y to x = 4. Therefore it is required to change the order of integration.

The limits are **x = 4y, x = 4** (limits over strip PQ) **y = 0, y = 1** for which the region of integration is as shown in the Fig. 14.8.

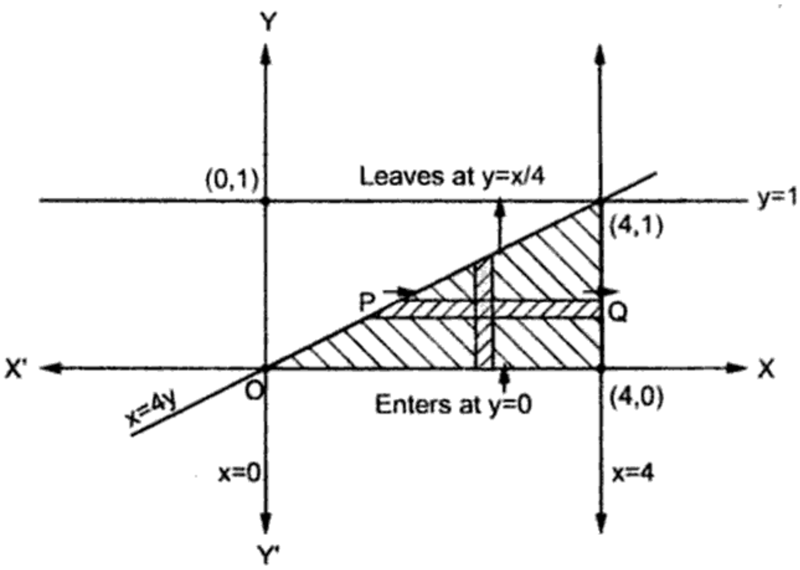


Fig. 14.8

To change the order, consider a vertical strip (parallel to y-axis) over which y varies from **y = 0 to** $\mathbf{y = \frac{x}{4}}$ and moving the strip from **x = 0 to x = 4.** We get the shaded region of integration.

From (1)

$$I = \int_{x=0}^{4} \int_{y=0}^{x/4} e^{x^2}\, dx dy$$

$$= \int_{x=0}^{4} e^{x^2} \left[\int_{y=0}^{x/4} dy \right] dx$$

$$= \int_{0}^{4} e^{x^2} (y)_0^{x/4}\, dx$$

$$= \int_{0}^{4} e^{x^2} \frac{x}{4}\, dx$$

$$= \frac{1}{8}\int_{0}^{4} e^{x^2} (2x)\, dx$$

$$\boxed{I = \frac{1}{8}\left(e^{x^2}\right)_0^4 = \frac{1}{8}\left(e^{16} - 1\right)} \quad \text{... Ans}$$

$$= 2\int_0^m \frac{t\,dt}{\sqrt{m^2 - t^2}\,t} \qquad \text{(Assume } m = \sqrt{a - y}\text{)}$$

$$I_1 = 2\left(\sin^{-1}\frac{t}{m}\right)_0^m = 2\frac{\pi}{2} = \pi$$

Now, from (1)

$$I = \int_0^a \frac{1}{y + a}\pi\, dy$$

$$= \pi\left[\log(y + a)\right]_0^a$$

$$\boxed{I = \pi\left[\log 2a - \log a\right] = \pi \log 2} \qquad \text{... Ans}$$

Example 14.13 : *Evaluate* $\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{\cos^{-1} x\,dx\,dy}{\sqrt{(1 - x^2 - y^2)(1 - x^2)}}$ **(May - 1998)**

Solution : Let

$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} \frac{\cos^{-1}x\,dx\,dy}{\sqrt{1 - x^2 - y^2}\sqrt{1 - x^2}} \qquad \text{... (1)}$$

The integrand in (1) is **complicated to integrate w.r.t. x** first, but **easy to integrate w.r.t y,** therefore, it is required to change the order of integration.

The limits are $\mathbf{x = 0}$, $\mathbf{x = \sqrt{1 - y^2}}$ or $x^2 = 1 - y^2$

$\Rightarrow x^2 + y^2 = 1$ (circle) centre = (0, 0) Radius = 1 and $\mathbf{y = 0}$, $\mathbf{y = 1}$.

Region of integration is as shown in the Fig. 14.10.

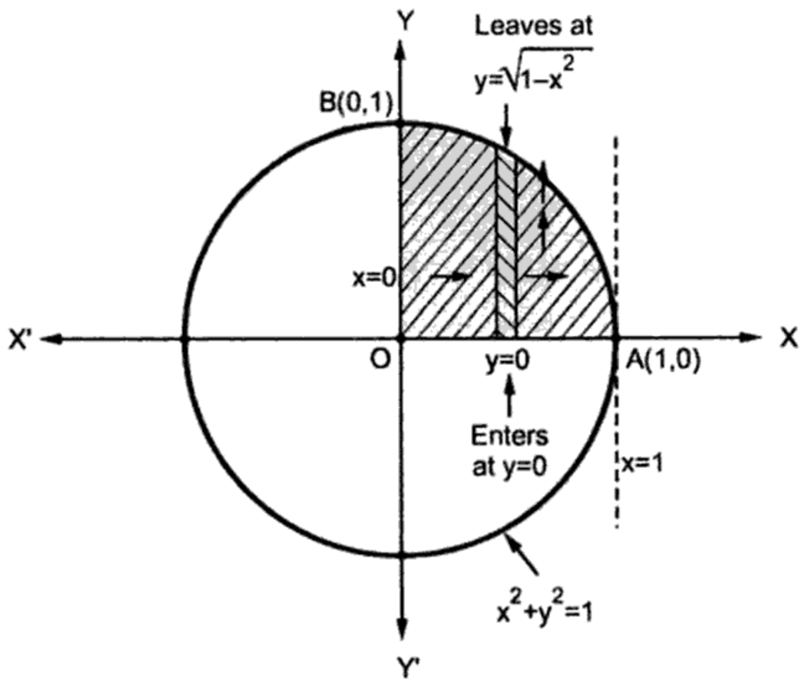


Fig. 14.10

Imagine vertical strip in the region of integration over which y varies from **y = 0 to** $y = \sqrt{1-x^2}$ and moving the strip from **x = 0 to x = 1** we get complete shaded quadrant of the circle (Region of integration).

From (1)

$$I = \int_{x=0}^{1} \frac{\cos^{-1}x}{\sqrt{1-x^2}} \left[\int_{y=0}^{\sqrt{1-x^2}} \frac{dy}{\sqrt{1-x^2-y^2}} \right] dx$$

$$= \int_{0}^{1} \frac{\cos^{-1}x}{\sqrt{1-x^2}} (I_1)\, dx \qquad \text{... (2)}$$

Where

$$I_1 = \int_{0}^{\sqrt{1-x^2}} \frac{dy}{\sqrt{1-x^2-y^2}}$$

$$= \int_{0}^{m} \frac{dy}{\sqrt{m^2-y^2}} \qquad \text{(Assume } m = \sqrt{1-x^2}\text{)}$$

$$I_1 = \left(\sin^{-1}\frac{y}{m} \right)_0^m = \sin^{-1}(1) - \sin^{-1}0 = \frac{\pi}{2} - 0 = \frac{\pi}{2} \qquad \text{... (3)}$$

From (2)

$$I_1 = \frac{\pi}{2} \int_{0}^{1} \frac{\cos^{-1}x}{\sqrt{1-x^2}}\, dx$$

Put $\cos^{-1} x = u$

$$\therefore \frac{-dx}{\sqrt{1-x^2}} = dy$$

Limits :

x	0	1
u	$\frac{\pi}{2}$	0

$$= \frac{\pi}{2} \int_{\pi/2}^{0} -u\, du$$

$$= \frac{\pi}{2} \int_{0}^{\pi/2} u\, du$$

$$= \frac{\pi}{2} \left(\frac{u^2}{2} \right)_0^{\pi/2}$$

$$\boxed{I = \frac{\pi^3}{16}} \qquad \text{... Ans}$$

Example 14.14 : *Evaluate* $\displaystyle\int_0^a \int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx\,dy$ **(May-2004)**

Solution : The integrand is difficult to integrate **w.r.t. x first** while it is handy to integrate **w.r.t. y first.** Therefore, it is required to change the order of integration.

The limits are

$x = 0,\ x = a - \sqrt{a^2 - y^2}$

or $x - a = -\sqrt{a^2 - y^2} < 0$ and $(x-a)^2 + y^2 = a^2$

is a circle with centre = (a, 0) and radius = a

$x = a - \sqrt{a^2 - y^2}$ represents a semicircle on left side of the line x = a as shown in the Fig. 14.11.

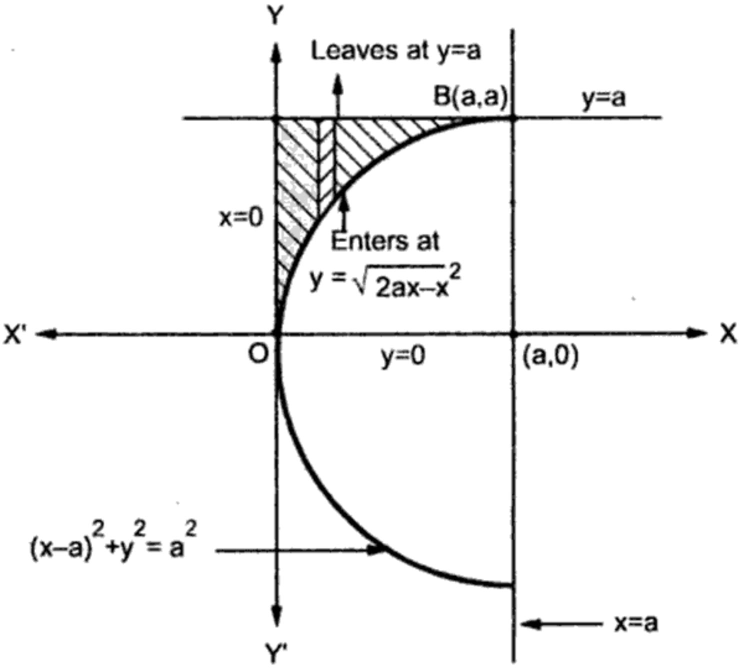


Fig. 14.11

To find new limits for x, y consider a vertical strip in the region of integration over which y varies from $\mathbf{y = \sqrt{2ax - x^2}}$ **to y = a** and moving the strip from **x = 0 to x = a.** We get the shaded region covered .

∴ From the given integral

$$I = \int_{x=0}^{a} \int_{y=\sqrt{2ax-x^2}}^{a} \frac{xy\log(x+a)}{(x-a)^2}\,dx\,dy$$

$$= \int_{x=0}^{a} \frac{x\log(x+a)}{(x-a)^2}\left[\int_{y=\sqrt{2ax-x^2}}^{y=a} y\,dy\right] dx$$

$$= \int_0^a \frac{x\log(x+a)}{(x-a)^2}\left(\frac{y^2}{2}\right)_{\sqrt{2ax-x^2}}^{a} dx$$

$$= \frac{1}{2}\int_0^a \frac{x\log(x+a)}{(x^2-2ax+a^2)}(a^2-2ax+x^2)\,dx$$

$$= \frac{1}{2}\int_0^a \underset{v}{x}\,\underset{u}{\log(x+a)}\,dx$$

$$= \frac{1}{2}\left[\left\{\log(x+a)\frac{x^2}{2}\right\}_0^a - \int_0^a \frac{x^2}{2}\frac{1}{x+a}\,dx\right]$$

$$= \frac{a^2}{4}\log 2a - \frac{1}{4}\int_0^a \frac{x^2}{x+a}\,dx$$

$$= \frac{a^2}{4}\log 2a - \frac{1}{4}\int_0^a \left[(x-a) + \frac{a^2}{x+a}\right] dx$$

$$= \frac{a^2}{4}\log 2a - \frac{1}{4}\left[\frac{x^2}{2} - ax + a^2\log(x+a)\right]_0^a$$

$$\boxed{I = \frac{a^2}{8} + \frac{a^2}{4}\log a = \frac{a^2}{8}(1 + 2\log a)} \quad \text{.... Ans}$$

Example 14.15 : *Evaluate* $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$ **(May-2002, Dec.-2007)**

Solution : Let $$I = \int_0^\infty \left[\int_{y=x}^\infty \frac{e^{-y}}{y}\,dy\right] dx \quad \text{... (1)}$$

The integral is difficult to integrate w.r.t. y first but easy to integrate w.r.t. x. Therefore, we change the order of integration.

Limits are $\left.\begin{array}{ll} y = x, & y = \infty \\ x = 0, & x = \infty \end{array}\right\}$

for which the region is as shown in Fig. 14.12.

To find new limits, draw a horizontal strip (parallel to x-axis) over which x varies from **x = 0 to x = y** and moving the strip vertically from **y = 0 to y = ∞** we get shaded region.

$\therefore$ from (1)

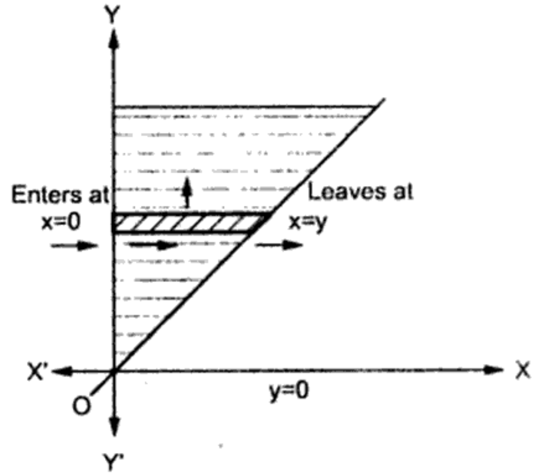


Fig. 14.12

$$I = \int_{y=0}^\infty \int_{x=0}^y \frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_{y=0}^\infty \frac{e^{-y}}{y}\left[\int_{x=0}^y dx\right] dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}(y)\,dy = \left(\frac{e^{-y}}{-1}\right)_0^\infty = 1$$

$$\boxed{I = 1} \quad \text{... Ans}$$

Example 14.16 : *Evaluate* $\int_0^a \int_{y^2/a}^{y} \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}}$ (May - 1994)

Solution : Let

$$I = \int_0^a \left[y \int_{x=y^2/a}^{x=y} \frac{dx}{(a-x)\sqrt{ax-y^2}} \right] dy \quad \text{... (1)}$$

The integrand is difficult to integrate w.r.t. x first but easy to integrate w.r.t. y. Therefore, we change the order of integration by considering reverse strip (parallel to y-axis).

The limits are

$$\left.\begin{array}{ll} x = \frac{y^2}{a}, & x = y \\ y = 0, & y = a \end{array}\right\}$$

for which the region of integration is as shown in the Fig. 14.13.

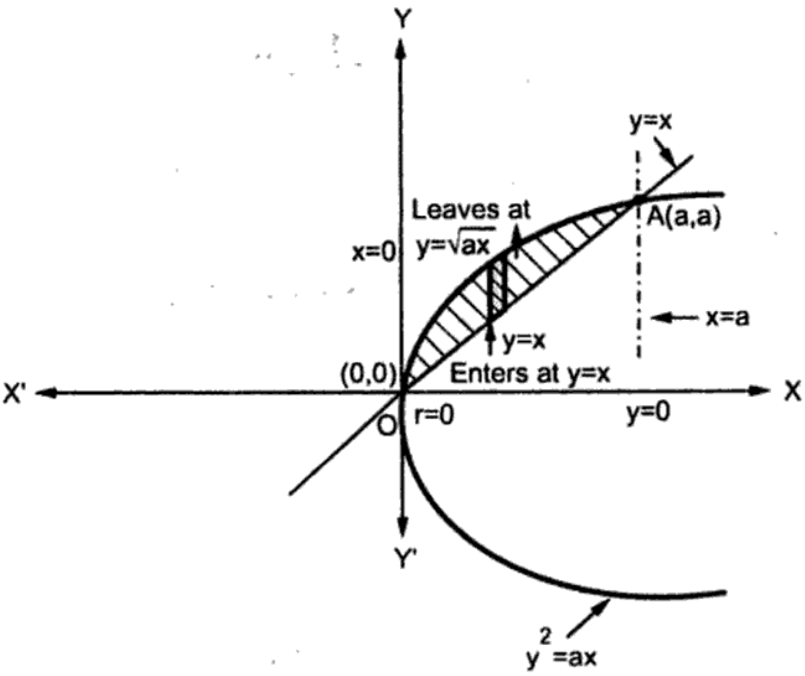


Fig. 14.13

$y^2 = ax$, put $y = x$ to get co-ordinates of point A

$x^2 = ax$

$x(x-a) = 0$

$\Rightarrow x = 0, x = a$

$\downarrow \quad \downarrow$

$y = 0, y = 0$

O (0, 0), A (a, a)

To find new limits, consider vertical strip over which y varies from **y = x, to y = √ax** and moving the strip from **x = 0 to x = a.**

From (1),

$$I = \int_0^a \frac{1}{a-x} \left[\int_x^{\sqrt{ax}} \frac{y\,dy}{\sqrt{ax-y^2}} \right] dx$$

$$= \int_0^a \frac{1}{a-x} [I_1]\, dx \quad \text{... (2)}$$

Example 14.20 : *Change the order of integration* $\int\limits_0^1 \int\limits_x^{a^2/x} f(x, y)\, dx dy$

Solution : Let
$$I = \int\limits_{x=0}^{a} \int\limits_{y=x}^{a^2/x} f(x, y)\, dx dy \qquad \dots (1)$$

The limits are $y = x$,

$y = \dfrac{a^2}{x} \Rightarrow xy = a^2$

(Hyperbola)

and $x = 0$, $x = a$

The region of integration for these limits is as shown in the Fig. 14.17.

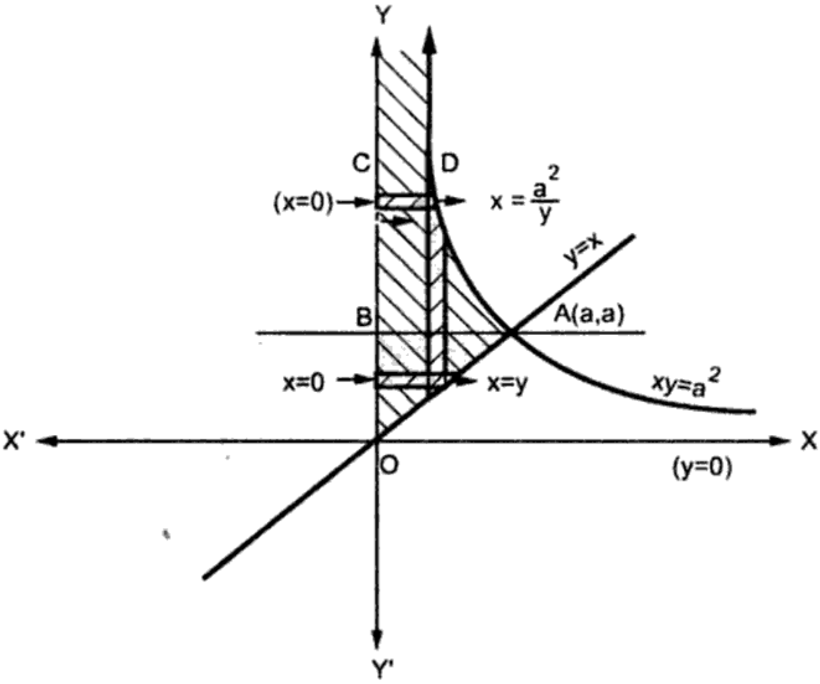


Fig. 14.17

To change the order of integration, the region of integration is to be divided in two parts namely ABOA and ABCDA

i) For the region ABOA, the limits over the vertical strip are **x = 0 to x = y.** Moving the strip vertically from **y = 0 to y = a** we get ABOA region.

ii) For the region ABCDA, the limits over the vertical strip are $\mathbf{x = 0}$ **to** $\mathbf{x = \dfrac{a^2}{y}}$ and moving the strip vertically from **y = a to y = ∞**, we get upper region ABCDA covered.

∴ From (1)

$$\boxed{I = \underbrace{\int\limits_0^a \int\limits_0^y f(x, y)\, dxdy}_{\text{ABOA}} + \underbrace{\int\limits_a^{\infty} \int\limits_0^{a^2/y} f(x, y)\, dxdy}_{\text{ABCDA}}} \qquad \dots \text{Ans}$$

boundaries viz. $x^2 + y^2 = a^2$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(x-a)^2 + y^2 = a^2$ etc. by putting $x = r\cos\theta$, $y = r\sin\theta$ $\therefore x^2 + y^2 = r^2$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ and

$$dxdy = |J|\, d\theta dr = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| d\theta dr = r\, d\theta dr$$

$\therefore$ From (1), we have

$$I = \iint_R f(r\cos\theta, r\sin\theta)\, r\, d\theta dr$$

$$= \iint_R F(r,\theta)\, r\, d\theta dr \qquad \text{... (2)}$$

To find corresponding limits for r, θ. Draw a radial strip OPQ in the region of integration R from the pole (r = 0) over which 'r' varies from $r_1 = f_1(\theta)$ to $r_2 = f_2(\theta)$ (for region R)

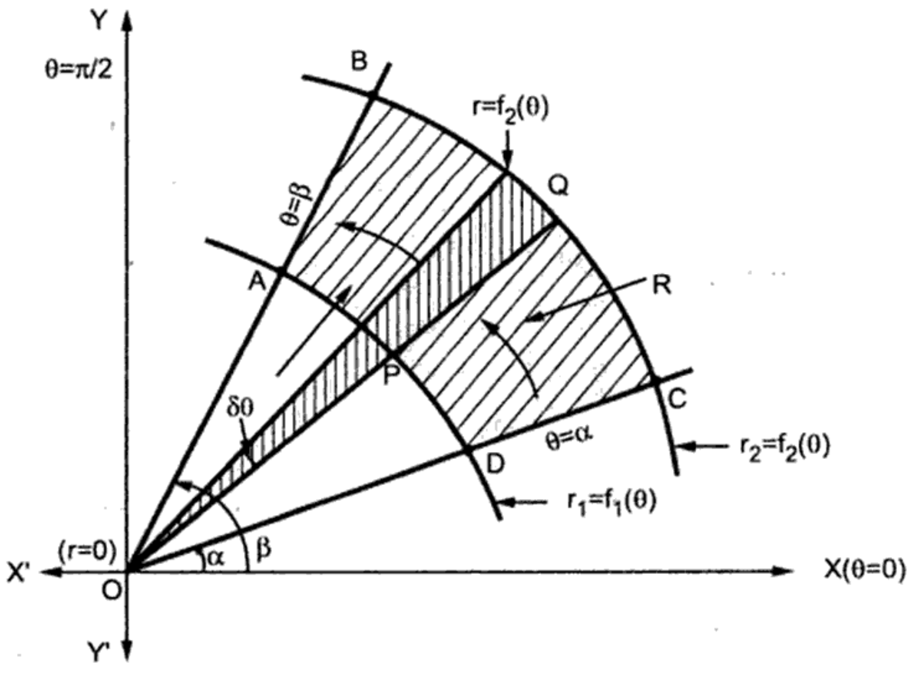


Fig. 14.20

To sweepout the complete region of integration. Rotate the strip OPQ from CD to AB (In anticlockwise direction from θ = α to θ = β)

$\therefore$ From (2)

$$I = \int_{\theta=\alpha}^{\theta=\beta} \int_{r_1 = f_1(\theta)}^{r_2 = f_2(\theta)} F(r,\theta)\, r\, d\theta dr \qquad \text{... (3)}$$

It is clear that the integrand to be integrated w.r.t. r first over the limits $r_1 = f_1(\theta)$ to $r_2 = f_2(\theta)$ (keeping θ constant)

$$\therefore \quad \boxed{I = \int_{\theta=\alpha}^{\theta=\beta} \boxed{\int_{r_1 = f_1(\theta)}^{r_2 = f_2(\theta)} F(r,\theta)\, r\, dr}\, d\theta} \qquad \dots (4)$$

Example 14.22 : *Evaluate* $\iint_R \frac{x^2 y^2}{x^2 + y^2} dxdy$, *where R is annulus between* $x^2 + y^2 = 4$ *and* $x^2 + y^2 = 9$ **(May-2005)**

Solution : Let $$I = \iint_R \frac{x^2 y^2}{x^2 + y^2} dxdy \qquad \dots (1)$$

The integrand is difficult to integrate w.r.t. x or w.r.t. y and it becomes handy if we transform into polar co-ordinates.

∴ Put $x = r\cos\theta$, $y = r\sin\theta$

$dxdy = r\, d\theta\, dr$

and $x^2 + y^2 = r^2$

$x^2 + y^2 = 4 \Rightarrow r = 2$

$x^2 + y^2 = 9 \Rightarrow r = 3$

To find limits for r and θ, draw a strip OPQ in the region of integration over which 'r' varies from **r = 2 to r = 3** (Since at P, r = 2 and at Q, r = 3)

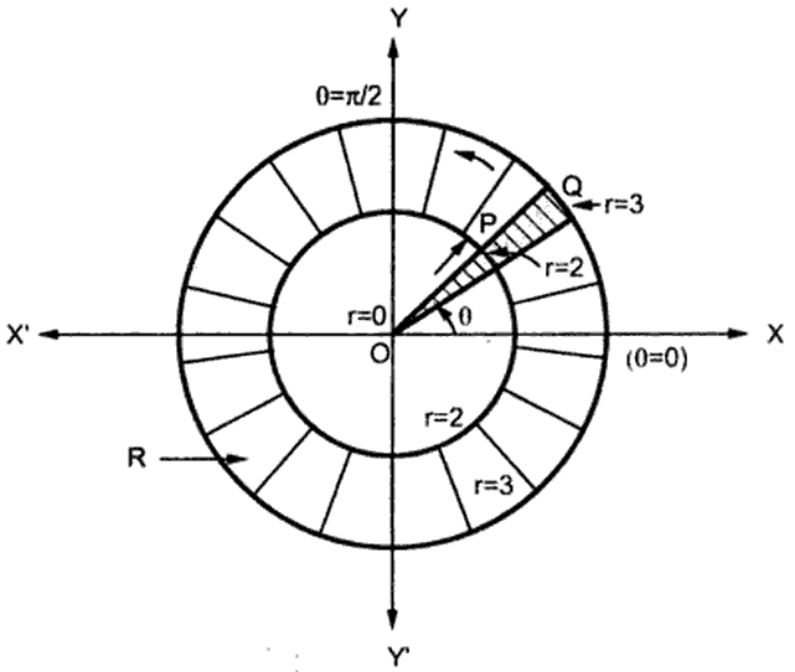


Fig. 14.21

Now rotating the strip from **θ = 0 to θ = 2π,'** we get entire shaded region R which is region of integration.

∴ From (1),

$$I = \int_{\theta=0}^{2\pi} \int_{r=2}^{3} \frac{r^2\cos^2\theta \; r^2\sin^2\theta}{r^2}\, r\, d\theta dr$$

$$= \int_{0}^{2\pi} \int_{r=2}^{3} r^3 \sin^2\theta \cos^2\theta \, d\theta dr$$

$$= \int_{0}^{2\pi} \sin^2\theta \cos^2\theta \left[\int_{r=2}^{3} r^3\, dr \right] d\theta$$

$$= \int_0^{2\pi} \sin^2\theta \cos^2\theta \left(\frac{r^4}{4}\right)_2^3 d\theta$$

$$= 4\int_0^{\pi/2} \sin^2\theta \cos^2\theta \left[\frac{(3)^4}{4} - \frac{(2)^4}{4}\right] d\theta$$

$$= 65\int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta$$

$$\boxed{I = 65\left[\frac{(2-1)(2-1)}{4\cdot 2}\cdot\frac{\pi}{2}\right] = \frac{65\pi}{16}} \quad \text{... Ans}$$

Example 14.23 : *Evaluate* $\iint_R xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}} dxdy$, *where R is positive quadrant of ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.

Solution : Let $$I = \iint_R xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}} dxdy \quad \text{... (1)}$$

Simplicity, we transform the integral (1) into elliptical polar co-ordinates by putting :

$x = a\, r \cos\theta$, $y = b\, r \sin\theta$, $dxdy = ab\, r\, d\theta\, dr$

$\therefore$ Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ gets transformed to $r^2 = 1 \Rightarrow r = 1$ as shown below.

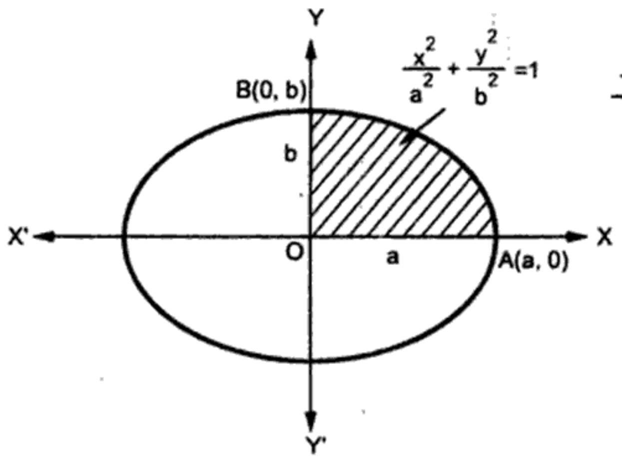


Fig. 14.22

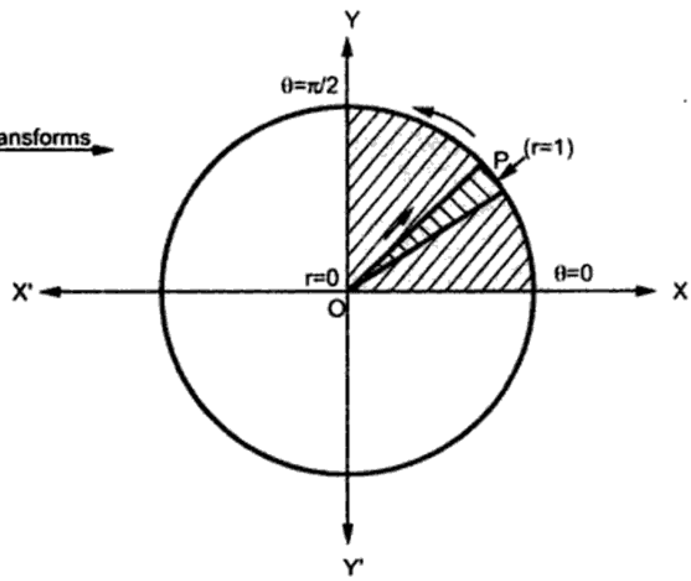


Fig. 14.23

'r' varies from **r = 0 to r = 1** and 'θ' varies from **θ = 0 to $\frac{\pi}{2}$**.

∴ From (1)

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} a\,r\cos\theta\;\; b\,r\sin\theta\,(r^2)^{\frac{n}{2}}\; ab\,r\,d\theta dr$$

$$= a^2b^2\int_{\theta=0}^{\pi/2}\frac{2\sin\theta\cos\theta\,d\theta}{2}\int_0^1 r^{n+3}\,dr$$

$$= \frac{a^2b^2}{2}\int_0^{\pi/2}\sin 2\theta\,d\theta\left(\frac{r^{n+4}}{n+4}\right)_0^1$$

$$= \frac{a^2b^2}{2(n+4)}\left(\frac{-\cos 2\theta}{2}\right)_0^{\pi/2}$$

$$= \frac{-a^2b^2}{4(n+4)}[-1-1]$$

$$\boxed{I = \frac{a^2b^2}{2(n+4)}} \quad \text{... Ans}$$

Example 14.24 : ***Evaluate $\iint y^2\,dxdy$ over the area outside $x^2+y^2-ax=0$ and inside $x^2+y^2-2ax=0$.***

Solution : Let $$I = \iint_R y^2\,dxdy \quad \text{... (1)}$$

where R is area outside

$x^2 + y^2 - ax = 0 \Rightarrow$

$\left(x-\frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2 \Rightarrow$

r = a cos θ circle with centre $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$.

and inside the circle

$x^2 + y^2 - 2ax = 0 \Rightarrow$

$x^2 - 2ax + y^2 + a^2 = a^2 \Rightarrow$

r = 2 a cos θ

$\Rightarrow (x-a)^2 + (y-0)^2 = a^2$

with centre at (a, 0) and radius a.

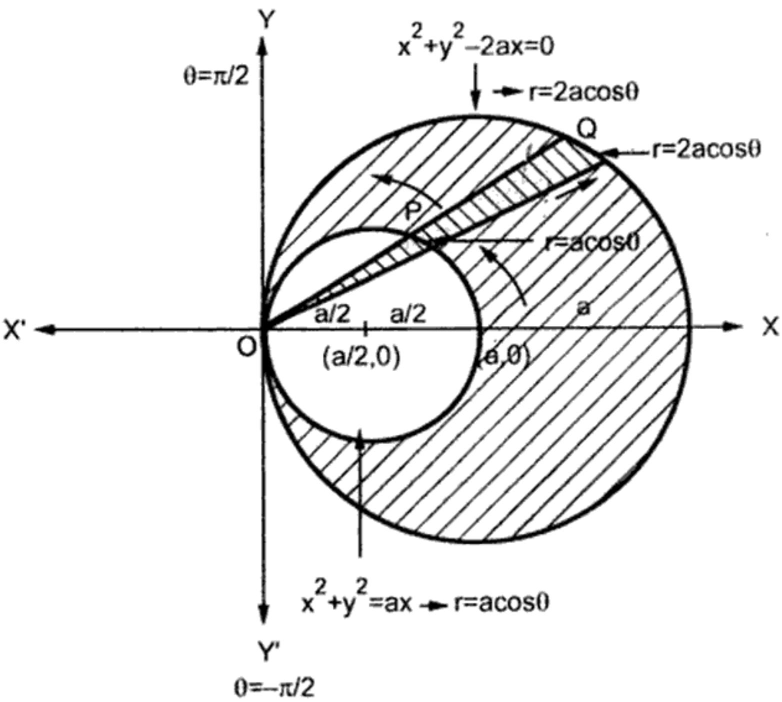


Fig. 14.24

It is convenient to tranform into polar form.

$\therefore$ Put $x = r\cos\theta$, $y = r\sin\theta$

$dxdy = r\,d\theta\,dr$

$x^2 + y^2 = r^2$.

To find limits, consider a radial strip OPQ over which 'r' varies from $r = a\cos\theta$ to $r = 2a\cos\theta$ and 'θ' varies from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.

$\therefore$ From (1)

$$I = \int_{\theta=-\frac{\pi}{2}}^{\pi/2} \int_{r=a\cos\theta}^{2a\cos\theta} r^2\sin^2\theta \; r\,d\theta\,dr$$

$$I = 2\int_{0}^{\pi/2} \int_{a\cos\theta}^{2a\cos\theta} r^3\sin^2\theta \; d\theta$$

$$= 2\int_{0}^{\pi/2} \sin^2\theta \left(\frac{r^4}{4}\right)_{a\cos\theta}^{2a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/2} \sin^2\theta\,[16a^4\cos^4\theta - a^4\cos^4\theta]\,d\theta$$

$$= \frac{15a^4}{2}\int_{0}^{\pi/2} \cos^4\theta\,\sin^2\theta\,d\theta$$

$$= \frac{15a^4}{2}\,\frac{[(4-1)(4-3)]\,[(2-1)]}{6\,(6-2)\,(6-4)}\cdot\frac{\pi}{2}$$

$$\boxed{I = \frac{15\pi a^5}{64}}$$ ... Ans

Example 14.25 : *Evaluate* $\iint\limits_R \frac{r\,dr\,d\theta}{\sqrt{a^2 + r^2}}$ *over one loop of Laminscate* $r^2 = a^2\cos 2\theta$.

(Dec.-2002)

Solution : Let
$$I = \iint\limits_R \frac{r\,d\theta\,dr}{\sqrt{a^2 + r^2}} \quad \text{... (1)}$$

which R is region of the loop of the laminscate as shown in the Fig. 14.25.

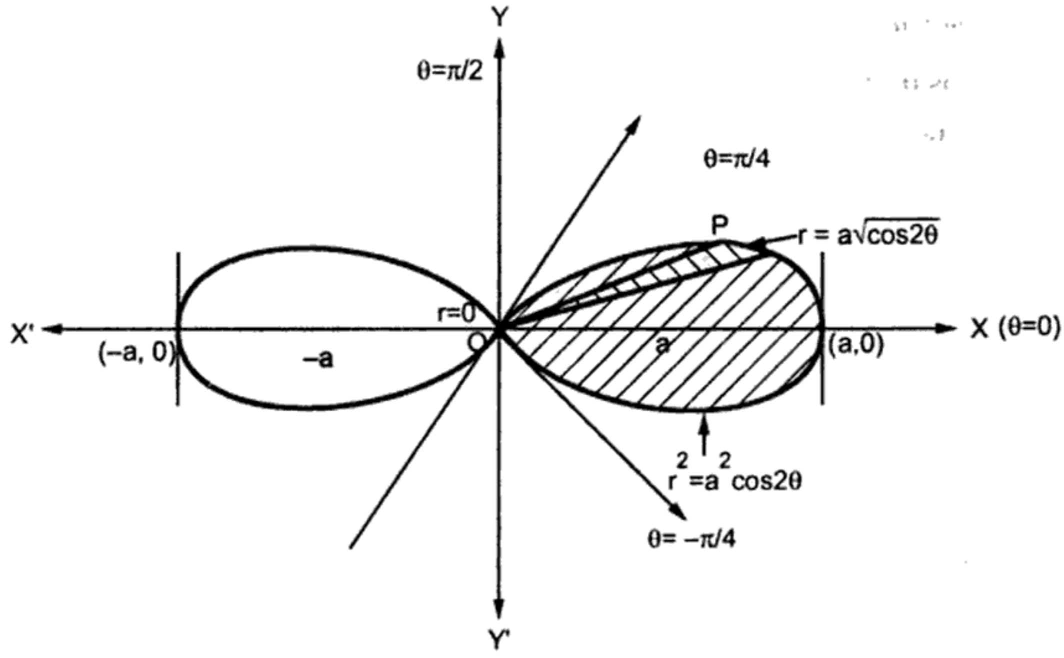


Fig. 14.25

Limits over the strip OP are $r = 0$ to $r = a\sqrt{\cos 2\theta}$

Rotating the strip OP from $\theta = \frac{-\pi}{4}$ to $\theta = \frac{\pi}{4}$

we get entire loop (Region R covered)

$\therefore$ From (1)

$$I = \int_{\theta=-\pi/4}^{\pi/4} \left[\int_{r=0}^{a\sqrt{\cos 2\theta}} \frac{r\,dr}{\sqrt{a^2+r^2}} \right] d\theta$$

(w.r.t. r first keeping θ as a constant)

$$= 2\int_{0}^{\pi/4} \left[\int_{r=0}^{a\sqrt{\cos 2\theta}} \frac{r\,dr}{\sqrt{a^2+r^2}} \right] d\theta$$

$$= 2\int_{0}^{\pi/4} \left[\sqrt{a^2+r^2} \right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2a\int_{0}^{\pi/4} \left[\sqrt{1+\cos 2\theta} - 1 \right] d\theta$$

$$= 2a\int_{0}^{\pi/4} \left[\sqrt{2}\cos\theta - 1 \right] d\theta$$

$$= 2a\left[\sqrt{2}\sin\theta - \theta \right]_0^{\pi/4}$$

$$= 2a\left[\sqrt{2}\cdot\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right]$$

$$\boxed{I = 2a\left(1 - \frac{\pi}{4}\right)} \qquad \text{... Ans}$$

Example 14.26 : *Evaluate* $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dxdy$ **(Dec.-1998)**

Solution : Let

$$I = \int_{x=0}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}} (x^2+y^2)\,dxdy \qquad \text{... (1)}$$

Here the limits are

$y = 0,\ y = \sqrt{2ax - x^2},\ x = 0,\ x = 2a$

$\Rightarrow y^2 = 2ax - x^2$

$\Rightarrow x^2 + y^2 = 2ax$

$\Rightarrow r^2 = 2\,a\,r\cos\theta$

or $\boxed{r = 2a\cos\theta}$

circle with centre at (a, 0) and radius is a. The region of integration is as shown in the Fig. 14.26.

Transforming the integral (1) into polar by putting $x = r\cos\theta,\ y = r\sin\theta$,

$\therefore\ x^2 + y^2 = r^2,\ dx\,dy = r\,d\theta\,dr$

and $x^2 + y^2 = 2ax \Rightarrow r^2 = 2a\,r\cos\theta \Rightarrow r = 2a\cos\theta$

From figure : 'r' varies from **r = 0 to r = 2 a cos θ** and 'θ' varies from $\theta = 0$ to $\theta = \frac{\pi}{2}$

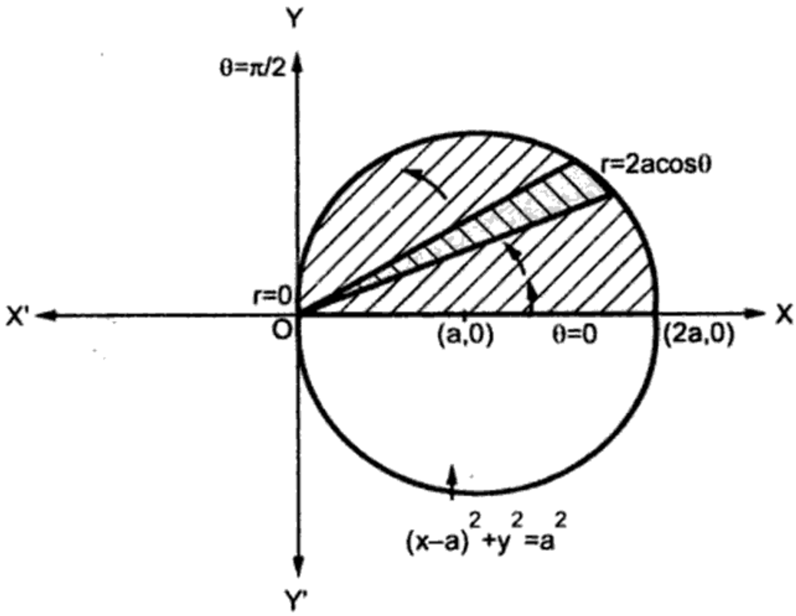


Fig. 14.26

$$\therefore \qquad I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r^2\cdot r\,d\theta\,dr$$

$$I = \int_0^{\pi/2}\left(\frac{r^4}{4}\right)_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} 16\, a^4 \cos^4\theta \; d\theta$$

$$= 4a^4 \int_0^{\pi/2} \cos^4\theta \; d\theta = 4a^4 \cdot \frac{4-1}{4} \cdot \frac{4-3}{4-2} \cdot \frac{\pi}{2}$$

$$\boxed{I = \frac{3\pi a^4}{4}}$$ ... Ans

Example 14.27 : *Evaluate* $\int_0^{a/\sqrt{2}} \int_x^{\sqrt{a^2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$ **(May-2002)**

Solution : Limits are $y = x$, $y = \sqrt{a^2 - x^2}$ or $x^2 + y^2 = a^2$ and $x = 0$, $x = \frac{a}{\sqrt{2}}$

The region of integration is as shown in the Fig. 14.27.

Transforms the given integral into polar form by putting $x = r\cos\theta$, $y = r\sin\theta$,

$\therefore$ $dxdy = r\, d\theta\, dr$ and $x^2 + y^2 = r^2$

'r' varies from $r = 0$ to $r = a$ and 'θ ' from varies from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$

From given integral;

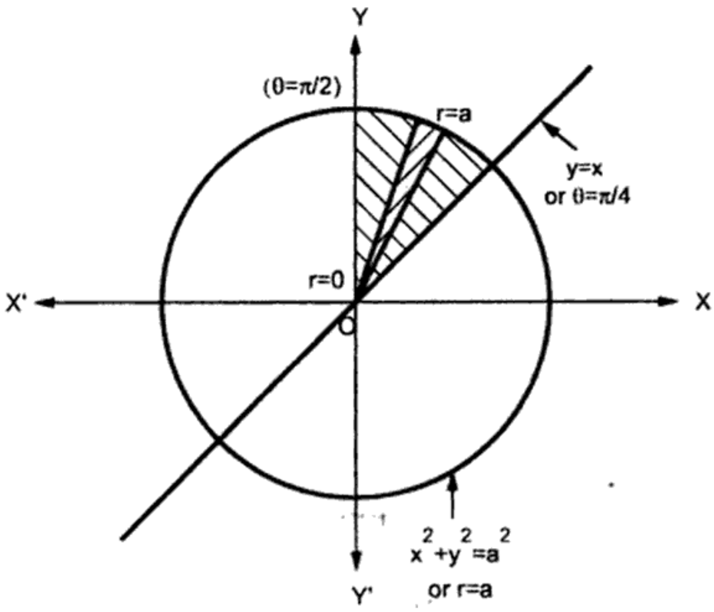


Fig. 14.27

$$I = \int_{\theta=\frac{\pi}{4}}^{\pi/2} \int_{r=0}^{a} \frac{r\cos\theta}{\sqrt{r^2}}\, r\, d\theta\, dr$$

$$I = \int_{\pi/4}^{\pi/2} \cos\theta \left[\int_{r=0}^{a} r\; dr\right] d\theta$$

$$= \int_{\pi/4}^{\pi/2} \cos\theta \left(\frac{r^2}{2}\right)_0^a d\theta = \frac{a^2}{2}\left(\sin\theta\right)_{\pi/4}^{\pi/2}$$

$$\boxed{I = \frac{a^2}{2}\left[1 - \frac{1}{\sqrt{2}}\right]}$$ ... Ans

Example 14.28 : *Evaluate* $\int_0^{4a}\int_{y^2/4a}^{y}\left(\frac{x^2-y^2}{x^2+y^2}\right)dxdy$

Solution : Here the limits are

$x = \frac{y^2}{4a}, \; x = y, \; y = 0, \; y = 4a$

or $y^2 = 4ax, \; x = y, \; y = 0, \; y = 4a$

The region of integration is as shown in the Fig. 14.28.

$y^2 = 4ax, \; y = 4a$

At A, $(4a)^2 = 4ax$

$\Rightarrow 16a^2 = 4ax$

$\Rightarrow 4a\,(4a - x) = 0$

$\Rightarrow x = 4a$

$\downarrow$

$y = 4a$

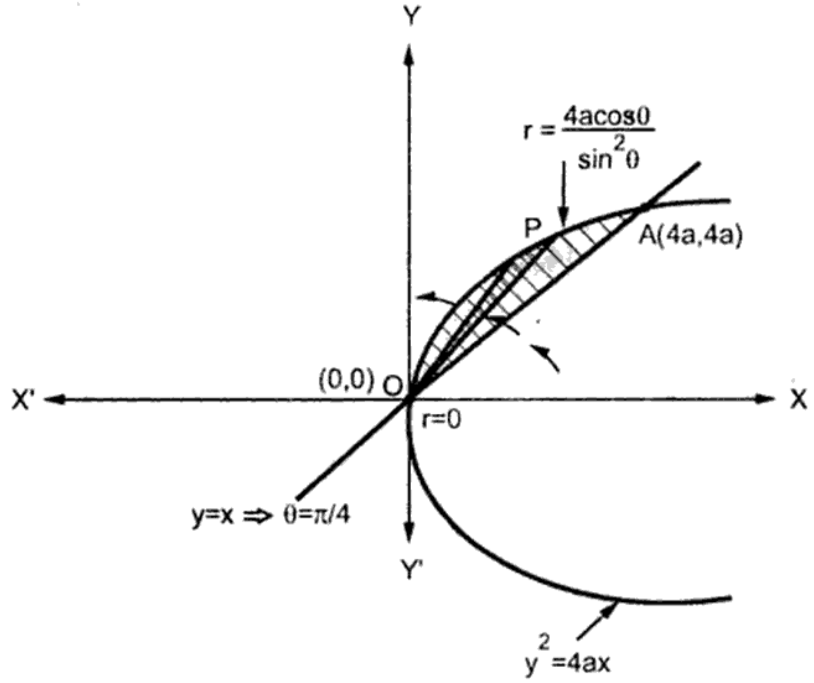


Fig. 14.28

It is convenient to transform the double integral into polar.

$\therefore$ Put $x = r\cos\theta, \; y = r\sin\theta,$

$\therefore dxdy = r\,d\theta\,dr$

and $x^2 + y^2 = r^2$

$x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta)$

'r' varies from $\mathbf{r = 0}$ **to** $\mathbf{r = \frac{4\cos\theta}{\sin^2\theta}}$ and θ varies from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$

Given integral will have the form

$$I = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{4\frac{\cos\theta}{\sin^2\theta}} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2}\, r\, d\theta dr$$

$$= \int_{\pi/4}^{\pi/4}(\cos^2\theta - \sin^2\theta)\left(\frac{r^2}{2}\right)_0^{\frac{4a\cos\theta}{\sin^2\theta}} d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2}(\cos^2\theta - \sin^2\theta)\frac{16a^2\cos^2\theta}{\sin^4\theta}\, d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} \left(\cot^4\theta - \cot^2\theta\right) d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} \left(\cot^2\theta \operatorname{cosec}^2\theta - 2\operatorname{cosec}^2\theta + 2\right) d\theta$$

$$= 8a^2 \left\{\frac{-\cot^3\theta}{3} + 2\cot\theta + 2\theta\right\}_{\pi/4}^{\pi/2}$$

$$= 8a^2 \left[0 + 0 + \pi + \frac{1}{3} - 2 - \frac{\pi}{2}\right]$$

$$\boxed{I = 8a^2\left[\frac{\pi}{2} - \frac{5}{3}\right]}$$ ... Ans

Example 14.29 : *Evaluate* $\displaystyle\int_0^1 \int_{y=\sqrt{x-x^2}}^{\sqrt{1-x^2}} \frac{xy\, e^{-x^2-y^2}}{x^2+y^2}\, dx dy$

Solution : Here the limits are

$y = \sqrt{x - x^2}$, $y = \sqrt{1 - x^2}$, $x = 0$ and $x = 1$

$\Downarrow$ $\Downarrow$

$x^2 + y^2 = x$, $x^2 + y^2 = 1$, $x = 0$ and $x = 1$

$r^2 = r\cos\theta$, $r^2 = 1$

$\mathbf{r = \cos\theta}$, $\mathbf{r = 1}$

The region of integration is as shown in the Fig. 14.29.

Transforming into polar by putting

$x = r\cos\theta$, $y = r\sin\theta$

$dxdy = r\, d\theta$

'r' varies from $\mathbf{r = \cos\theta}$ (At P) to $\mathbf{r = 1}$ (At Q) and 'θ' varies from $\theta = \mathbf{0}$ **to** $\theta = \frac{\pi}{2}$

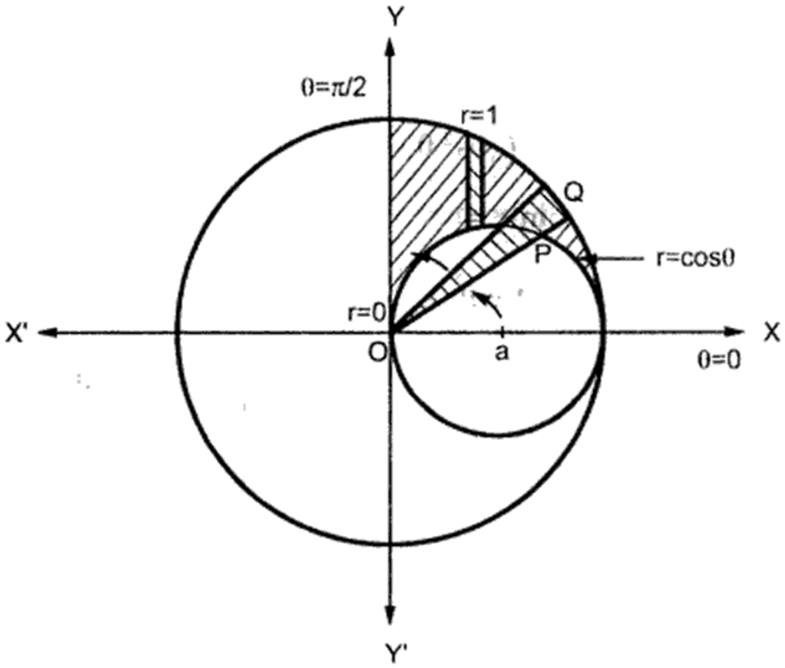


Fig. 14.29

From given integral :

$$I = \int_{\theta=0}^{\pi/2} \int_{r=\cos\theta}^{1} \frac{r\cos\theta \; r\sin\theta \; e^{-r^2}}{r^2} \, r \, d\theta \, dr$$

$$= \int_{0}^{\pi/2} \sin\theta \; \cos\theta \left[\int_{r=\cos\theta}^{1} r \; e^{-r^2} dr \right] d\theta$$

$$= \int_{0}^{\pi/2} \sin\theta \cos\theta \left(-\frac{e^{-r^2}}{2} \right)_{\cos\theta}^{1} d\theta$$

$$= -\frac{1}{2} \int_{0}^{\pi/2} \sin\theta \cos\theta \left[e^{-1} - e^{-\cos^2\theta} \right] d\theta$$

$$= -\frac{1}{2e} \int_{0}^{\pi/2} \sin\theta \cos\theta \, d\theta + \frac{1}{2} \int_{0}^{\pi/2} \sin\theta \cos\theta \; e^{-\cos^2\theta} d\theta$$

$$= -\frac{1}{2e} \left(\frac{\sin 2\theta}{2} \right)_{0}^{\pi/2} + \frac{1}{2} \left[\frac{e^{-\cos^2\theta}}{2} \right]_{0}^{\pi/2}$$

$$= -\frac{1}{2e} \left[\frac{1}{2} - 0 \right] + \frac{1}{4} \left[e^{0} - e^{-1} \right]$$

$$\boxed{I = \frac{1}{4}\left(1 - \frac{2}{e}\right)} \quad \text{... Ans}$$

Example 14.30 : *Evaluate* $\displaystyle\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}} \cdot dxdy$

Solution : Here the limits are

$y = 0$, $y = \sqrt{1-x^2}$, $x = 0$ to $x = 1$, $x^2 + y^2 = 1$

Region of integration is positive quadrant of the circle $x^2 + y^2 = 1$ and it is convenient to transform the integral into polar co-ordinates by putting $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = r\,d\theta\,dr$

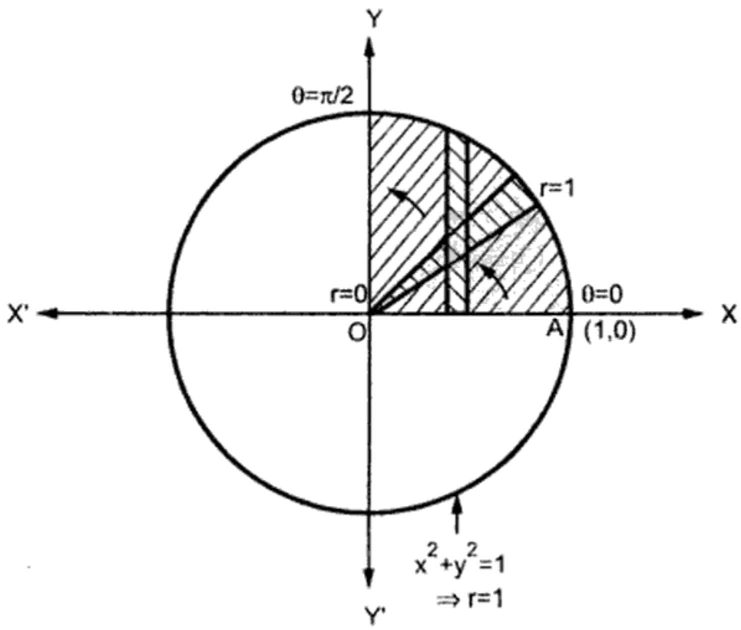


Fig. 14.30

$x^2 + y^2 = r^2$, **'r' varies from 0 to 1 and 'θ' varies from 0 to $\frac{\pi}{2}$.**

From given integral

$$I = \int_0^{\pi/2} \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\ r\, d\theta dr$$

$$= \int_0^{\pi/2} \left[\int_{r=0}^{1} \sqrt{\frac{1-r^2}{1+r^2}}\ r dr \right] d\theta$$

Put $r^2 = t \quad \therefore 2rdr = dt$,

Limits :

r	0	1
t	0	1

$\therefore$
$$I = \frac{1}{2} \int_0^{\pi/2} \left[\int_{t=0}^{1} \sqrt{\frac{1-t}{1+t}} \cdot dt \right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[\int_0^1 \frac{1-t}{\sqrt{1-t^2}} dt \right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[\int_0^1 \frac{dt}{\sqrt{1-t^2}} - \int_0^1 \frac{t\, dt}{\sqrt{1-t^2}} \right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[\left(\sin^{-1} t\right)_0^1 + \left(\sqrt{1-t^2}\right)_0^1 \right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left(\frac{\pi}{2} - 1\right) d\theta$$

$$= \frac{\pi}{4}\left(\frac{\pi}{2} - 1\right)$$

$$\boxed{I = \frac{\pi^2}{8} - \frac{\pi}{4}} \qquad \text{... Ans}$$

Example 14.31 : *Evaluate $\iint_R r\, e^{-r^2/a^2} \cos\theta \sin\theta\, dr d\theta$, where R is the area of the upper half of the circle $r = 2a \cos\theta$.*

Solution : Let
$$I = \iint_R r\, e^{-r^2/a^2} \cos\theta\ \sin\theta\ dr\, d\theta \qquad \text{... (1)}$$

Where R is upper half of the circle

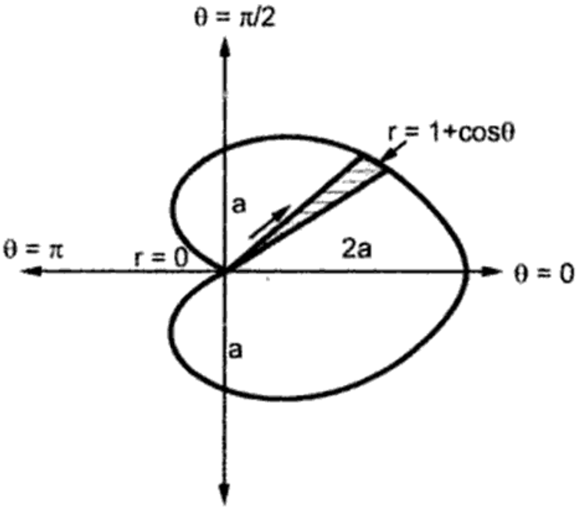


Fig. 14.35

Hint : $I = \iint\limits_R dxdy$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1+\cos\theta} r\, d\theta dr = \frac{3\pi}{2}$$

4) *Evaluate* $\iint\limits_R \frac{dxdy}{\sqrt{xy}}$, *where R is region bounded by* $x^2 + y^2 - x = 0$, $y = 0$, $y > 0$

Hint : *Transform to polar :*

$\therefore x^2 + y^2 = x \Rightarrow r^2 = r\cos\theta \Rightarrow r = \cos\theta$

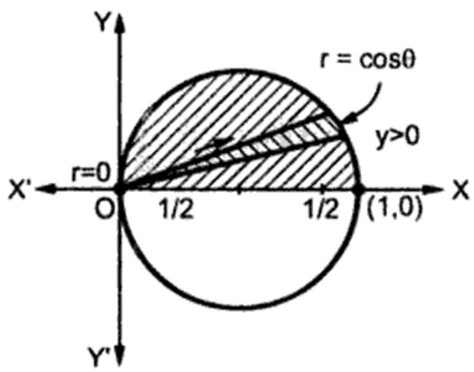


Fig. 14.36

$r = 0$, $r = \cos\theta$, $\theta = 0$, $\theta = \frac{\pi}{2}$

$$\therefore I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\cos\theta} \frac{r\, d\theta dr}{\sqrt{r^2 \sin\theta \cos\theta}}$$

$$\boxed{I = \frac{\pi}{\sqrt{2}}}$$

... *Ans*

5) *Evaluate* $\iint_R \frac{(x^2+y^2)^2}{x^2y^2}dxdy$, *where R is common to* $x^2+y^2=ax$, $x^2+y^2=by$, $(a>b>0)$

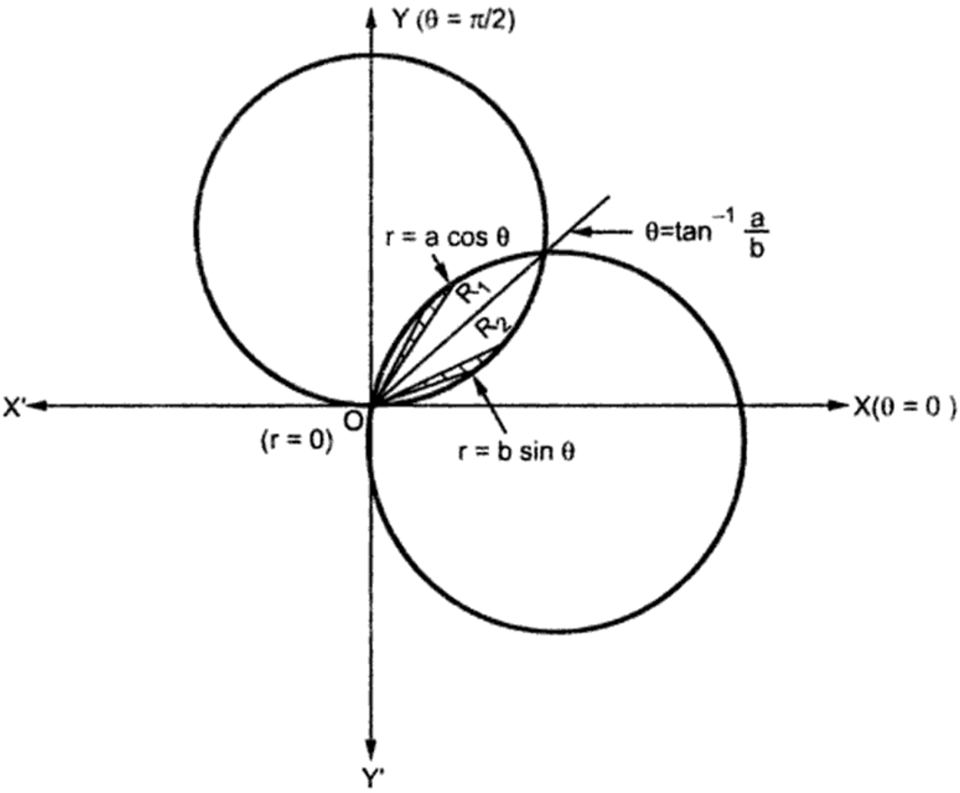


Fig. 14.37

Hint :

The equation of OA is

$ax = by \Rightarrow \frac{y}{x} = \frac{a}{b} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{a}{b}\right)$

The equation of circles $\left.\begin{matrix} r = a\cos\theta \\ r = b\sin\theta \end{matrix}\right\}$

Limits for R_1 : $r = 0$ *to* $r = b\sin\theta$, $\theta = 0$ *to* $\theta = \tan^{-1}\left(\frac{a}{b}\right)$

Limits for R_2 : $r = 0$ *to* $r = a\cos\theta$, $\theta = \tan^{-1}\left(\frac{a}{b}\right)$ *to* $\theta = \frac{\pi}{2}$

$$\therefore I = \iint_{R_1} + \iint_{R_2}$$

$$= \int_{\theta=0}^{\tan^{-1}\left(\frac{a}{b}\right)} \int_{r=0}^{b\sin\theta} \frac{(r^2)^2 r\,d\theta dr}{r^4\cos^2\theta\sin^2\theta} + \int_{\theta=\tan^{-1}\left(\frac{a}{b}\right)}^{\theta=\frac{\pi}{2}} \int_{r=0}^{a\cos\theta} \frac{(r^2)^2 r\,d\theta dr}{r^4\cos^2\theta\sin^2\theta}$$

$$\boxed{I = \frac{ab}{2} + \frac{ab}{2} = ab}$$ *... Ans*

$$\therefore I = \iint_{R_1} + \iint_{R_2}$$

$$= \int_{\theta=0}^{\pi/4} \int_{r=0}^{\sec\theta} \frac{r\,d\theta dr}{\left(1+r^2\right)^{3/2}} + \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{\operatorname{cosec}\theta} \frac{r\,d\theta dr}{\left(1+r^2\right)^{3/2}}$$

$$\boxed{I = \frac{\pi}{6}}$$ ... *Ans*

II) LIMITS ARE PROVIDED

Evaluate the following integral by transforming into polar co-ordinates :

1) $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-\left(x^2+y^2\right)}\,dxdy$

***Hint :** Positive quadrant of the circle $x^2 + y^2 = a^2$*

$$\therefore I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} e^{-r^2} r\,d\theta dr$$

$$\boxed{I = \frac{\pi}{4}\left(1 - e^{-a^2}\right)}$$ ... *Ans*

2) $\displaystyle\int_0^{a/\sqrt{2}} \int_0^{\sqrt{a^2-y^2}} \log_e\left(x^2+y^2\right) dxdy$

***Hint :** Change to polar :*

$$I = \int_{\theta=0}^{\pi/4} \int_{r=0}^{a} \log\left(r^2\right) r\,d\theta dr$$

$$\boxed{I = \frac{\pi}{2}\left[\frac{a^2}{2}\log a - \frac{a^2}{4}\right]}$$ ... *Ans*

3) $\displaystyle\int_0^2 \int_{1-\sqrt{2x-x^2}}^{1+\sqrt{2x-x^2}} \frac{dxdy}{\left(x^2+y^2\right)^2}$ **(May-97)**

***Hint :** Transform to polar*

$$I = \int_{\theta=0}^{\pi/2} \int_{(\sin\theta+\cos\theta)-\sqrt{\sin 2\theta}}^{(\sin\theta+\cos\theta)+\sqrt{\sin 2\theta}} \frac{r\,d\theta dr}{r^4}$$

$$\boxed{I = \pi}$$ ... *Ans*

$$= \frac{1}{a}\int_0^{\pi/2}\left[\int_{r=0}^{a\sin\theta}\left(a^2r - r^3\right)dr\right]d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2}\left(a^2\frac{r^2}{2} - \frac{r^4}{4}\right)_0^{a\sin\theta} d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2}\left[a^2\frac{a^2\sin^2\theta}{2} - \frac{a^4\sin^4\theta}{4}\right]d\theta$$

$$= \frac{a^3}{2}\int_0^{\pi/2}\sin^2\theta \; d\theta - \frac{a^3}{4}\int_0^{\pi/2}\sin^4\theta \; d\theta$$

$$= \frac{a^3}{2}\,\frac{2-1}{2}\cdot\frac{\pi}{2} - \frac{a^3}{4}\,\frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi a^3}{8} - \frac{3a^3\pi}{64}$$

$$\boxed{I = \frac{5\pi a^3}{64}}$$ .... Ans

Example 14.37 : *Evaluate* $\int_0^1\int_{y^2}^1\int_0^{1-x} x\, dz\, dx\, dy$

Solution : Let $I = \int_0^1\int_{y^2}^1 x\left[\int_{z=0}^{1-x} dz\right]dx\;dy$ (w.r.t. 'z' first keeping x, y constant)

$$= \int_0^1\int_{y^2}^1 x\left[(z)_0^{1-x}\right]dx\;dy$$

$$= \int_0^1\int_{x=y^2}^1 x\,(1-x)\;dx\,dy$$

$$= \int_0^1\left[\int_{x=y^2}^1 \left(x - x^2\right)dx\right]dy$$

$$= \int_0^1\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_{y^2}^1 dy$$

$$= \int_0^1\left[\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{y^4}{2} - \frac{y^6}{3}\right)\right]dy$$

iii) For ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Put $x = a\, r \sin\theta \cos\phi,$

$y = b\, r \sin\theta \sin\phi,$

$z = c\, r \cos\theta$

$dx\, dy\, dz = abc\, r^2 \sin\theta\, dr\, d\theta\, d\phi$

r varies from **r = 0 to r = 1**

θ varies from **θ = 0 to θ = π**

φ varies from **φ = 0 to φ = 2π**

b) Cylindrical polar co-ordinates :

Put $x = \rho \cos\phi$

$y = \rho \sin\phi$

$z = z$

$\therefore \quad x^2 + y^2 = \rho^2$

and $\quad dx\, dy\, dz = |J|\, d\rho\, d\phi\, dz$

$$= \left|\frac{\partial(x\, y\, z)}{\partial(\rho, \phi, z)}\right| d\rho\, d\phi\, dz$$

$$dx\, dy\, dz = \rho\, d\rho\, d\phi\, dz$$

c) Dirichlet's Theorem :

i) For two variables x and y,

$$\iint x^{a-1} y^{b-1}\, dx\, dy = \frac{\Gamma a \cdot \Gamma b}{\Gamma a + b + 1} \qquad \text{where } x + y \le 1$$

ii) For three variables x, y, z

$$\iiint x^{a-1}, y^{b-1}, z^{c-1}\, dx\, dy\, dz = \frac{\Gamma a \cdot \Gamma b \cdot \Gamma c}{\Gamma 1 + a + b + c} \qquad \text{where } x + y + z \le 1$$

iii) For n variables $x_1, x_2, \ldots, x_n$

$$\iiint \ldots \int x_1^{a_1 - 1} \cdot x_2^{a_2 - 1} \cdot x_3^{a_3 - 1} \ldots x_n^{a_n - 1}\, dx_1\, dx_2 \ldots dx_n$$

$$= \frac{\Gamma a_1\, \Gamma a_2\, \Gamma a_3 \ldots \Gamma a_n}{\Gamma 1 + a_1 + a_2 + \ldots + a_n}$$

(where $x_1 + x_2 + x_3 + \ldots + x_n \le 1$)

14.7 Illustrated Examples

Example 14.38 : *Evaluate* $\iiint\limits_V \dfrac{dx\,dy\,dz}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}$; *where V is annulus between the spheres* $x^2+y^2+z^2=a^2$ *and* $x^2+y^2+z^2=b^2$ $(a > b > 0)$ **(Dec.-1999)**

Solution : It is convenient to transform the triple integral into spherical polar co-ordinate by putting

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\cos\phi$

$z = r\cos\theta$

$\therefore\ x^2+y^2+z^2 = r^2,$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$x^2+y^2+z^2 = a^2$$

$\Rightarrow$ **r = a**

and $x^2+y^2+z^2 = b^2$

$\Rightarrow$ **r = b** $(b < r < a)$

For the positive octant,

r varies from **r = b to r = a**

θ varies from θ **= 0 to** $\theta = \dfrac{\pi}{2}$

ϕ varies from ϕ **= 0 to** $\dfrac{\pi}{2}$

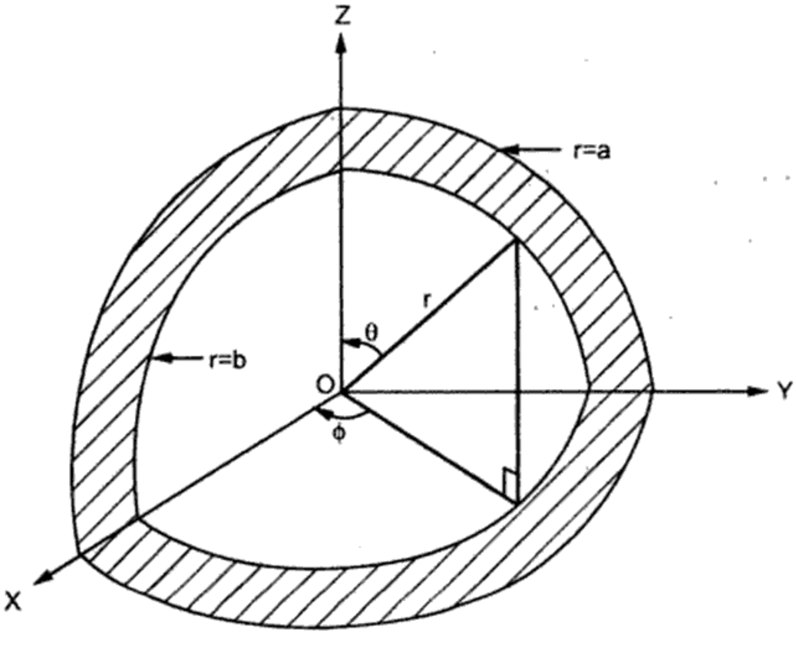


Fig. 14.40

$$\therefore I = \iiint\limits_V \frac{dx\,dy\,dz}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$= 8\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=b}^{a}\frac{r^2\sin\theta\,d\theta\,d\phi\,dr}{\left(r^2\right)^{3/2}}$$

$$= 8\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\left[\int_{r=b}^{a}\frac{1}{r}\,dr\right]\sin\theta\,d\theta\,d\phi$$

$$= 8\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\left[\log r\right]_b^a\sin\theta\,d\theta\,d\phi$$

$$= 8\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}(\log a - \log b)\sin\theta\,d\theta\,d\phi$$

$$= 8\log\frac{a}{b}\int_{\phi=0}^{\pi/2}\left[\int_{\theta=0}^{\pi/2}\sin\theta\,d\theta\right]d\phi$$

$$= 8 \log \frac{a}{b} \int_{\phi=0}^{\pi/2} (-\cos\theta)_0^{\pi/2} \, d\phi$$

$$\boxed{I = 8 \log \frac{a}{b}\left(\frac{\pi}{2}\right) = 4 \log\left(\frac{a}{b}\right)} \quad \text{... Ans}$$

Example 14.39 : *Evaluate* $\iiint\limits_V \dfrac{dx\,dy\,dz}{\left(1 + x^2 + y^2 + z^2\right)^2}$ *over the entire positive octant of the space.* **(May-2000, Dec.-2001)**

Solution : Transforming the integral into spherical polar form using

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \cos\phi$

$z = r \cos\theta$;

$x^2 + y^2 + z^2 = r^2$

$\therefore dx\,dy\,dz =$

$r^2 \sin\theta\, dr\, d\theta\, d\phi,$

Over the positive octant :

r varies from **r to 0 to r = ∞**

θ varies from **θ = 0 to θ = $\frac{\pi}{2}$**

ϕ varies from **ϕ = 0 to ϕ = $\frac{\pi}{2}$**

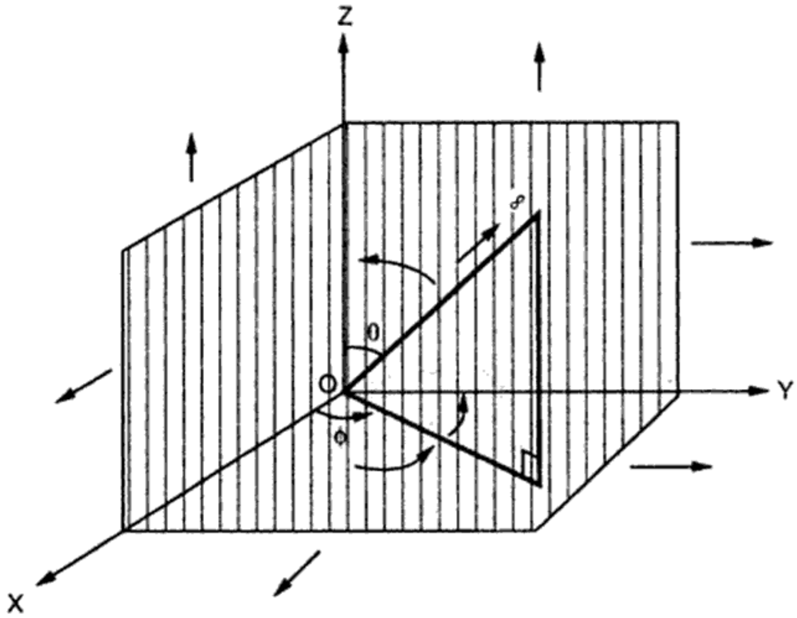


Fig. 14.41

From the given integral

$$I = \iiint\limits_V \frac{dx\,dy\,dz}{\left(1 + x^2 + y^2 + z^2\right)^2}$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} \frac{r^2 \sin\theta\, d\theta\, d\phi\, dr}{\left(1 + r^2\right)^2}$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \sin\theta \left[\int_{r=0}^{\infty} \frac{r^2 dr}{\left(1 + r^2\right)^2}\right] d\theta\, d\phi$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \sin\theta\, [I_1]\, d\theta\, d\phi \quad \text{... (1)}$$

where $$I_1 = \int_0^{\infty} \frac{r^2\, dr}{\left(1 + r^2\right)^2}$$

Put $r^2 = u$

$\therefore 2rdr = du$

$rdr = \frac{du}{2}$

$r^2dr = \frac{\sqrt{u}\,du}{2}$

Limits :

r	0	∞
u	0	∞

$$\therefore \quad I_1 = \int_0^\infty \frac{1}{(1+u)^2} \frac{u^{1/2}du}{2}$$

$$= \int_0^\infty \frac{u^{3/2}}{(1+u)^{3/2+1/2}} du$$

$$= \frac{1}{2}\beta\left(\frac{3}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma 3/2 \, \Gamma 1/2}{\Gamma 2}$$

$$= \frac{1}{2} \frac{\frac{1}{2}\Gamma 1/2 \, \Gamma 1/2}{1} = \frac{\pi}{4} \qquad \text{... (2)}$$

From (1)

$$I = \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta \left(\frac{\pi}{4}\right) d\theta \, d\phi$$

$$= \frac{\pi}{4} \int_0^{\pi/2} \left[\int_0^{\pi/2} \sin\theta \, d\theta\right] d\phi$$

$$= \frac{\pi}{4} \int_0^{\pi/2} (-\cos\theta)_0^{\pi/2} \, d\phi$$

$$\boxed{I = \frac{\pi^2}{8}} \qquad \text{... Ans}$$

Example 14.40 : *Evaluate* $\iiint (x^2y^2 + y^2z^2 + z^2x^2)\,dx\,dy\,dz$ *throughout the volume of sphere* $x^2 + y^2 + z^2 = a^2$ **(May-2005, May-1998)**

Solution : Let $I = \iiint (x^2y^2 + y^2z^2 + z^2x^2)\,dx\,dy\,dz$ ... (1)

$$= \frac{1}{c} \frac{\Gamma b \, \Gamma c+1}{\Gamma b+c+1} \times \frac{\Gamma b \cdot \Gamma b+c+1}{\Gamma a+b+c+1}$$

$$\boxed{I = \frac{\Gamma a \cdot \Gamma b \cdot \Gamma c}{(a+b+c)\Gamma a+b+c}} \quad \text{... Ans}$$

Example 14.42 : *Evaluate* $\iiint_V \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\, dxdydz$ *throughout the volume of ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Solution : Using $x = a\, r \sin\theta \cos\phi$, $y = b\, r \sin\theta \sin\phi$, $z = c\, r \cos\theta$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = r^2$, $dx\, dy\, dz = abc\, r^2 \sin\theta\, dr\, d\theta\, d\phi$ ellipsoid gets transformed to a unit sphere $r = 1$.

r varies from $r = 0$ to $r = 1$

θ varies from $\theta = 0$ to $\theta = \pi$

φ varies from $\theta = 0$ to $\phi = 2\pi$

From the given integral

$$I = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{1} \sqrt{1-r^2}\, a\, b\, c\, r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$= abc \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \left[\int_{r=0}^{1} \sqrt{1-r^2}\, r^2 dr \right] d\theta\, d\phi$$

Put $r = \sin t$, $dr = \cos t\, dt$

$$= abc \int_{\phi=0}^{\pi} \sin\theta\, d\theta \int_{\phi=0}^{2\pi} d\phi \times \int_{t=0}^{\pi/2} \cos t \sin^2 t \cos t\, dt$$

$$= abc(2)\,(2\pi) \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\boxed{I = \frac{\pi^2\, abc}{4}} \quad \text{... Ans}$$

Example 14.43 : *Evaluate* $\iiint_V \sqrt{x^2 + y^2}\, dxdydz$ *over the volume bounded by the right circular cone* $x^2 + y^2 = z$, $(z > 0)$ *and the planes* $z = 0$ *and* $z = 1$.

Solution : Transforming the given integral to cylindrical polar co-ordinates by putting

$x = \rho \cos\phi$, $y = \rho \sin\phi$, $z = z$ and $dx\, dy\, dz = \rho\, d\rho\, d\phi\, dz$

ρ varies from $\rho = 0$ to $\rho = 1$

ϕ varies from $\phi = 0$ to $\phi = 2\pi$

z varies from $z = \rho$ to $z = 1$

$$\therefore \quad I = \iiint \sqrt{\rho^2}\,\rho\ d\rho\ d\phi\ dz$$

$$= \int_0^1 \rho^2 d\rho \cdot \int_0^{2\pi} d\phi \cdot \int_\rho^1 dz$$

$$= \int_0^1 \rho^2 d\phi\,(2\pi)(1-\rho)$$

$$= 2\pi\left[\frac{\rho^3}{3} - \frac{\rho^4}{4}\right]_0^1$$

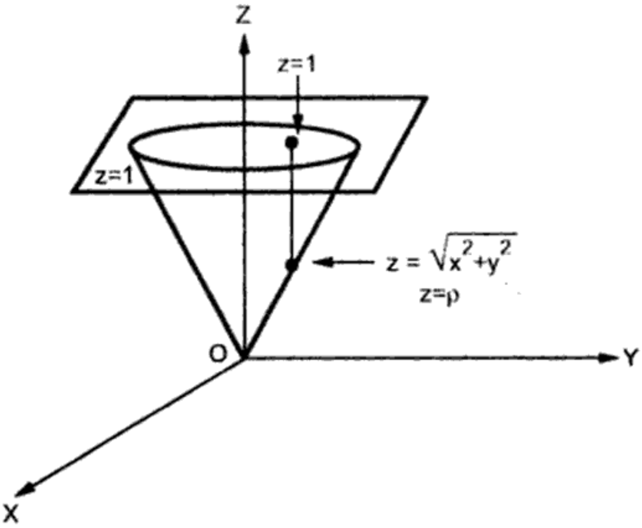


Fig. 14.43

$$\boxed{I = 2\pi\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{\pi}{6}}$$ ... Ans

Example 14.44 : *Evaluate $\iiint z^2\,dx\,dy\,dz$ over the volume bounded by the cylinder $x^2 + y^2 = a^2$ and the elliptical paraboloid $x^2 + y^2 = z$, and the plane $z = 0$*

Solution : Let $I = \iiint z^2\ dx\ dy\ dz$

$$= \iiint dx\ dy \int_{z=0}^{x^2+y^2} z^2\ dz$$

$$= \iint dx\ dy\left(\frac{z^3}{3}\right)_0^{x^2+y^2}$$

$$= \frac{1}{3}\iint (x^2 + y^2)^3\ dx\ dy$$

Transforming into Polar, by putting

$x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$ and $dx\ dy = r\,d\theta\ dr$

$$\therefore I = \frac{1}{3}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} (r^2)^3\ r\,d\theta\ dr$$

$$= \frac{1}{3}\int_\theta^{2\pi}\left(\frac{r^8}{8}\right)_0^a d\theta$$

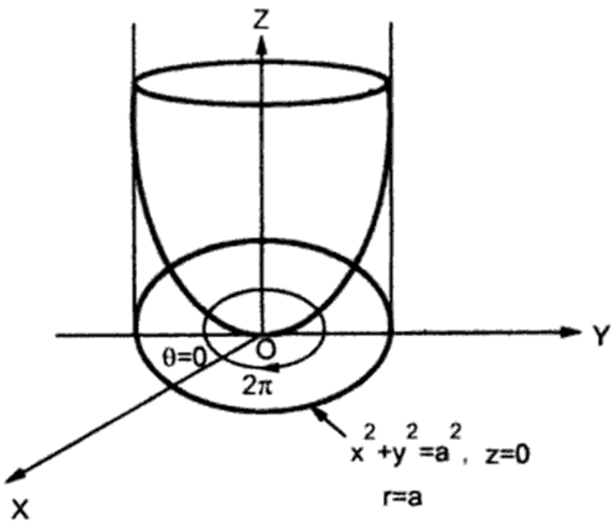


Fig. 14.44

$$= \frac{a^8}{24} \int_{\theta=0}^{2\pi} d\theta$$

$$= \frac{a^8}{24}(2\pi - 0)$$

$$\boxed{I = \frac{\pi a^8}{12}}$$ ... Ans

Exercise 14.5

I) Evaluate the following triple integrals.

1) $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z \, dz \, dy \, dz$ [Ans. : $\frac{1}{2}$]

2) $\int_1^e dy \int_1^{\log y} dx \int_1^{e^x} \log z \, dz$ [Ans. $\frac{e^2 - 8e + 13}{4}$]

3) $\int_0^1 dx \int_0^1 dy \int_{\sqrt{x^2+y^2}}^2 xyz \, dz$ [Ans. : $\frac{3}{8}$]

4) $\int_0^{\pi/2} \int_0^{a\cos\theta} \int_0^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta$ (Dec. - 2002) [Ans. : $\frac{a^3}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right)$]

5) $\int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2 \, dx \, dy \, dz$ [Ans. : $\frac{a^5}{60}$]

6) $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dx \, dy \, dz}{(1+x+y+z)^3}$ [Ans. : $\frac{1}{2}\left(\log 2 - \frac{5}{8}\right)$]

7) $\int_0^1 dx \int_0^2 dy \int_1^2 x^2yz \, dy \, dz$ [Ans. : 1]

8) $\int_0^{\log 2} \int_0^x \int_0^{x+y} e^{x+y+z} dx \, dy \, dz$ (May-2003) [Ans. : $\frac{5}{8}$]

9) $\int_2^2 \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dx \, dy \, dz$ [Ans. : $8\sqrt{2}\pi$]

5) *Evaluate* $\iiint \sqrt{x^2 + y^2}\, dx\, dy\, dz$ *over the volume of solid bounded by surfaces* $x^2 + y^2 = z^2$, $z = 0$, $z = 1$. **[Ans. :** $\frac{\pi}{6}$**]**

6) *Evaluate* $\iiint_V (x^2 + y^2)\, dx\, dy\, dz$, *where V is volume bounded by* $x^2 + y^2 = 2z$ *and the plane* $z = 2$. **[Ans. :** $\frac{16\pi}{3}$**]**

7) *Evaluate* $\iiint_V \frac{z^2\, dx\, dy\, dz}{x^2 + y^2 + z^2}$ *over the volume of the sphere* $x^2 + y^2 + z^2 = 2$

(May - 2003) [Ans. : $\frac{8\sqrt{2}\pi}{9}$**]**

8) *Evaluate* $\iiint z^2\, dx\, dy\, dz$, *over the volume common to the sphere* $x^2 + y^2 + z^2 = a^2$ *and the cylinder* $x^2 + y^2 = ax$. **[Ans. :** $\frac{4}{15} a^5 \left[\frac{\pi}{2} - \frac{8}{15}\right]$**]**

9) *Evaluate* $\iiint \frac{dx\, dy\, dz}{\sqrt{a^2 - x^2 - y^2 - z^2}}$ *taken throughout the volume of the sphere* $x^2 + y^2 + z^2 = a^2$. **(May - 2001)**

University Questions

May - 2003

1) *Evaluate* $\iint \sqrt{4x^2 - y^2}\, dx\, dy$ *over the region bounded by* $y = 0$, $y = x$ *and* $x = 1$

2) *Evaluate* $\iiint \frac{z^2\, dx\, dy\, dz}{x^2 + y^2 + z^2}$ *over the volume of the sphere* $x^2 + y^2 + z^2 = 2$ **[6 Marks]**

3) *Express as single integral and hence evaluate* $\int_0^{a/\sqrt{2}} \int_0^{x} x\, dx\, dy + \int_{a/\sqrt{2}}^{a} \int_0^{\sqrt{a^2 - x^2}} x\, dx\, dy$ **[6 Marks]**

4) *Evaluate* $\int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz$ **[5 Marks]**

May - 2004

1) *Evaluate* $\int_0^{a} \int_0^{a - \sqrt{a^2 - y^2}} \frac{xy \log(x + a)}{(x - a)^2}\, dx dy.$ **[5 Marks]**

2) *Evaluate* $\iiint \frac{z^2}{x^2 + y^2 + z^2}\, dx\, dy\, dz$ *where V is the volume bounded by* $x^2 + y^2 + z^2 = z$.

Dec. - 2004

1) Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-x^2(1+y^2)}\, x\, dx\, dy$.

2) Evaluate $\iiint x^2yz\, dx\, dy\, dz$ throughout the volume bounded by the plane $x = 0,\ y = 0,\ z = 0,$ $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. **[5 Marks]**

May - 2005

1) Evaluate $\iint_R \frac{x^2y^2}{x^2+y^2} dx\, dy$, where R is annulus between $x^2+y^2=4$ and $x^2+y^2=9$. **[5 Marks]**

2) Evaluate $\iiint (x^2y^2+y^2z^2+z^2x^2) dx\, dy\, dz$ throughout the volume of the sphere $x^2+y^2+z^2=a^2$. **[5 Marks]**

Dec. - 2005

1) Evaluate $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{xy}{x^2+y^2} e^{-(x^2+y^2)}\, dx\, dy$. **[6 Marks]**

May - 2006

1. Express the following integral as a single term double integral and evaluate $\int_0^1\int_0^y (x^2+y^2) dx dy + \int_1^2\int_0^{2-y} (x^2+y^2)\, dx dy$. Show the region of integration. **[6 Marks]**

2. Find the valume of the surface bounded by $z = 4 - x^2 - y^2$ and the xy plane. **[5 Marks]**

Dec. - 2006

1. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{dx\, dy}{(1+e^y)\sqrt{1-x^2-y^2}}$. **[6 Marks]**

May - 2007

1. Evaluate $\iint_R (x^2+y^2)\, dx\, dy$ over the area of triangle whose vertices are (0, 1) (1, 1) and (1, 2). **[5 Marks]**

2. Evaluate $\iiint_V \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\, dx\, dy\, dz$ throughout the volume of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. **[6 Marks]**

3. Evaluate $\int_0^a \int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\, dx\, dy$ **[5 Marks]**

Dec. - 2007

1. *Evaluate* $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dxdy.$

[5 Marks]

2. *Evaluate* $\iint_R \frac{x^2y^2}{x^2+y^2}\,dxdy$, *where R is annulus between* $x^2+y^2=4$ *and* $x^2+y^2=9.$

[5 Marks]

May - 2008

1. *Evaluate* $\int.\int y\,dx\,dy$ *over the area bounded by* $y = x^2$ *and* $x + y = 2$, *integrating w.r.t. x first.*

[6 Marks]

2. *Evaluate* $\int\int\int \frac{dxdydx}{\sqrt{a^2-x^2-y^2-z^2}}$ *over the volume of the sphere* $x^2+y^2+z^2=a^2$ *in the positive octant.*

Dec. - 2008

[5 Marks]

1. *Express the following integral as a single term double integral and evaluate :*

$$\int_0^1\int_0^y (x^2+y^2)\,dxdy + \int_1^2\int_0^{2-y} (x^2+y^2)\,dxdy$$

[6 Marks]

2. *Find the volume enclosed between the cylinders* $x^2+y^2=2ax$ *and* $z^2=2ax.$ **[6 Marks]**

❑❑❑

15.2 Illustrated Examples

I) Cartesian form :

Example 15.1 : *Find area bounded by the curves $y^2 = 4\ ax$ and $x^2 = 4\ ay$.*

Solution :

$$\text{Area} = \iint_A dx\,dy \qquad \dots (1)$$

Where A ($y^2 = 4$ ax, $x^2 = 4$ ay) as shown in the Fig. 15.2.

At A, $\quad y^2 = 4\text{ ax}, \; x^2 = 16\,a^2 y^2$

$$\frac{x^4}{16\,a^2} - 4\,ax = 0 \;\Rightarrow x\left(x^3 - 64\,a^3\right) = 0$$

$$\Rightarrow \quad x = 0, \; x^3 = 64a^3$$

$$\Rightarrow \quad x = 0, \; x = 4a$$

$\therefore$ 0 (0, 0) and A (4a, 4a) are the point of intersection of two curves.

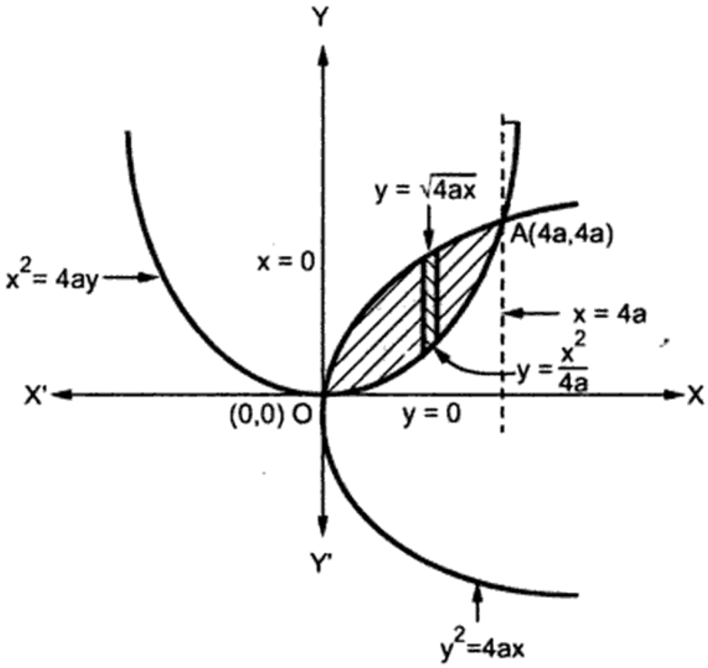


Fig. 15.2

In the shaded area, over the strip y varies from

$$y = \frac{x^2}{4a} \text{ to } y = \sqrt{4\,ax}$$

and x varies from

$$x = 0 \text{ to } x = 4a \qquad \text{from (1)}$$

$$\therefore \quad \text{Area} = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{\sqrt{4ax}} dx\,dy$$

$$= \int_0^{4a} (y)_{\frac{x^2}{4a}}^{\sqrt{4ax}} dx$$

$$= \int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a} \right) dx$$

$$= \left[\sqrt{4a} \frac{x^{3/2}}{3/2} - \frac{1}{4a} \frac{x^3}{3} \right]_0^{4a}$$

$$= \left[\sqrt{4a}\,(4a)^{3/2} \frac{2}{3} - \frac{1}{4a \times 3} (4a)^3 \right]$$

$$= \frac{32\,a^2}{3} - \frac{16\,a^2}{3}$$

$$\boxed{\text{Area} = \frac{16\,a^2}{3}} \quad \text{... Ans.}$$

Example 15.2 : *Find area between the curves $y^2 = 4x$ and $2x - 3y + 4 = 0$.*

Solution :

$$\text{Area} = \iint_A dx\,dy \quad \text{... (1)}$$

Where A is area between $y^2 = 4x$, and $2x - 3y + 4 = 0$ as shown in the Fig. 15.3.

To find the point of intersection B and C.

$$y^2 = 4x, \quad 2x - 3y + 4 = 0.$$

$$y^2 = 2\,(2x)$$

$$= 2\,(-4 + 3y)$$

$$\Rightarrow \quad y^2 - 6y + 8 = 0$$

$$\Rightarrow \quad y = 2, y = 4$$

When $y = 2, x = 1$

$y = 4, x = 4,$

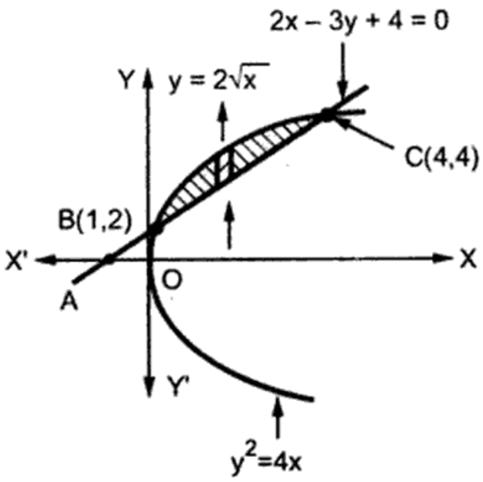


Fig. 15.3

$\therefore$ B (1, 2) and C(4, 4)

Limits over the strip $y = \dfrac{2x + 4}{3}$ to $y = 2\sqrt{x}$ and moving the strip from $x = 1$ to $x = 4$.

$$= 16a^2 \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\boxed{\text{Area} = 3\pi a^2} \qquad \text{... Ans.}$$

Example 15.5 : *Find the area between the curve $y^2 x = 4a^2(2a - x)$ and it's asymtote.*

Solution : $x = 0$ i.e. y-axis an asymtote to the curve.

Put $y = 0 \Rightarrow 2a - x = 0$

$\Rightarrow x = 2a$

$(2a, 0)$ is the point where, the curve meet x-axis.

By symmetry,

$$\text{Total area} = 2\iint_A dx\,dy$$

$$= 2 \times \text{Area of upper Half}$$

$$= 2 \times \int_{x=0}^{2a} \int_0^{2a\sqrt{\frac{2a-x}{x}}} dx\,dy$$

$$= 2 \times \int_0^{2a} \sqrt{\frac{2a-x}{x}}\, dx,$$

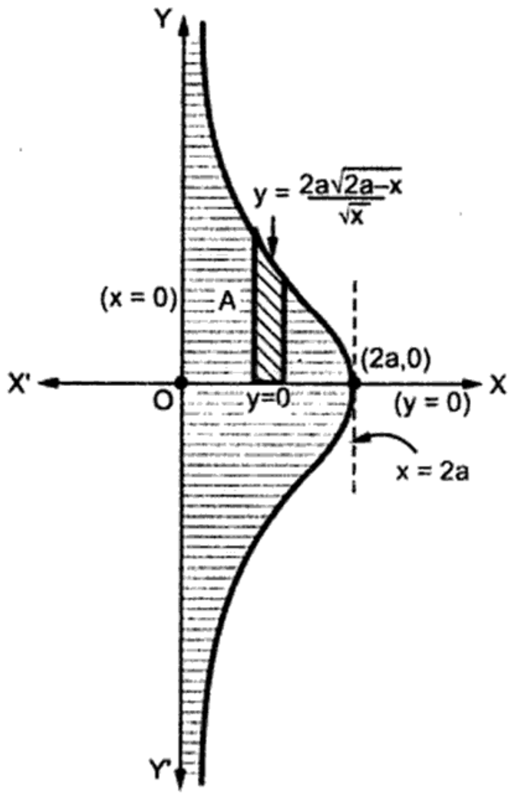


Fig. 15.6

PUT $x = 2at$

$dx = 2a\,dt$

Limits :

x	0	2a
t	0	1

$\therefore$

$$\text{Area} = 2\int_0^1 (2a - 2at)^{1/2} (2at)^{-1/2}\, 2a\, dt$$

$$= 8a^2 \int_0^1 t^{-1/2} (1 - t)^{1/2} dt = 8a^2 \beta\left(\frac{1}{2}, \frac{3}{2}\right)$$

$$= 8a^2 \frac{\Gamma 1/2 \; \Gamma 3/2}{\Gamma 2}$$

$$\boxed{\text{Area} = 4a^2 \pi} \qquad \text{... Ans.}$$

Example 15.6 : *Find area enclosed by the curves* $x(x^2 + y^2) = a(x^2 - y^2)$ *and it's asymtote.*

Solution : The shape of the curve is as shown in the Fig. 15.7.

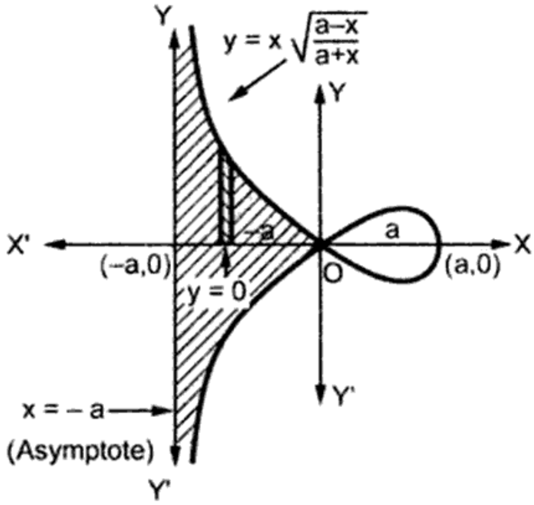


Fig. 15.7

x = – a is the parallel asymtote

$$\text{Total area} = 2 \times \text{Area of upper half shaded in the Fig. 15.7.}$$

$$= 2 \times \iint_A dx\,dy = 2\int_{-a}^{0}\int_{0}^{x\sqrt{\frac{a-x}{a+x}}} dx\,dy$$

$$= 2\int_{-a}^{0} \frac{x(a-x)}{\sqrt{a^2 - x^2}}\,dx$$

PUT $x = -a \sin\theta$

$\therefore\ dx = -a \cos\theta\ d\theta$

x	– a	0
0	π/2	0

$$\therefore \quad \text{Area} = 2\int_{\pi/2}^{0} \frac{-a \sin\theta(a + a\sin\theta)}{a\cos\theta} - a\cos\theta\ d\theta$$

$$= 2a^2 \int_{0}^{\pi/2} \left(\sin\theta + \sin^2\theta\right) d\theta$$

$$= -2a^2\left[1 + \frac{1}{2}\frac{\pi}{2}\right]$$

$$\boxed{\text{Area} = +2a^2\left(1 + \frac{\pi}{4}\right) \text{(Numerically)}}$$ ... Ans.

Example 15.8 : *Find area inside the cardiode $r = 2a\,(1 + \cos\theta)$ and outside the parabola* $r = \dfrac{2a}{1 + \cos\theta}$

Solution :

$$\text{Area} = 2 \iint_A r\, d\theta\, dr$$

$$= 2 \times \text{Area of the upper half.}$$

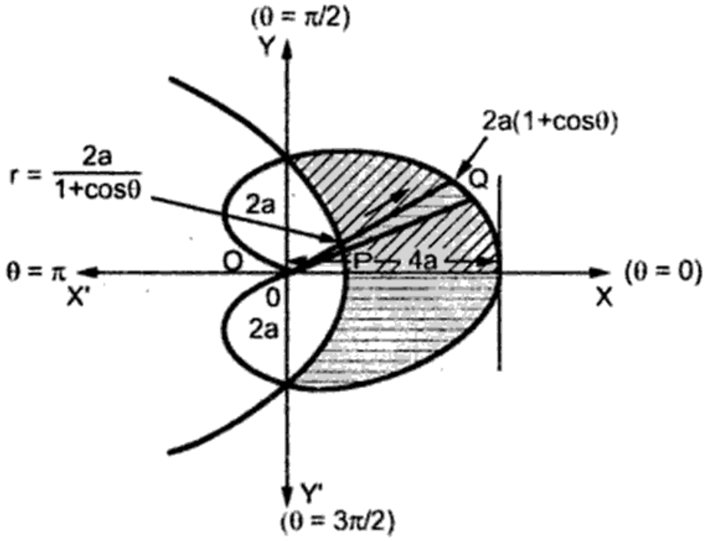


Fig. 15.9

$$= 2 \int_{\theta=0}^{\pi/2} \int_{\frac{2a}{1+\cos\theta}}^{2a(1+\cos\theta)} r\, d\theta\, dr$$

$$= 2 \int_0^{\pi/2} \left(\frac{r^2}{2}\right)_{\frac{2a}{1+\cos\theta}}^{2a(1+\cos\theta)} d\theta\, dr$$

$$= \int_0^{\pi/2} \left[4a^2 (1 + \cos\theta)^2 - \frac{4a^2}{(1 + \cos\theta)^2}\right] d\theta$$

$$= 4a^2 \int_0^{\pi/2} \left[1 + 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right] d\theta - 4a^2 \int_0^{\pi/2} \frac{d\theta}{\left(2 \cos \frac{\theta}{2}\right)^2}$$

$$= 4a^2 \left[\frac{3}{2}\theta + 2 \sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} - \frac{4a^2}{4} \int_0^{\pi/2} \sec^4 \frac{\theta}{2}\, d\theta$$

$$= 4a^2\left[\frac{3\pi}{4} + 2\right] - a^2 \int_0^{\pi/2} \left(1 + \tan^2\frac{\theta}{2}\right) \sec^2\frac{\theta}{2}\, d\theta$$

Put $\tan\frac{\theta}{2} = t$

$\because \quad \frac{1}{2}\sec^2\frac{\theta}{2}\, d\theta = dt$

Limits :

θ	0	π/2
t	1	1

$$= 4a^2\left[\frac{3\pi}{2} + 2\right] - a^2 \int_0^1 \left(1 + t^2\right) 2\, dt$$

$$= 4a^2\left[\frac{3\pi}{2} + 2\right] - \frac{8a^2}{3} = 4a^2\left[\frac{3\pi}{2} + 2 - \frac{2}{3}\right]$$

$$\boxed{\text{Area} = a^2\left(4\pi + \frac{16}{3}\right)} \qquad \text{... Ans}$$

Example 15.9 : *Find area outside the circle $x^2 + y^2 = a^2$ and inside the cardiode $r = a(1 + \cos\theta)$.* **(Dec.-2007)**

Solution : Total Area = 2 × Area of upper half

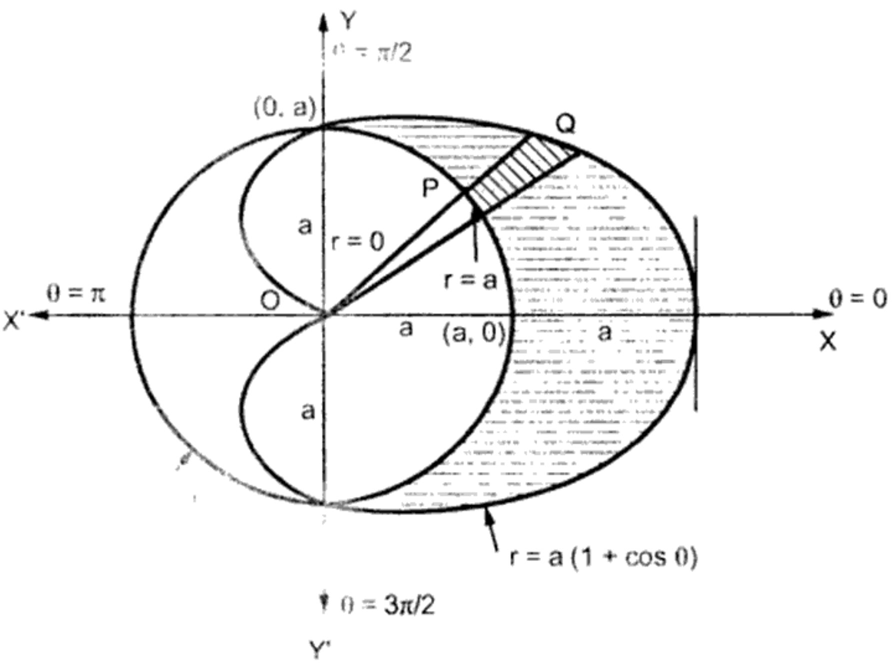


Fig. 15.10

$\iint r\, d\theta\, dr$

$$\therefore \quad \frac{dy}{d\theta} = 3a \sin^2\theta \cos\theta \; d\theta \qquad \text{from (1)}$$

$$\text{Total area} = 4\int_0^{\pi/2} a \cos^3\theta \; 3a \sin^2\theta \cos\theta \; d\theta$$

$$= 12a^2 \int_0^{\pi/2} \sin^2\theta \cos^4\theta \; d\theta$$

$$= 12a^2 \frac{(2-1)\left[(4-1)(4-3)\right]}{6(6-2)(6-4)} \frac{\pi}{2}$$

$$\boxed{\text{Area} = \frac{3\pi a^2}{8}} \qquad \text{... Ans}$$

Example 15.12 : *Find area of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ between it's base and portion of the curve from cusp to cusp.*

Solution : The curve is symmetrical about x-axis as shown in Fig. 15.13.

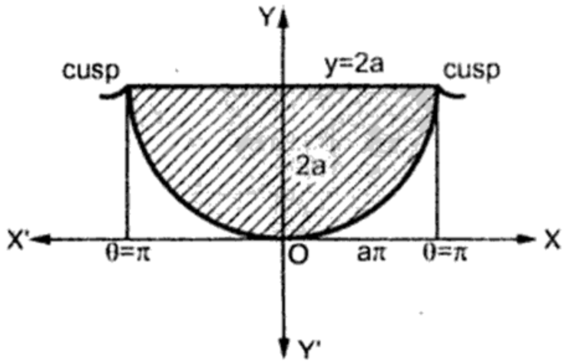


Fig. 15.13

$$\text{Total area} = 2 \times \text{Area in first quadrant.}$$

$$= 2\int x\,dy$$

$$= 2\int_0^{\pi} x \frac{dy}{d\theta} \; d\theta$$

$$= 2\int_0^{\pi} a(\theta + \sin\theta)\, a \sin\theta \; d\theta$$

$$= 2a^2 \int_0^{\pi} (\theta \sin\theta + \sin^2\theta) \; d\theta$$

$$= 2a^2 \left[(-\theta \cos\theta + \sin\theta)_0^{\pi} + 2 \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right]$$

$$= 2a^2 \left[\pi + \frac{\pi}{2}\right]$$

$$\boxed{\text{Area} = 3\pi a^2} \qquad \text{... Ans}$$

Example 15.13 : *Find area of the loop of the curve $x^3 + y^3 = 3axy$ (Folium of Descartes)*

$$\text{Area} = \frac{3a^2}{2}\int_1^\infty \frac{du}{u^2}$$

$$= \frac{3a^2}{2}\left(-\frac{1}{u}\right)_1^\infty = \frac{3a^2}{2}(0+1)$$

$$\boxed{\text{Area} = \frac{3a^2}{2}}$$ ... Ans

Example 15.14 : *Find the total area of the curve $r = a\cos 2\theta$.*

Solution : The given curve is **Four leaved rose. n is even** (2)

∴ It contains **2n leaves.** (4)

Note : For the curve r = a cos n θ....

1) If n is even then total area $= 2n\int_{-\frac{\pi}{2n}}^{\frac{\pi}{2n}} \frac{r^2}{2}\, d\theta$

2) If n is odd then total area $= n\int_{-\frac{\pi}{2n}}^{\frac{\pi}{2n}} \frac{r^2}{2}\, d\theta$

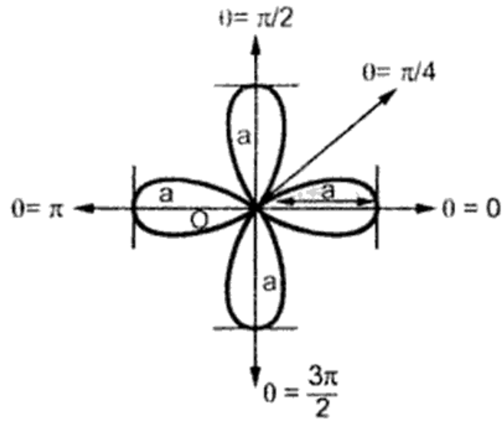


Fig.15.15

$$\text{Required area} = 2\cdot 2\int_{-\pi/4}^{\pi/4} \frac{r^2}{2}\, d\theta = 2\int_{-\pi/4}^{\pi/4} a^2\cos^2 2\theta\, d\theta$$

$$= 2 \times 2a^2\int_0^{\pi/4} \cos^2 2\theta\, d\theta$$

Put $2\theta = u$

$d\theta = \frac{du}{2}$

θ	0	π/4
u	0	π/2

$$= 4a^2\int_0^{\pi/2} \cos^2 u \frac{du}{2}$$

$$= 2a^2\int_0^{\pi/2} \cos^2 u\, du = 2a^2\,\frac{2-1}{2}\cdot\frac{\pi}{2}$$

$$\boxed{\text{Area} = \frac{\pi a^2}{2}}$$ ... Ans

15.4 Illustrated Examples

Example 15.17 : *Find the mass of the lamina bounded by the curve $16y^2 = x^3$ and the line $2y = x$ assuming that the density at a point of the area varies as the distance of the point from x - axis.*

Solution : Given curves are,

$$16y^2 = x^3 \text{ and } 2y = x \qquad \dots (1)$$

To find the point of intersection of the curves we have,

$$16\left(\frac{x}{2}\right)^2 = x^3$$

$$\Rightarrow \quad 4x^2 = x^3$$

$$\therefore \quad x = 0, \; x = 4$$

If $\quad x = 0, \; y = 0$

and $\quad x = 4, \; y = 2$

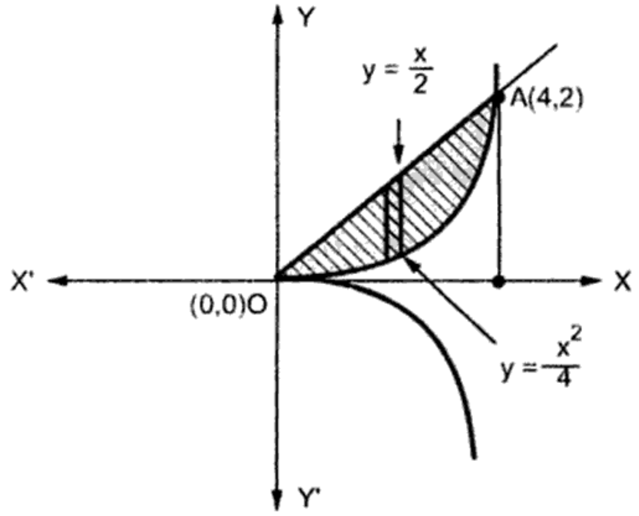


Fig. 15.18

Given density of any point (x, y) is **ky**.

$$\text{Mass of lamina} = \iint_A \rho \; dx \; dy$$

$$= \iint_A ky \; dx \; dy$$

$$= \int_{x=0}^{4} \int_{y=\frac{x^{3/2}}{4}}^{x/2} ky \; dx \; dy$$

$$= k\int_0^4 \left(\frac{y^2}{2}\right)_{\frac{x^{3/2}}{4}}^{x/2} dx$$

$$= \frac{k}{2}\int_0^4 \left(\frac{x^2}{4} - \frac{x^3}{16}\right) dx$$

$$= \frac{k}{2}\left(\frac{x^2}{12} - \frac{x^4}{64}\right)_0^4$$

$$= \frac{k}{2}\left(\frac{4^3}{12} - 4\right)$$

$$\boxed{\text{Mass} = \frac{2}{3}k} \qquad \dots \text{Ans.}$$

Example 15.18 : *Find the mass of a lamina in the form of the cardiode $r = a(1 + \cos\theta)$ whose density at any point varies as the square of it's distance from the initial line.*

Solution :
$$\text{Mass} = \iint_R F(r,\theta)\, r\, d\theta\, dr \qquad \dots (1)$$

Where R is cardiode r = a (1 + cos θ) as shown in the Fig. 15.19.

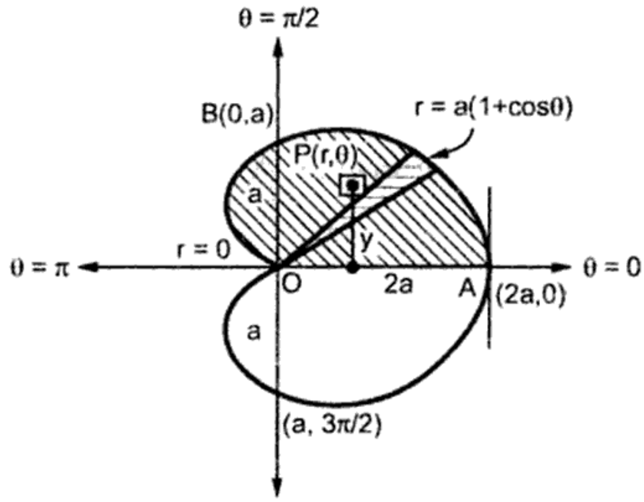


Fig. 15.19

The distance of any point P (r, θ) from the initial line is r sin θ.

$\therefore$ Density $= \rho = k(r \sin\theta)^2 = kr^2\sin^2\theta$

By symmentry,

$$\text{Mass} = 2\int_{\theta=0}^{\pi}\int_{0}^{a(1+\cos\theta)} \rho\, r\, d\theta\, dr$$

$$= 2\int_{0}^{\pi}\int_{0}^{a(1+\cos\theta)} k\, r^2\sin^2\theta\; r d\theta\, dr$$

$$= 2k\int_{0}^{\pi}\left(\int_{0}^{a(1+\cos\theta)} r^3 dr\right)\sin^2\theta\, d\theta$$

$$= 2k\int_{0}^{\pi}\left(\frac{r^4}{4}\right)_0^{a(1+\cos\theta)}\sin^2\theta\, d\theta$$

$$= \frac{k}{2}\int_{0}^{\pi} a^4(1+\cos\theta)^4\sin^2\theta d\theta$$

$$= \frac{ka^4}{2}\int_{0}^{\pi}\left(2\cos^2\frac{\theta}{2}\right)^4\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2 d\theta \qquad \text{Put } \frac{\theta}{2} = u$$

$$= 64\ k\ a^4 \int_0^{\pi/2} \sin^2 u\ \cos^{10} u (2du) \qquad \therefore\ d\theta = 2\ dv$$

$$= 64\ k\ a^4 \frac{(1)\ (9 \cdot 7 \cdot 5 \cdot 3 \cdot 1)}{12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2 \cdot} \frac{\pi}{2}$$

$$\boxed{\text{Mass} = \frac{21.k\ \pi\ a^4}{32}} \qquad \text{... Ans.}$$

Example 15.19 : *Find the mass of the octant of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ *if density at any point being kxyz.*

Solution :

$$\text{Mass of solid} = M = \iiint_V \rho\ dx\ dy\ dx \qquad \text{... (1)}$$

ρ is density at the point P (x, y, z) = kxyz

$$\therefore \quad M = \iiint_V k\ xyz\ dx\ dy\ dx \qquad \text{... (2)}$$

Where V is volume of the octant of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Put $\quad \frac{x}{a} = X, \frac{y}{b} = Y, \frac{z}{c} = Z$

$$\therefore \quad x^2 + y^2 + z^2 = 1 \text{ (Sphere)}$$

$$\therefore \quad M = \iiint ka^2b^2c^2\ XYZ\ dx\ dy\ dz$$

Transforming to spherical polar co-ordinates

$$X = r\ \sin\theta\ \cos\phi,\ Y = r\ \sin\theta\ \sin\phi,\ Z = r\ \cos\theta$$

$$dX\ dY\ dZ = r^2\ \sin\theta\ d\theta\ d\phi\ dr$$

r varies from $\quad r = 0$ to $r = 1$

θ varies from $\quad \theta = 0$ to $\theta = \pi/2$

ϕ varies from $\quad \phi = 0$ to $\phi = \pi/2$

$$\therefore \quad M = ka^2b^2c^2 \int_0^{\pi/2} \int_0^{\pi/2} \int_{r=0}^{1} r^5 \sin^3\theta\ \cos\theta\ \sin\phi \cos\phi\ d\theta\ d\phi\ dr$$

$$= ka^2b^2c^2 \left(\frac{r^6}{6}\right)_0^1 \left(\frac{\sin^4\theta}{4}\right)_0^{\pi/2} \left(\frac{\sin^2\phi}{2}\right)_0^{\pi/2}$$

$$= ka^2b^2c^2 \frac{1}{6} \cdot \frac{1}{4} \cdot \frac{1}{2}$$

$$\boxed{\text{Mass} = \frac{ka^2b^2c^2}{48}}$$ ... Ans.

Excercise 15.2

1) *If the density at any point on a circular lamina is k times the square of it's distance from a fixed, point on it's circumference find it's mass.* **[Ans. : $\frac{3}{2}\pi\ ka^4$]**

2) *A lamina in the from of a parabolic segment of mass M, height h, and base 2k has density at a point is $\lambda\ pq^3$ per unit area. Where p, q are distances from the base and axes respectively. Find the value of λ.* **[Ans. : $\lambda = \frac{24m}{k^4h^2}$]**

Hint :

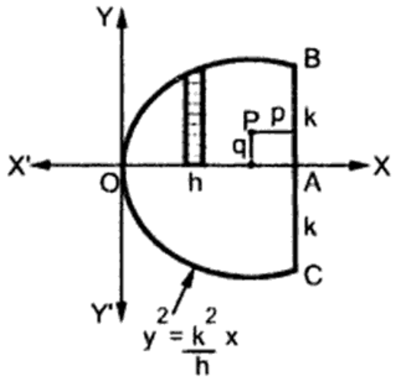


Fig. 15.20

$$\text{Mass} = M = 2\int_0^h \int_0^{k\sqrt{\frac{x}{h}}} \lambda\ pq^3 dx\ dy = \frac{\lambda\ k^4 h^2}{24}$$

3) *Find the mass of the plate in the form of the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$, the density being $\rho = kxy$.*

[Ans. : $\frac{ka^2b^2}{20}$]

4) *The density at any point of a non-uniform circular lamina of radius unity is k times it's distance froma given diameter. Find the total mass of lamina.* **[Ans. : $\frac{4\ k}{3}$]**

5) *The boundaries of a plate can be given as $x = 0$, $y = 0$, $x = 1$ and $y = e^x$. If density at any point varies as the square of it's distance from the origin. Find the mass of the plate.*

[Ans. : $k\left(e + \frac{e^2}{9} - \frac{19}{9}\right)$]

6) *Find mass of lamina bounded by $y = x^2 - 3x$ and $y = 2x$, ($\rho = \lambda xy$).* **(May-1998)**

15.5 Volume of Solids

1) The volume of solid by triple integration is given by

$$\text{Volume} = V = \iiint_V dv = \iiint_V dx\ dy\ dz \quad \text{... (1)}$$

2) In spherical polar co-ordinates

$$V = \iiint_V r^2 \sin\theta\ dr\ d\theta\ d\phi \quad \text{... (2)}$$

3) In cylindrical polar co-ordinates

$$V = \iiint_V \rho\ d\rho\ d\phi\ dz \quad \text{... (3)}$$

15.6 Illustrated Examples

Example 15.20 : *Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by triple integration.*

Solution : By triple integration

$$\text{Total volume} = 8\iiint_V dx\ dy\ dx \text{ (By symmetry)} \quad \text{... (1)}$$

Where 'V' is the volume of the ellipsoid . $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ located in the first octant.

Put $\frac{x}{a} = X,\ \frac{y}{b} = Y,\ \frac{z}{c} = Z$

$\therefore \quad X^2 + Y^2 + Z^2 = 1$ (sphere)

and $dx = a\ dx,\ dy = bdy,\ dz = c\ dz$ from equation (1)

$$\text{Total volume} = 8\iiint_V abc\ dX\ dY\ dZ \quad \text{... (2)}$$

Transforming into shperical polar co-ordinates by

Putting $X = r \sin\theta \cos\phi$ and $Y = r \sin\theta \sin\phi$, $Z = r\cos\theta$

And $dX\ dY\ dZ = r^2 \sin\theta\ dr\ d\theta\ d\phi$ from equation (2)

$$\text{Volume} = 8\,abc \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r^2 \sin\theta\ d\theta\ d\phi\ dr$$

$$= 8\ abc \int_{\phi=0}^{\pi/2} d\phi \int_{\theta=0}^{\pi/2} \sin\theta\ d\theta \int_{r=0}^{1} r^2 dr$$

$$= 8\ abc\left(\frac{\pi}{2}\right) \times (-\cos\theta)_0^{\pi/2} \times \left(\frac{r^3}{3}\right)_0^1$$

$$= 8\ abc \times \frac{\pi}{2} \times 1 \times \frac{1}{3}$$

$$\boxed{\text{Volume} = \frac{4abc\ \pi}{3}} \qquad \text{... Ans.}$$

Example 15.21 : *Find volume of the tetrahedron bounded by the co-ordinates planes and the plane* $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$

Solution : $\text{Volume} = \iiint dx\ dy\ dz \qquad \text{... (1)}$

Put $\frac{x}{2} = u,\ \frac{y}{3} = u,\ \frac{z}{4} = w$

$\Rightarrow$ $x = 2u,\ y = 3v,\ z = 4w$ from equation (1)

$$V = \iiint 24\ du\ dv\ dw$$

$$= 24 \iiint du\ dv\ dw$$

$$= 24 \iiint u^{1-1} v^{1-1} w^{1-1}\ du\ dv\ dw,\ (u + v + w = 1)$$

$$= 24\ \frac{\lceil 1 \cdot \lceil 1 \cdot \lceil 1}{\lceil 1 + 1 + 1 + 1}$$

$$= \frac{24}{\lceil 4} = \frac{24}{3!}$$

$$\boxed{\text{Volume} = \frac{24}{6} = 4} \qquad \text{... Ans.}$$

Example 15.22 : *Find the volume enclosed by the cone* $x^2 + y^2 = z^2$ *and the paraboloid* $x^2 + y^2 = z$ **(May-1999, May-2004))**

Solution :

$$V = \iint \int_{z = x^2 + y^2}^{\sqrt{x^2 + y^2}} dx\ dy\ dz$$

$$= \iint \left(\int_{z = x^2 + y^2}^{\sqrt{x^2 + y^2}} dz \right) dx\ dy$$

$$V = \iint \left[\sqrt{x^2 + y^2} - (x^2 + y^2)\right] dx\ dy \qquad \text{... (1)}$$

The intersection of the cone and the paraboloid is

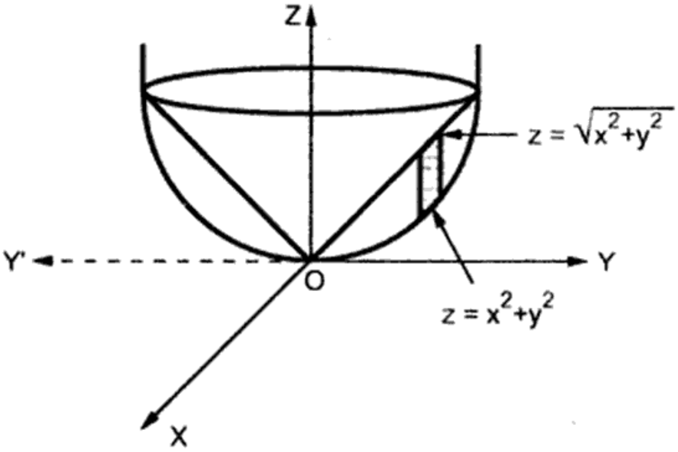


Fig. 15.21

$$\sqrt{x^2 + y^2} = x^2 + y^2 \text{ i.e.}$$

$$x^2 + y^2 = 1$$

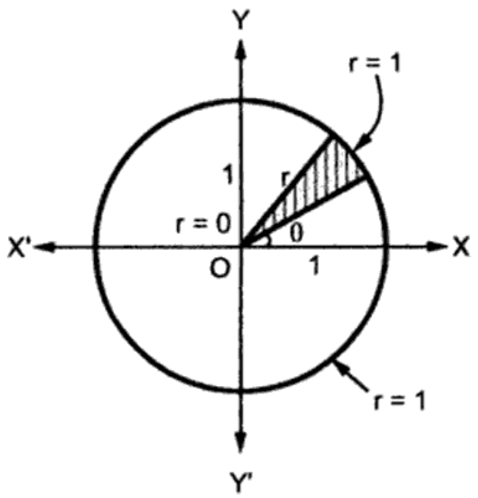


Fig. 15.22

Changing equation (1) into polar co-ordinates by

Putting $x = r\cos\theta$

$y = r\sin\theta$, $dx\,dy = r\,d\theta\,dr$

$$V = 4\int_0^{\pi/2}\int_0^1 (r - r^2) r\,d\theta\,dr$$

$$= 4\int_0^{\pi/2}\left[\int_0^1 (r^2 - r^3)\,dr\right] d\theta$$

$$= 4\int_0^{\pi/2}\left(\frac{r^3}{3} - \frac{r^4}{4}\right)_0^1 d\theta$$

$$= 4\int_0^{\pi/2}\left(\frac{1}{3} - \frac{1}{4}\right) d\theta$$

$$= 4\int_0^{\pi/2}\left(\frac{1}{12}\right) d\theta$$

$$= 4\cdot\frac{1}{12}\cdot\frac{\pi}{2}$$

$$\boxed{\text{Volume} = \frac{\pi}{6}}$$ ... Ans.

Example 15.23 : *Find volume of the region bounded by paraboliod $x^2 + y^2 = 2z$ and the cylinder $x^2 + y^2 = 4$* **(Dec.-2001)**

Solution : By using cylindrical polar co-ordinates.

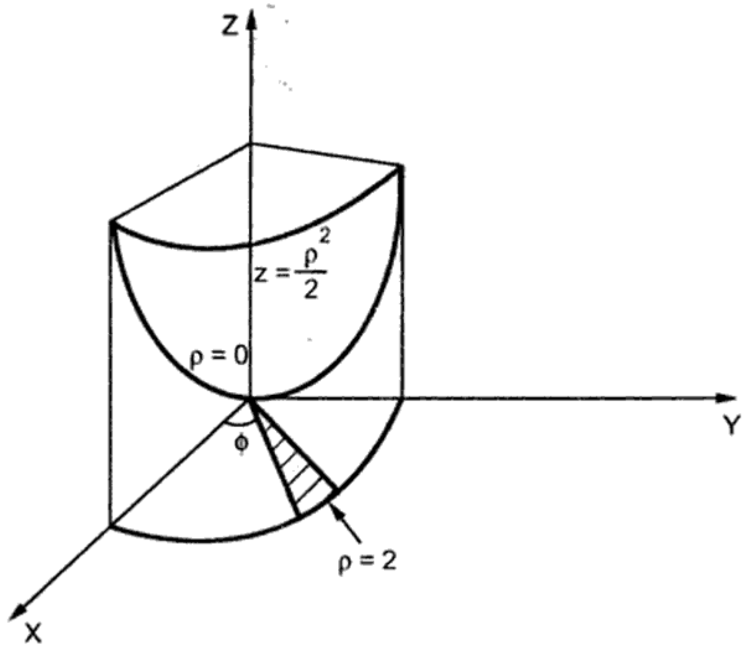


Fig. 15.23

$$x = \rho \cos \theta, \quad y = \rho \sin \theta, \ z = z$$

and
$$x^2 + y^2 = \rho^2$$

$$dx\, dy\, dz = \rho\, d\rho\, d\phi\, dz$$

$$\therefore \quad V = 4 \iiint dx\, dy\, dz$$

$$= 4 \iiint \rho\, d\rho\, d\phi dz$$

$$= 4 \int_0^{\pi/2} d\phi \int_0^2 \rho \left(\int_{z=0}^{\rho^2/2} dz \right) d\rho = 4 \times \frac{\pi}{2} \times \int_0^2 \frac{\rho^3}{2}\, d\rho$$

$$= 4 \cdot \frac{\pi}{2} \cdot \frac{1}{2} \left(\frac{e^4}{4} \right)_0^2$$

$$= \pi \frac{(2)^4}{4}$$

$$\boxed{\text{Volume} = 4\pi}$$ ... Ans.

z varies from $z = x^2 + y^2$ to $z = 2x$

$$\therefore \text{ Volume } = \iiint_V dx\, dy\, dz$$

$$= \iint_R \left(\int_{x^2+y^2}^{2x} dz \right) dx\, dy$$

$$= \iint_R (2x - x^2 - y^2)\, dx\, dy \quad \dots (2)$$

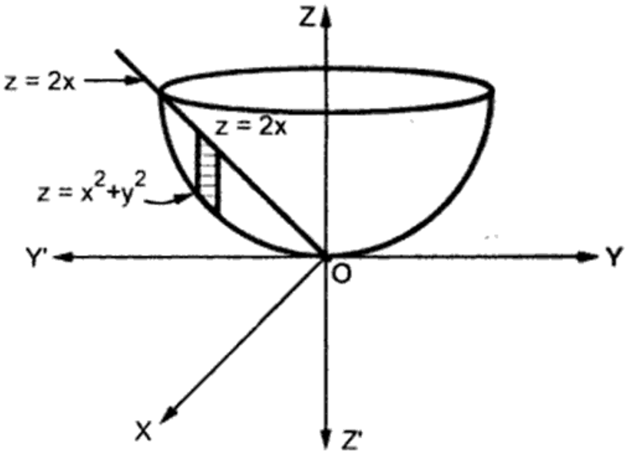


Fig. 15.24

R is circle $r = 2\cos\theta$

Transforming (2) into polar

'r' varies from $r = 0$ to $r = 2\cos\theta$

and 'θ' varies from $\theta = \frac{-\pi}{2}$ to $\theta = \frac{\pi}{2}$

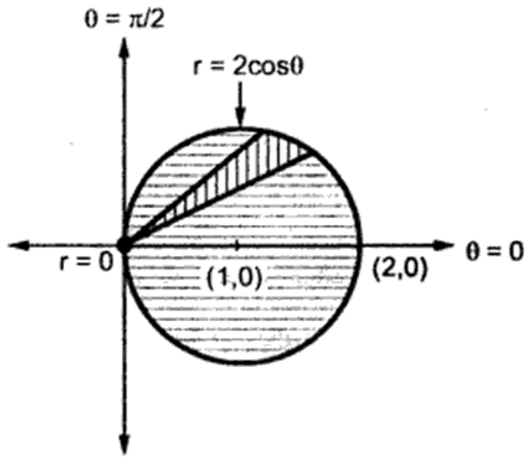


Fig. 15.25

From equation (1)

$$\text{Volume} = 2\int_0^{\pi/2}\int_0^{2\cos\theta} (2\, r\cos\theta - r^2)\, r\, d\theta\, \partial r$$

$$= 2\int_0^{\pi/2} \left(2\cos\theta\, \frac{r^3}{3} - \frac{r^4}{4}\right)_0^{2\cos\theta} d\theta$$

$$= 2\int_0^{\pi/2} \left[2\cos\theta\, \frac{(2\cos\theta)^3}{3} - \frac{(2\cos\theta)^4}{4}\right] d\theta$$

$$= 2\cdot\frac{16}{12}\int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= \frac{8}{3}\,\frac{4-1}{4}\cdot\frac{4-3}{4-2}\,\frac{\pi}{2}$$

$$\boxed{\text{Volume} = \frac{\pi}{2}} \quad \dots \text{Ans.}$$

Example 15.26 : *Find volume of the cylinder $x^2 + y^2 = 2ax$, intercepted between paraboloid $x^2 + y^2 = 2az$ and the XY - plane* **(May - 2005)**

Solution :

$$\text{Volume} = \iint \int_0^{\frac{x^2+y^2}{2a}} dx\, dy\, dz$$

$$= \iint_R \frac{x^2+y^2}{2a}\, dx\, dy$$

Where R is $x^2 + y^2 = 2ax$ as shown in the Fig. 15.26.

$$\text{Volume} = \frac{1}{2a} 2\int_0^{\pi/2} \int_0^{2a\cos\theta} r^2\, r\, d\theta\, dr$$

$$= \frac{1}{a}\int_0^{\pi/2} \left(\frac{r^4}{4}\right)_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{4a}\int_0^{\pi/2} (2a\cos\theta)^4\, d\theta$$

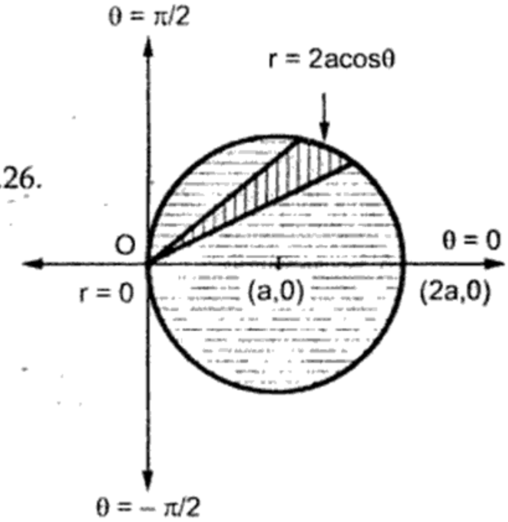


Fig. 15.26

$$\boxed{\text{Volume} = \frac{3\pi a^3}{4}}$$ ... Ans.

Example 15.27 : *Cylindrical hole of radius b is bored symmetrically through a sphere of radius a. Find the volume of remaining solid.* **(May - 2001)**

Solution : Let the equation of sphere be $x^2 + y^2 + z^2 = a^2$

$$\text{Volume} = V = \iint \int_0^{\sqrt{a^2-x^2-y^2}} dx\, dy\, dz$$

$$= 2\iint_R \sqrt{a^2-x^2-y^2}\, dx\, dy$$

Where R is region bounded by the circles r = a, r = b (a > b) in polars.

From equation (1)

$$V = 2\int_0^{2\pi}\int_b^a \sqrt{a^2-r^2}\, r\, dr\, d\theta$$

$$= -\int_0^{2\pi} d\theta \left[\frac{2}{3}\left(a^2-r^2\right)^{3/2}\right]_b^a$$

$$= \frac{2}{3}(2\pi)\left(a^2-b^2\right)^{3/2}$$

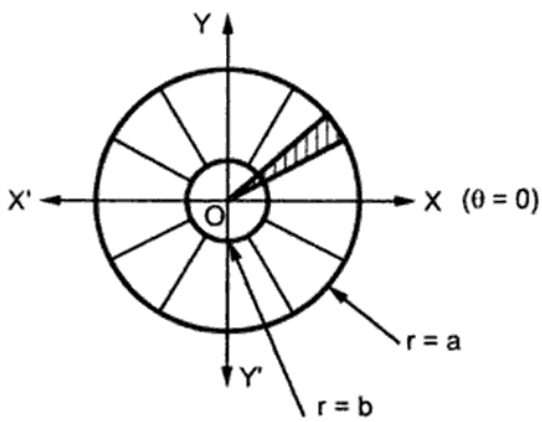


Fig. 15.27

$$\boxed{\text{Volume} = \frac{4\pi}{3}\left(a^2 - b^2\right)^{3/2}}$$ ... Ans.

Example 15.28 : *Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$* **(Dec. - 1998)**

Solution :

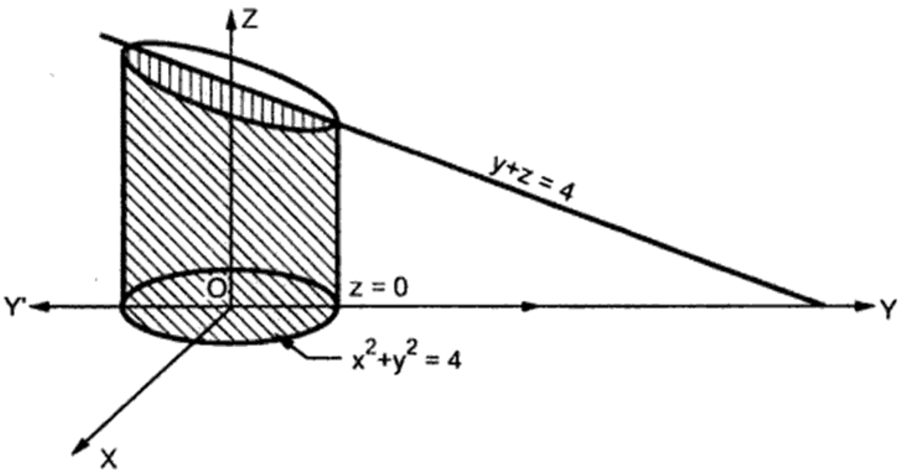


Fig. 15.28

For cylinder $x^2 + y^2 = 4$ and the plane $y + z = 4$

$\therefore$ $y = \pm\sqrt{4 - x^2}$ and $z = 4 - y$, Given $z = 0$

z varies from $z = 0$ to $z = 4 - y$

y varies from $y = -\sqrt{4 - x^2}$ to $y = \sqrt{4 - x^2}$

x varies from $x = -2$ to $x = 2$

Required volume $= V = \iiint dx\, dy\, dz$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{4-y} dx\, dy\, dz$$

$$= \int_{-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y)\, dx\, dy$$

$$= \int_{-2}^{2} \left(4y - \frac{y^2}{2}\right)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= 8\cdot\int_{-2}^{2} \sqrt{4 - x^2}\, dx$$

$$= 8\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]$$

$$\boxed{\text{Volume} = 16\pi} \quad \text{... Ans.}$$

Excercise 15.3

1) *Find the volume cut off from the paraboloid $x^2 + \frac{1}{9}y^2 + z = 1$ by the plane $z = 0$.*

Hint : $V = \iint\limits_R z\ dx\ dy = \iint\limits_{\substack{R \\ (\text{ellipse})}} \left(1 - x^2 - \frac{y^2}{9}\right) dx\ dy$

$$= \int_{-1}^{1} \int_{-3\sqrt{1-x^2}}^{+3\sqrt{1-x^2}} \left(1 - x^2 - \frac{y^2}{9}\right) dx\ dy$$

[Ans. : $\frac{3\pi}{2}$]

2) *Find the volume bounded by $y^2 = x$, $x^2 = y$ and the planes $z = 0$ and $x + y + z = 1$*

(Dec. - 1999, Nov. - 2002)

Hint : $V = \iint\limits_R z\ dx\ dy = \iint\limits_R (1 - x - y)\ dx\ dy$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (1 - x - y)\ dx\ dy$$

$= \left(\frac{1}{30}\right)$ *on simplification cubic units.*

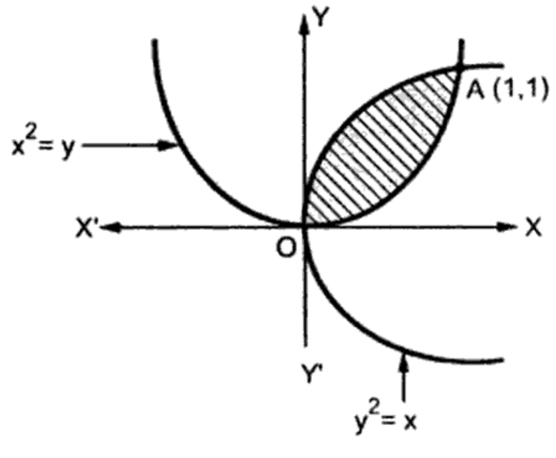


Fig. 15.29

3) *FInd the volume of the solid bounded by the cylinder $x^2 + y^2 = 2ay$, the paraboloid $x^2 + y^2 = az$ and the plane $z = 0$.*

Hint : $V = \iint\limits_R \int_0^{\frac{x^2+y^2}{a}} dx\ dy\ dz$

$$= 2\int_0^{\pi/2} \int_0^{2a\sin\theta} \frac{r^3}{a}\ r\ d\theta\ dr$$

[Ans. : $\frac{3}{2}\pi a^3$]

$$= \left(\frac{t}{2} - \frac{\sin 2\,pt}{4p} \right)_0^{2\pi} = \pi \qquad \text{... (3)}$$

$$I_2 = \int_0^{2\pi} \frac{(1 + \cos 2\,qt)}{2}\, dt$$

$$= \left(\frac{t}{2} + \frac{\sin 2\,qt}{4q} \right)_0^{2\pi}$$

$$= \pi \qquad \text{... (4)}$$

and $$I_3 = \int_0^{2\pi} \left[\sin(p+q)\,t + \sin(p-q)\,t \right] dt$$

$$= \left[\frac{-\cos(p+q)\,t}{p+q} - \frac{\cos(p-q)\,t}{p-q} \right]_0^{2\pi}$$

$$= 0 \qquad \text{... (5)}$$

$\therefore$ $$I = a^2(\pi) + b^2\pi + ab(0)$$

$$= \pi\left(a^2 + b^2\right) \qquad \text{... (6)}$$

Substituting from equations (2), (3), (4), (5), (6) in equation (1), we get

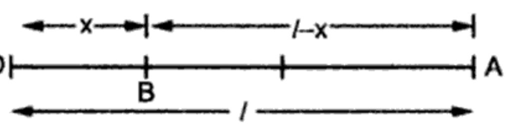
R.M.S. value of the function $$f\,(t) = \sqrt{\frac{\pi\left(a^2 + b^2\right)}{2\pi}}$$

$$\boxed{\text{R.M.S. value} = \sqrt{\frac{a^2 + b^2}{2}}} \qquad \text{... Ans.}$$

Example 15.30 : ***A rod length l is divided into two parts at random. Find average sum of squares of these parts. Also find mean value of rectangle contained by these two segments.***

Solution : Let $OA = l,\ OB = x$

$\therefore$ $BA = l - x$



Sum of square of these parts $= x^2 + (l-x)^2$

$$\text{Average value} = \frac{\int_0^l \left[x^2 + (l-x)^2 \right] dx}{\int_0^l dx}$$

$$= \frac{1}{l} \left[\frac{x^3}{3} + \frac{(l-x)^3}{-3} \right]_0^l$$

Example 15.33 : *Find the average density of the sphere of radius a whose density at a distance r from the centre is given by* $\rho = \rho_0 \left(1 + k\dfrac{r^3}{a^3}\right)$

Solution : Average density $= \dfrac{\text{Total Mass}}{\text{Volume}} = \dfrac{\text{Total Mass}}{\frac{4}{3}\pi a^3}$ ... (1)

To find total mass, we divide the sphere into a large number of concentric spherical shells and consider the shells of radii r and r + dr. The volume enclosed is $4\pi r^2 dr$

The mass of this shell is $= 4\pi r^2 dr \rho$.

$$= 4\pi r^2 dr \rho_0 \left(1 + k\frac{r^3}{a^3}\right) \qquad \text{from equation (1)}$$

$$\text{Average density} = \frac{\int_0^a 4\pi \rho_0 \left(1 + \frac{k r^3}{a^3}\right) r^2 dr}{\frac{4}{3}\pi a^3}$$

$$= \frac{4\pi \rho_0 \int_0^a \left(r^2 + \frac{k r^5}{a^3}\right) dr}{\frac{4\pi a^3}{3}}$$

$$= \frac{3}{4\pi a^3} 4\pi \rho_0 \left(\frac{r^3}{3} + \frac{k}{a^3}\frac{r^6}{6}\right)_0^a$$

$$= \frac{3\rho_0}{a^3}\left(\frac{a^3}{3} + \frac{k}{a^3}\frac{r^6}{6}\right)$$

$$= \rho_0 \left(1 + \frac{k}{2}\right) \qquad \text{... Ans.}$$

Example 15.34 : *Find the mean value of* $e^{-(x^2+y^2)}$ *over the area within the circle* $x^2 + y^2 = 1$. **(May - 1998)**

Solution : Let $u = f(x\, y) = e^{-x^2 - y^2}$

$$\text{M.V. of } u = u_m = \frac{\iint_R e^{-x^2 - y^2} dx\, dy}{\iint_R dx\, dy} \qquad \text{... (1)}$$

Let P (r, θ) be any point with in the square. Equation of line AB is

$$x = a \Rightarrow r\cos\theta = a = \Rightarrow r = a\sec\theta$$

Equation of OB is

$$y = x \Rightarrow r\sin\theta = r\cos\theta \Rightarrow \theta = \pi/4$$

$$\text{Mean distance} = \frac{\iint\limits_{R(OAB)} f(x,y)\,dx\,dy}{\iint\limits_{R(OAB)} dx\,dy}$$

$$= \frac{\iint r\cdot r\,d\theta\,dr}{\text{Area of }\Delta\,OAB}$$

$$= \frac{\int\limits_{\theta=0}^{\pi/4}\int\limits_{r=0}^{a\sec\theta} r^2\,d\theta\,dr}{\frac{1}{2}a^2}$$

$$= \frac{2}{a^2}\int\limits_{\theta=0}^{\pi/4}\left(\frac{r^3}{3}\right)_0^{a\sec\theta} d\theta$$

$$= \frac{2}{3a^2}\int\limits_0^{\pi/4} a^3\sec^3\theta\,d\theta$$

$$= \frac{2a}{3}\int\limits_0^{\pi/4}\sqrt{1+\tan^2\theta}\,\sec^2\theta\,d\theta \qquad \text{Put } \tan\theta = u$$

$$= \frac{2a}{3}\int\limits_0^1\sqrt{1+u^2}\,du \qquad \therefore\ \sec^2\theta\,d\theta = du$$

Limits :			
	θ	0	π/4
	u	0	1

$$= \frac{2a}{3}\left[\frac{u}{2}\sqrt{1+u^2}+\frac{1}{2}\log\left(1+\sqrt{1+u^2}\right)\right]_0^1$$

$$= \frac{2a}{3}\left[\frac{\sqrt{2}}{2}+\frac{1}{2}\log\left(1+\sqrt{2}\right)\right]$$

$$\boxed{\text{Mean distance} = \frac{a}{3}\left[\sqrt{2}+\log\left(1+\sqrt{2}\right)\right]}$$ .... Ans.

$$= \frac{\int v ds}{\int ds}$$

$$= \frac{\int v \frac{v}{f} dv}{\int \frac{v}{f} dv}$$

$$= \frac{\int_{v_0}^{v_1} v^2 dv}{\int_{v_0}^{v_1} v dv}$$

$$= \frac{(v_1^3 - v_0^3)/3}{(v_1^2 - v_0^2)/2}$$

$$\boxed{\text{S.A.V.} = \frac{2}{3}\left(\frac{(v_1^2 + v_1 v_0 + v_0^2)}{v_1 + v_0}\right)} \quad \text{.... Ans.}$$

Example 15.38 : *A rod of length 'a' is divided into three parts at random. Find M.V. of the sum of the square of these parts.*

Solution : If OC be a rod of length i.e. OC = a., Dividing randomly into three parts.

Let $\quad OA = x, \; AB = y,$

$\therefore \quad BC = a - x - y$

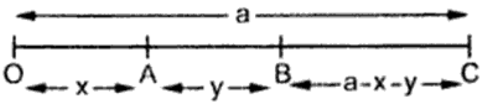


Fig. 15.34

The required mean value is

$$\text{M.V.} = \frac{\int_0^a \int_0^{a-x} \left[x^2 + y^2 + (a-x-y)^2\right] dx\, dy}{\int_0^a \int_0^{a-x} dx\, dy}$$

$$= \frac{\int_0^a \left[x^2 y + \frac{y^3}{3} + \frac{(a-x-y)^3}{-3}\right]_0^{a-x} dx}{\int_0^a (a-x)\, dx}$$

$$= \frac{6\,abc}{\pi} \times \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{6}$$

$$\boxed{\text{M.V.} = \frac{abc}{8\pi}} \quad \text{.... Ans.}$$

Excecise 15.4

1) *Find the R.M.S. value of "a sin pt + b cos qt + c sin rt + d cos st + ... "*

[Ans. : $\sqrt{\frac{(a^2 + b^2 + c^2 - - -)}{2}}$**]**

2) *The law of density ρ of a sphere of radius a is* $\rho = \rho_0 \frac{\sin(kr)}{nr}$, *where r is the distance from the centre,* ρ_0, *k and n are constants. Find average density.* **(Dec. - 1999)**

Hint : Refer example 15.33.

[Ans. : $\frac{3r_0}{nk^2a^2}$ (sin ak − ak cos ak)**]**

3) *Find R.M.S. value of distances from origin of points within* $x^2 + y^2 - 2ax = 0$ *cut off by the line* $x = a\ (x \geq a)$.

[Ans. : $a\sqrt{\frac{3}{2} + \frac{8}{3\pi}}$**]**

4) *Find the mean height of portion of parabola* $y = (x-2)(3-x)$ *which lies above x - axis.*

[Ans. : $\frac{1}{6}$**]**

5) *The surface density of an electrified disc of radius 'a' varies as* $k(a^2 - r^2)^{-1/2}$ *where 'r' is the distance from the centre. Find the mean density and also the ratio of the mean density to the density at the centre.*

[Ans. : Mean density = $\frac{2k}{a}$

Density = $\frac{k}{a}$**, Ratio = 2 : 1]**

6) *Find mean value of '*$x^2y^2z^2$*' over the*

i) *Positive octant of the sphere* $x^2 + y^2 + z^2 = a^2$.

[Ans. : $\frac{a^3}{8\pi}$**]**

ii) *Positive octant of the ellespoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

(May - 1999, Dec. - 2004) [Ans. : $\frac{abc}{8\pi}$**]**

7) *Find the mean value of '*r^2*' over the area of the cardiode* $r = a(1 + \cos\theta)$.

[Ans. : $\frac{35a^2}{24}$**]**

8) *Find the R.M.S. value of reciprocal of fourth power of the distance of a point of a circle from the centre.*

[Ans. : $\frac{1}{a^2}$**]**

Hint : $\int_0^{\pi/2}\int_0^a \frac{1}{r^4}\, r\, d\theta\, dr = -\frac{\pi}{a^2}$

9) *Find the mean value of the square of the distances from the origin of the points of the tringles whose vertices are* (0, 1), (1, 1), (1, 2).

[Ans. : $\frac{7}{12}$**]**

10) *Find R.M.S. value of the distances from origin of the points within the first quadrant of the circle of radius a.* **[Ans. : $\frac{a}{\sqrt{2}}$]**

11) *Find the mean value of xy over the area of the loop of Folium of Descartes "$x^3 + y^3 = 3axy$" ($a > 0$).* **[Ans. : $\frac{3a^2}{4}$]**

12) *A partical moves from rest with uniform acceleration, prove that in any interval of time from starting point the space average of velocity is $\frac{2}{3}$ V, where V is find velocity.* **[Ans. : $\frac{2}{3}$ V]**

$$\textbf{Hint : } \textit{Space average velocity} = \frac{\int v\,ds}{\int ds} = \frac{\int_0^V v^2 ds}{\int_0^V v\,dv}$$

13) *If in a spherical mass whose density ρ is a function of r where 'r' is distance from the centre. If D denotes mean density of matter of the mass included within a concentric sphere of radius r, prove that* $\rho = D + \frac{r}{3}\frac{dD}{dr}$.

14) *Find the mean value of the rectangle formed by the abscissa and ordinate at any point in the first quadrant of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **[Ans. : $\frac{ab}{2\pi}$]**

15) *Find the mean value of the product 'i.e.' for complete period, when* $i = I \sin\left(\frac{2\pi t}{T} + \alpha\right)$ *and* $e = E \sin\left(\frac{2\pi t}{T} + \beta\right)$ **[Ans. : $\frac{IE}{2} \cos(\alpha - \beta)$]**

16) *Find the R.M.S. value of an electric current given by the relation* $I = I_0 + I_1 \sin\left(\frac{2\pi t}{T} + \alpha_1\right) + I_2 \sin\left(\frac{4\pi t}{T} + \alpha_2\right)$ **(May - 2000, Dec. - 2005)**

17) *Find the mean value of the reciprocals of distances of points on a circular area of radius from a fixed point on it is circumference.* **(May - 2001)**

15.9 Centre of Gravity and Moment of Inertia

Centre of Gravity : Definition

The centroid or centre of mass or centre of Gravity (C. G.) of a body is defined as the point through which resultant weight of the body acts.

If $m_1, m_2 \ldots m_n$ are the point masses situated at the points $(x_1, y_1, z_1), (x_2, y_2, z_2) \ldots\ldots (x_n, y_n, z_n)$ respectively and $(\bar{x}, \bar{y}, \bar{z})$ are the co-ordinates of centre of gravity of the system, then

$$\boxed{\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}; \quad \bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}; \quad \bar{z} = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i}} \quad \ldots (1)$$

Where $m_1 + m_2 + \ldots + m_n = \sum_{i=1}^{n} m_i$ and $m_1 x_1 + m_2 x_2 + \ldots . m_n x_n = \sum_{i=1}^{n} m_i x_i$ .... etc.

$$\therefore \quad \int x\,ds = \int_0^{\pi} a\,(1 + \cos\theta)\,\cos\theta\;2a\,\cos\frac{\theta}{2}\,d\theta$$

$$= 2a^2 \int_0^{\pi} 2\,\cos^2\frac{\theta}{2}\,\cos\theta\,\cos\frac{\theta}{2}\,d\theta$$

$$= 4a^2 \int_0^{\pi} \cos^3\frac{\theta}{2}\left[2\cos^2\frac{\theta}{2} - 1\right] d\theta, \quad \text{Put } \frac{\theta}{2} = t$$

$$\therefore \quad \int x\,ds = 4a^2\left[4\int_0^{\pi/2} \cos^5 t\,dt - 2\int_0^{\pi/2} \cos^3 t\,dt\right]$$

$$\int x\,ds = 4a^2\left[4\,\frac{4\cdot 2\cdot 1}{5\cdot 3} - 2\cdot\frac{2}{3}\right]$$

$$= \frac{16\,a^2}{5}$$

$$\text{and} \quad \int ds = \int_0^{\pi} 2a\,\cos\frac{\theta}{2}\,d\theta$$

$$= 2a\cdot 2\left(\sin\frac{\theta}{2}\right)_0^{\pi}$$

$$= 4a$$

$$\therefore \quad \bar{x} = \frac{16a^2}{5}\cdot\frac{1}{4a} = \frac{4a}{5}$$

$$\text{Now,} \quad \int y\,ds = \int_0^{\pi} r\,\sin\theta\,ds$$

$$= \int_0^{\pi} a(1 + \cos\theta)\,\sin\theta\,ds$$

$$= \int_0^{\pi} a\,(1 + \cos\theta)\,\sin\theta\cdot 2a\,\cos\frac{\theta}{2}\,d\theta$$

$$= 2a^2 \int_0^{\pi} 2\,\cos^2\frac{\theta}{2}\;2\,\sin\frac{\theta}{2}\,\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\,d\theta$$

$$= 8a^2 \int_0^{\pi} \cos^4\frac{\theta}{2}\,\sin\frac{\theta}{2}\,d\theta$$

$$\therefore \quad \int y\,ds = 4a^2 \int_0^{\pi} \sin^3 t\ 2dt$$

$$= 16a^2 \int_0^{\pi/2} \sin^3 dt = 16a^2 \times \frac{2}{3}$$

$$= \frac{32a^2}{3}$$

and
$$ds = \int_0^{2\pi} 2a \sin\frac{\theta}{2}\, d\theta$$

$$= 2a\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi}$$

$$= 4a\,[1 + 1]$$

$$= 8a$$

$$\therefore \quad \bar{y} = \frac{\frac{32a^2}{3}}{8a} = \frac{4}{3}a$$

$\therefore$ Hence C.G. is $\boxed{(\bar{x}, \bar{y}) = \left(a\pi, \frac{4a}{3}\right)}$ .... Ans.

Example 15.43 : *If s is the length of the arc of the catenary $y = c\cosh\left(\frac{x}{c}\right)$ from the vertex to a point P (x, y). Prove that the C.G. of the arc is given by*
$$\bar{x} = x - \frac{c\,(y - c)}{s}, \quad \bar{y} = \frac{y}{2} + \frac{c\,x}{2\,s}$$

Solution : Let $(\bar{x}, \bar{y})$ be C.G., where

$$\bar{x} = \frac{\int x\,ds}{\int ds};\ \bar{y} = \frac{\int y\,ds}{\int ds}$$

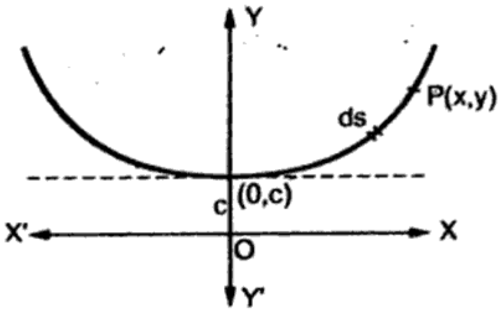


Fig. 15.40

$$y = c\cosh\left(\frac{x}{c}\right),$$

$$\frac{dy}{dx} = \sinh\left(\frac{x}{c}\right), \text{ so that}$$

$$dy = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$= \sqrt{1 + \sinh^2\frac{x}{c}}\ dx$$

$$= \cosh\left(\frac{x}{c}\right) dx$$

$$\therefore \quad \int x\,ds = \int_0^x x \cosh\left(\frac{x}{c}\right) dx = u\,v_1 - u'v_2 + u''v_3 + \ldots \quad \text{... (IBP)}$$

$$= \left[x \cdot c \sinh\left(\frac{x}{c}\right) - (1)\, c^2 \cosh\left(\frac{x}{c}\right)\right]_0^x$$

$$= cx \sinh\left(\frac{x}{c}\right) - c^2 \cosh\left(\frac{x}{c}\right) + c^2 \quad \text{... (a)}$$

and
$$\int y\,ds = \int_0^x c \cosh\left(\frac{x}{c}\right) \cdot \cosh\left(\frac{x}{c}\right) dx$$

$$= \frac{c}{2}\int_0^x \left[1 + \cosh\left(\frac{2x}{c}\right)\right] dx$$

$$= \frac{c}{2}\left[x + \frac{c}{2}\sinh\left(\frac{2x}{c}\right)\right]_0^x$$

$$= \frac{c}{2}\left[x + c \sinh\left(\frac{x}{c}\right) \cosh\left(\frac{x}{c}\right)\right] \quad \text{... (b)}$$

Also,
$$\int ds = \int_0^x \cosh\left(\frac{x}{c}\right) dx = \left[c \sinh\left(\frac{x}{c}\right)\right]_0^x = c \sinh\left(\frac{x}{c}\right) \quad \text{... (c)}$$

From equations (a), (b) and (c) we have,

$$\int x\,ds = xs - cy + c^2$$

$$\int y\,ds = \frac{c}{2}\left(x + s\,\frac{y}{c}\right)$$

$$= \frac{cx}{2} + \frac{sy}{2}$$

$$\therefore \quad \bar{x} = \frac{xs - cy + c^2}{s}$$

$$\boxed{\bar{x} = \frac{x - c(y - c)}{s}} \quad \text{.... Ans.}$$

and
$$\bar{y} = \frac{\frac{cx}{2} + \frac{sy}{2}}{s}$$

Now,
$$\bar{y} = \frac{\iint y\ dx\ dy}{\iint dx\ dy}$$

$$= \frac{\int_{-2}^{1}\int_{y^2}^{2-y} y\ dx\ dy}{9/2}$$

$$= \frac{2}{9}\int_{-2}^{1} y[(2-y)-y^2]\ dy$$

$$= \frac{2}{9}\int_{-2}^{1}(2y-y^2-y^3)\ dy$$

$$= \frac{2}{9}\left(y^2-\frac{y^3}{3}-\frac{y^4}{4}\right)_{-2}^{1}$$

$$= \frac{2}{9}\left[\left(1-\frac{1}{3}-\frac{1}{4}\right)-\left(4+\frac{8}{3}-\frac{16}{4}\right)\right]$$

$$= \frac{2}{9}\left(-\frac{27}{12}\right)=\frac{2}{9}\left(-\frac{9}{4}\right)$$

$$\bar{y} = -\frac{1}{2}$$

$\therefore$
$$\boxed{\bar{x} = \frac{8}{5},\ \bar{y} = -\frac{1}{2}}$$ .... Ans.

Example 15.46 : *Find the centre of Gravity of the area of the cardiode $r = (1+\cos\theta)$ which lies above the initial line.*

Solution : The C.G. is

$$\bar{x} = \frac{\iint x\ dx\ dy}{\iint dx\ dy},\ \bar{y} = \frac{\iint y\ dx\ dy}{\iint dx\ dy} \quad ... (1)$$

Using polar co-ordinates,

$$\iint x\ dx\ dy = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)}(r\cos\theta)\ r\,d\theta\,dr$$

$$= \int_{0}^{\pi}\cos\theta\left(\frac{r^3}{3}\right)_{0}^{a(1+\cos\theta)} d\theta$$

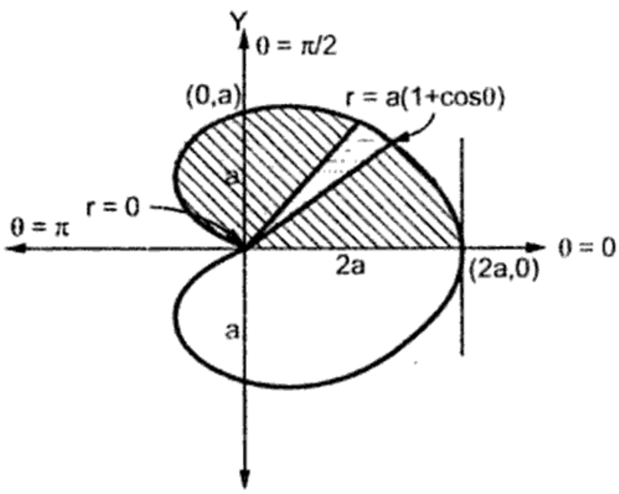


Fig. 15.43

$$= \int_0^{\pi} \left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2 (1+\cos\theta)^2 d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} \left(2\cos^2\frac{\theta}{2}\right)^2 d\theta$$

$$= 2a^2 \int_0^{\pi} \cos^4\frac{\theta}{2}\, d\theta, \quad \text{put } \frac{\theta}{2} = t, \quad d\theta = 2\,dt$$

$$= 2a^2 \int_0^{\pi/2} \cos^4 t\, 2\,dt$$

$$= 4a^2\, \frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot\frac{\pi}{2}$$

$$= \frac{3\pi a^2}{4} \qquad \dots (4)$$

$$\therefore \qquad \bar{x} = \frac{\frac{5a^3\pi}{8}}{\frac{3\pi a^2}{4}}, \quad \bar{y} = \frac{\frac{4a^3}{8}}{\frac{3\pi a^2}{4}}$$

or $\boxed{\bar{x} = \frac{5a}{6}, \quad \bar{y} = \frac{16a}{9\pi}}$ .... Ans.

Example 15.47 : *Find the centroid of the loop of the Laminscate $r^2 = a^2 \cos 2\theta$*

Solution : Since the loop is symmetrical about x - axis.

$$\therefore \qquad \bar{y} = 0$$

and
$$\bar{x} = \frac{\iint x\,dx\,dy}{\iint dx\,dy}$$

$$= \frac{\iint r\cos\theta\; r\,d\theta\,dr}{\iint r\,d\theta\,dr} = \frac{N}{D} \qquad \dots (1)$$

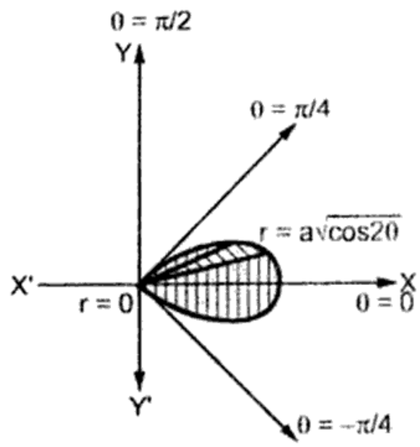


Fig. 15.44

For the loop, 'r' varies from **r = 0 to r = a** $\sqrt{\cos 2\theta}$,

'θ' varies from $\theta = \frac{-\pi}{4}$ to $\theta = \frac{\pi}{4}$

$$\therefore \quad N = \int_{\theta=-\frac{\pi}{4}}^{\pi/4} \int_{r=0}^{a\sqrt{\cos 2\theta}} r^2 \cos\theta \, d\theta \, dr$$

$$= \int_{\theta=-\frac{\pi}{4}}^{\pi/4} \cos\theta \left(\frac{r^3}{3}\right)_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \frac{2a^3}{3} \int_0^{\pi/4} \cos\theta \left(\sqrt{\cos 2\theta}\right)^3 d\theta$$

$$= \frac{2a^3}{3} \int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2} \cos\theta \, d\theta, \qquad \text{Put } \sqrt{2}\sin\theta = \sin t,$$

$$\sqrt{2}\cos\theta \, d\theta = \cos t \, dt$$

Limits :

θ	0	π/4
t	0	π/2

$$= \frac{2a^3}{3} \frac{1}{\sqrt{2}} \int_0^{\pi/2} \cos^3 t \cos t \, dt$$

$$= \frac{2a^3}{3\sqrt{2}} \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi a^3}{8\sqrt{2}} \qquad \dots (2)$$

$$D = \int_{-\pi/4}^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} r \, d\theta \, dr$$

$$= \int_{-\pi/4}^{\pi/4} \left(\frac{r^2}{2}\right)_0^{a\sqrt{\cos 2\theta}}$$

$$D = a^2 \int_0^{\pi/4} \cos 2\theta \, d\theta$$

$$= a^2 \left(\frac{\sin 2\theta}{2}\right)_0^{\pi/4}$$

$$= \frac{a^2}{2}[1 - 0] = \frac{a^2}{2} \qquad \dots (3)$$

$$= a^2\left(1 - \frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= a^2\left(\frac{4-\pi}{4}\right) \qquad \text{... (3)}$$

From equations (1) , (2) and (3)

$$x = \frac{a^3\,(3\pi - 8)}{12} \times \frac{4}{a^2\,(4-\pi)}$$

$$= \frac{a}{3}\,\frac{(3\pi - 8)}{(4-\pi)}$$

$\therefore$ C.G. is $\boxed{(x, y) = \left[\frac{a}{3}\,\frac{(3\pi-8)}{(4-\pi)}, 0\right]}$ .... Ans.

Example 15.50 : *Find the C.G. of one loop of r = sin* 2θ **(May-2001, May-2004)**

Solution : The curve r = a sin 2θ is four leaved rose lies within the circle r = a,

Consider a loop lies between θ = 0 to θ = π/2 and is symmetrical about the line θ = π/4 as shown in the Fig. 15.47.

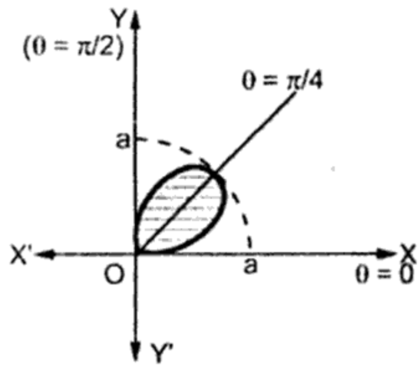


Fig. 15.47

$$\therefore \qquad \bar{x} = \bar{y},$$

$$\bar{x} = \frac{\iint x\,dx\,dy}{\iint dx\,dy} = \frac{N}{D} \qquad \text{... (1)}$$

Where

$$N = \iint x\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^{a\sin 2\theta} r\cos\theta\; r\,d\theta\,dr$$

$$= \int_0^{\pi/2} \cos\theta\,\frac{a^3\sin^3 2\theta}{3}\,d\theta$$

$$= \frac{8a^3}{3}\int_0^{\pi/2}\sin^3\theta\,\cos^4\theta\,d\theta$$

$$= \frac{8a^3}{3}\,\frac{[(3-1)]\cdot[(4-1)(4-3)]}{(7)(7-2)(7-4)(7-6)}\cdot 1$$

$$= \frac{16a^3}{105} \qquad \text{... (2)}$$

$$= \int_0^{2a} x \frac{x^{3/2}}{\sqrt{2a - x}} \, dx$$

$$N = \int_0^{2a} x \frac{x^{3/2}}{\sqrt{2a - x}} \, dx$$

Put $x = 2a \sin^2\theta$,

$dx = 4a \sin\theta \cos\theta \, d\theta$

Limits :

x	0	2a
θ	0	π/2

$$\therefore \quad N = \int_0^{\pi/2} \frac{2a \sin^2\theta \cdot (2a \sin^2\theta)^{3/2}}{\sqrt{2a - 2a \sin^2\theta}} \, 4a \sin\theta \cos\theta \, d\theta$$

$$= \frac{(2a)^{5/2}}{(2a)^{1/2}} \int_0^{\pi/2} \frac{\sin^5\theta \; 4a \sin\theta \cos\theta}{\cos\theta} \, d\theta$$

$$= (2a)^2 (4a) \int_0^{\pi/2} \sin^6\theta \, d\theta$$

$$= 16a^3 \, \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{5\pi a^3}{2} \qquad \text{... (2)}$$

and

$$D = \int_0^{2a} \int_0^{y} dx \, dy$$

$$= \int_0^{2a} y \, dx$$

$$= \int_0^{2a} \frac{x^{3/2}}{\sqrt{2a - x}} \, dx \qquad \text{... Put } x = 2a \sin^2\theta$$

$$= 8a^2 \int_0^{\pi/2} \sin^4\theta \, d\theta$$

$$= 8a^2 \, \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{3\pi a^2}{2} \qquad \text{... (3)}$$

$$= \int_0^{\pi} \sin\theta \left(\frac{r^3}{3}\right)_{a\sin\theta}^{2a\sin\theta} d\theta$$

$$= \frac{1}{3}\int_0^{\pi} \sin\theta\,[(2a\sin\theta)^3 - (a\sin\theta)^3]\,d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi} 7\sin^4\theta\,d\theta$$

$$= \frac{14a^2}{3}\int_0^{\pi/2} \sin^4\theta\,d\theta$$

$$= \frac{14a^3}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{7\pi a^3}{8} \qquad \text{... (2)}$$

and $\qquad D = \iint dx\,dy$

$$= \int_0^{\pi}\int_{a\sin\theta}^{2a\sin\theta} r\,d\theta\,dr$$

$$= \int_0^{\pi} \left(\frac{r^2}{2}\right)_{a\sin\theta}^{2a\sin\theta} d\theta$$

$$= \frac{3a^2}{2}\int_0^{\pi} \sin^2\theta\,d\theta$$

$$= \frac{3a^2}{2}\,2\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= 3a^2\cdot\frac{1}{2}\cdot\frac{\pi}{2} \quad \textbf{OR} \quad \text{Area} = \pi a^2 - \frac{\pi a^2}{4} = \frac{3\pi a^2}{4}$$

$$= \frac{3\pi a^2}{4} \qquad \text{... (3)}$$

From equations (1), (2) and (3)

$$\bar{y} = \frac{7\pi a^3}{8}\cdot\frac{4}{3\pi a^2}$$

$$\bar{y} = \frac{7a}{6}$$

$\therefore$ The C.G. is $\boxed{\left(0, \frac{7a}{6}\right)}$ Ans.

Example 15.53 : *Find the C.G. of the area which lies above the initial line bounded by the line $\theta = \pi/2$, the circle $r = 2a \cos \theta$ and the cardiode. $r = a(1 + \cos \theta)$*

Solution : Let the C.G. is

$$\bar{x} = \frac{\iint x \, dx \, dy}{\iint dx \, dy},$$

$$\bar{y} = \frac{\iint y \, dx \, dy}{\iint dx \, dy}$$

Area bounded by $\theta = \pi/2$, $r = 2a \cos \theta$, $r = a(1 + \cos \theta)$ is as shown in the Fig. 15.50.

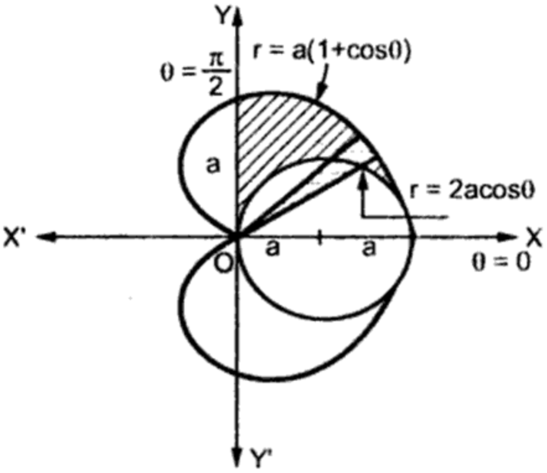


Fig. 15.50

Using polar co-ordinates

$$\iint x \, dx \, dy = \int_0^{\pi/2} \int_{2a\cos\theta}^{a(1+\cos\theta)} (r \cos \theta) \, r \, d\theta \, dr$$

$$= \int_0^{\pi/2} \left(\frac{r^3}{3}\right)_{2a\cos\theta}^{a(1+\cos\theta)} \cos\theta \, d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} [a^3 (1 + \cos\theta)^3 - (2a)^3 \cos^3\theta] \cos\theta \, d\theta$$

$$= \frac{1}{3} a^3 \int_0^{\pi/2} \left(\cos\theta + 3\cos^2\theta + 3\cos^3\theta - 7\cos^4\theta\right) d\theta$$

$$= \frac{a^3}{3} \left[1 + 3 \cdot \frac{1}{2} \cdot \frac{\pi}{2} + 3 \frac{2}{3} \cdot 1 - 7 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right]$$

$$= \frac{a^3}{16} (16 - 3\pi) \qquad \text{... (1)}$$

$$= \frac{a^2}{2}\left[(\theta + 2\sin\theta)_0^{\pi/2} - 3\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= a^2\left(\frac{8-\pi}{8}\right) \quad \text{... (3)}$$

From equations (1), (2) and (3),

$$\bar{x} = \frac{a^3}{16}(16-3\pi) \times \frac{8}{a^2(8-\pi)}$$

$$\boxed{\bar{x} = \frac{16-3\pi}{(8-\pi)}\cdot\frac{a}{2}} \quad \text{Ans.}$$

and $$\bar{y} = \frac{7a^3}{12} \times \frac{8}{(8-\pi)\,a^2}$$

$$\boxed{\bar{y} = \frac{14\,a}{3(8-\pi)}} \quad \text{Ans.}$$

Example 15.54 : ***The density at a point on a circular limina varies as the distance from a point O on the circumference. Show that the centre of gravity divides the diameter through the ratio 3 : 2.***

Solution : Let pole O is fixed point on the circumference of circle of radius 'a' and OA is diameter (initial line). By symmetry C.G. of the lamina lies on OA $\therefore y = 0$.

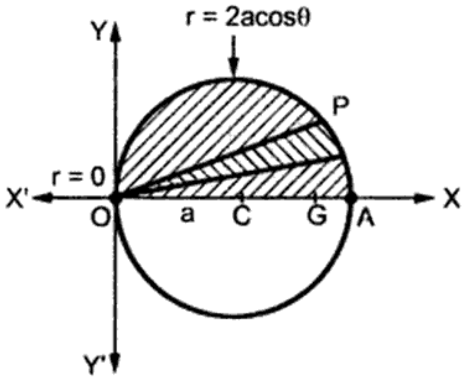


Fig. 15.51

and $$\bar{x} = \frac{\iint x\rho\, dx\, dy}{\iint \rho\, dx\, dy} \quad \text{... (1)}$$

The equation of circle is

$$(x-a)^2 + y^2 = a^2 \Rightarrow x^2 + y^2 = 2\,ax \Rightarrow r = 2\,a\cos\theta$$

and density, $\rho \propto r$ i.e. $\rho = kr$

From equations (2), (3) and (4)

$$\bar{x} = \frac{32a^4}{15} \times \frac{9}{16a^3}$$

$$= \frac{6a}{5}$$

$\therefore$ C.G. is $\boxed{(\bar{x}, \bar{y}) = \left(\frac{6a}{5}, 0\right)}$ Ans.

Now, $OA = 2a, \quad OG = \frac{6a}{5}$

$\therefore \quad GA = OA - OG$

$$= 2a - \frac{6a}{5}$$

$$= \frac{4a}{5}$$

$$\therefore \quad \frac{OG}{GA} = \frac{6a}{5} \cdot \frac{5}{4a} = \frac{3}{2}$$

$\therefore$ G divides OA in the ratio 3 : 2 .... Ans.

III) On C.G. of solid :

Example 15.55 : *Find the centroid of the region in the first octant bounded by* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \ (a > 0, b > 0, c > 0)$

Solution : Let the centroid be $(\bar{x}, \bar{y}, \bar{z})$

Where, $$\bar{x} = \frac{\iiint x \, dx \, dy \, dz}{\iiint dx \, dy \, dz},$$

$$\bar{y} = \frac{\iiint y \, dx \, dy \, dz}{\iiint dx \, dy \, dz}$$

and $$\bar{z} = \frac{\iiint z \, dx \, dy \, dz}{\iiint dx \, dy \, dz} \qquad \text{... (1)}$$

Put $x = au, \ y = bv, \ z = cw, \ \rho = \text{constant}$

$\therefore \quad dx \, dy \, dz = abc \, du \, dv \, dw$ and $u + v + w = 1$

$$\therefore \quad \iiint x \, dx \, dy \, dz = \iiint au \, du \, dv \, dw$$

$$= a^2bc \iiint u^{2-1}v^{1-1}w^{1-1}du\ dv\ dw$$

$$= a^2bc \frac{\lceil 2 \lceil 1 \lceil 1}{\lceil 1+2+1+1}$$

$$= \frac{a^2bc}{4\,!}$$

$$= \frac{a^2bc}{24} \quad \text{(Using Dirichlet's theorem)}$$

Similarly $\iiint y\ dx\ dy\ dz = \frac{ab^2c}{24}$

And $\iiint z\ dx\ dy\ dz = \frac{abc^2}{24}$

Also $\iiint dx\ dy\ dz =$ Volume of the tetrahydron

$$= \frac{abc}{6}$$

$$\bar{x} = \frac{\frac{a^2bc}{24}}{\frac{abc}{6}} = \frac{a}{4}$$

Similarly $\bar{y} = \frac{b}{4}$ and $\bar{z} = \frac{c}{4}$

$\therefore$ $\boxed{\bar{x} = \frac{a}{4},\ \bar{y} = \frac{b}{4},\ \bar{z} = \frac{c}{4}}$ ...Ans

Example 15.56 : *A solid is in the form of positive octant of the sphere $x^2 + y^2 + z^2 = a^2$. The density at any point is given by $\rho = k\,x\,y\,z$. Where k is constant. Find the Co-ordinates of the centroid of solid.*

Solution : We have

$$\bar{x} = \frac{\iiint \rho\ x\ dx\ dy\ dz}{\iiint \rho\ dx\ dy\ dz}$$

$$= \frac{\iiint k\,x\,y\,z \cdot x \cdot dx\ dy\ dz}{\iiint k\,x\,y\,z\ dx\ dy\ dz} = \frac{N}{D} \qquad \text{... (1)}$$

Where $N = \iiint k\,x^2yz\ dx\ dy\ dz$

$$= \frac{1}{2}\left(\frac{16r^6}{2} - 8\frac{r^4}{4} + \frac{r^6}{6}\right)_0^2 \cdot 2\pi$$

$$= \pi\left(32 - 32 + \frac{64}{6}\right)$$

$$= \frac{32\pi}{3} \qquad \text{... (3)}$$

and $D = \iiint dx\, dy\, dz$

$$= \iint \int_0^{4-x^2-y^2} dx\, dy\, dz$$

$$= \int_{\theta=0}^{2\pi}\int_0^2 (4-r^2)\, r\, d\theta\, dr$$

$$= 2\pi\left(\frac{4r^2}{2} - \frac{r^4}{4}\right)_0^2$$

$$= 2\pi(8-4) = 8\pi \qquad \text{... (4)}$$

From equations (1), (3) and (4)

$$\therefore \quad \bar{z} = \frac{\frac{32\pi}{3}}{8\pi}$$

$$\boxed{\bar{z} = \frac{4}{3}}$$

$\therefore$ C.G. is $\boxed{\left(0, 0, \frac{4}{3}\right)}$ ... Ans.

Example 15.58 : ***If the density of a hemisphere varies as the distance from the bounding plane. Show that the distance of the centre of mass from that plane is $\left(\frac{8}{15}\right)^{th}$ the radius of hemisphere.***

Solution : Let $x^2 + y^2 + z^2 = a^2$ be hemisphere above the XOY plane. and $\rho = k, z$ (k is constant)

$$\therefore \quad \bar{z} = \frac{\iiint \rho\, z\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz} \qquad \text{... (1)}$$

Transform the integrals into spherical polar co-ordinates by putting,

$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \; z = r\cos\theta$

$dx\, dy\, dz = r^2 \sin\theta\, d\theta\, d\phi\, dr$, Now for hemisphere :

r varies from **r = 0 to r = a**

θ varies from **θ = 0 to θ = π/2**

φ varies from **φ = 0 to φ = 2π**

From equation (1)

$$\bar{z} = \frac{\iiint kz^2 dx\, dy\, dz}{\iiint kz\, dx\, dy\, dz}$$

Where $$N = \iiint kz^2 dx\, dy\, dz$$

$$= k \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2 \cos^2\theta\ r^2 \sin\theta\ d\theta\ d\phi\ dr$$

$$= k \int_0^{2\pi} d\phi \cdot \int_0^{\pi/2} \cos^2\theta\ \sin\theta\ d\theta \int_0^a r^4 dr$$

$$= k\,(2\pi) \cdot \left(-\frac{\cos^3\theta}{3}\right)_0^{\pi/2} \cdot \left(\frac{r^5}{5}\right)_0^a$$

$$= \frac{2\,k\,a^5\pi}{15} \qquad \text{... (3)}$$

And $$D = \iiint k\, z\, dx\, dy\, dz$$

$$= k \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r\cos\theta\ r^2 \sin\theta\ d\theta\ d\phi\ dr$$

$$= k \int_0^{2\pi} d\phi \cdot \int_0^{\pi/2} \sin\theta\ \cos\theta\ d\theta \cdot \int_0^a r^3\ dr$$

$$= k(2\pi) \left(-\frac{\cos^2\theta}{2}\right)_0^{\pi/2} \cdot \left(\frac{r^4}{4}\right)_0^a$$

$$= \frac{ka^4\pi}{4} \qquad \text{... (4)}$$

From equations (2), (3) and (4)

$$\bar{z} = \frac{2ka^5\pi}{15} \cdot \frac{4}{ka^4\pi}$$

$$\boxed{\bar{z} = \frac{8a}{15}} \qquad \text{.... Ans.}$$

Excercise 15.5

C.G. of an arc or curve :

1) *Find the C.G. of an arc of the cycloid* $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ *measured from cusp to cusp.* **[Ans.** $\left(0, \frac{2a}{3}\right)$**]**

2) *Show that the centroid of a wire bent in the form of cardiode* $r = a(1 + \cos\theta)$ *and with line density* $k \cdot \sec\frac{\theta}{2}$ *(k being constant) is on the axis of cardiode at a distance* $\frac{a}{2}$ *from the cusp.*

3) *Find the C.G. of an arc of the catenary* $y = a\cosh\frac{x}{a}$ *from* $x = -a$ *to* $x = a$ **[Ans.** $\left(0, \frac{2\sinh 2}{4\sinh 1} \cdot a\right)$**]**

4) *If 0 is pole of the laminscate* $r^2 = a^2\cos 2\theta$ *and G is the Centre of Gravity of any arc PQ of the curve prove that OG biseets angle POQ.*

C.G. of plane area or lamina

1) *Find the C.G. of the lamina bounded by* $y = 2 - 3x^2$ *and the line* $3x + 2y = 1$. **(Dec.-99, May-99)** **[Ans. :** $\left(\frac{1}{4}, \frac{40}{27}\right)$**]**

2) *Find the C.G. of area enclosed in the positive quadrant of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **[Ans. :** $\left(\frac{4a}{3\pi}, \frac{4b}{3\pi}\right)$**]**

3) *ABCD is a square plate of side a and O is midpoint of AB. If the surface density varies as the square of the distance from O. Show that the C.G. of the plate is at a distance* $\frac{7a}{10}$ *from AB.*

4) *Find the co-ordinates of the C.G. of the area in the first quadrant bounded by the curve* $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$, *the density being given by* $\rho = k\,x\,y$ *(where k is a constant).* **[Ans. :** $\left(\frac{128\,a}{429}, \frac{128\,b}{429}\right)$**]**

5) *Find the C.G. of the plate cut from the parabola* $y^2 = 8x$ *by it's Latus rectum* $x = 2$, *if the density is numerically equal the distance from the Latus rectum.* **[Ans. :** $\left(\frac{6}{7}, 0\right)$**]**

6) *Find the X co-ordinate of the C.G. of the area bounded by the parabola* $y^2 = 4x$ *and the line* $2x - y - 4 = 0$. **[Ans. :** $\left(x = \frac{8}{5}\right)$**]**

7) *Find the C.G. of the area enclosed between the curves* $y^2 = ax$ *and* $x^2 + y^2 = 2ax$. **[Ans. :** $\left(a\frac{15\pi - 44}{15\pi - 40}, 0\right)$**]**

8) *Find the centroid of the smaller segment of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *cut off by the straight line* $\frac{x}{a} + \frac{y}{b} = 1$. **[Ans. :** $\left(\frac{2a}{3(\pi - 2)}, \frac{2b}{3(\pi - 2)}\right)$**]**

$$= \frac{\rho}{4} 2^4 \int_{\pi/4}^{\pi/2} \frac{\cos^4 \theta}{\sin^8 \theta} d\theta$$

$$= 4\rho \int_{\pi/4}^{\pi/2} \cot^4 \theta \operatorname{cosec}^4 \theta \, d\theta$$

$$= 4\rho \int_{\pi/4}^{\pi/2} \cot^4 \theta \left(1 + \cot^2 \theta\right) \operatorname{cosec}^2 \theta \, d\theta$$

Put $\cot \theta = t$, $-\operatorname{cosec}^2 \theta \, d\theta = dt$

θ	$\pi/4$	$\pi/2$
t	1	0

$$\text{M.I.} = 4\rho \int_{1}^{0} \left(t^4 + t^6\right) (-dt)$$

$$= -4\rho \left(\frac{t^5}{5} + \frac{t^7}{7}\right)_1^0$$

$$= 4 \frac{12}{35} \rho$$

$$= \frac{48}{35} \rho$$

Mass of the area is

$$M = \int_{\pi/4}^{\pi/2} \int_{0}^{\frac{2\cos\theta}{\sin^2\theta}} \rho \, r \, d\theta \, dr$$

$$= \rho \int_{\pi/4}^{\pi/2} \left(\frac{r^2}{2}\right)_0^{\frac{2\cos\theta}{\sin^2\theta}} d\theta$$

$$= \rho \int_{\pi/4}^{\pi/2} \frac{4 \cos^2\theta}{2 \sin^4\theta} d\theta$$

$$= 2\rho \int_{\pi/4}^{\pi/2} \cot^2 \theta \operatorname{cosec}^2\theta \, d\theta$$

$$= 2\rho \left\{\frac{-\cot^3 \theta}{3}\right\}_{\pi/4}^{\pi/2}$$

Now, $\quad \text{Mass} = M = \int \rho \, ds$

$$= \int_{-\alpha}^{\alpha} \rho \, a \, d\theta$$

$$= \rho \, a \, (\theta)_{-\alpha}^{\alpha}$$

$$M = \rho \, a(2\alpha)$$

$$\therefore \quad \rho = \frac{M}{2a\alpha} \qquad \text{... (3)}$$

From equation (2) and equation (3)

$$\text{M.I.} = \frac{M}{2\,a\alpha} \cdot 4a^3(\alpha - \sin\alpha)$$

$$\boxed{\text{M.I.} = 2a^2 M\left(1 - \frac{\sin\alpha}{\alpha}\right)} \qquad \text{... Ans}$$

ii) To find M.I. about radius bisecting angles 2α i.e. M.I. about OX line is

$$\text{M.I.} = \int_{-\alpha}^{\alpha} y^2 \rho \, ds \qquad (\because p = y)$$

$$= 2\,a\rho \int_0^{\alpha} a^2 \sin^2\theta \, d\theta$$

$$= \frac{2a^3}{2} \rho \int_0^{\alpha} (1 - \cos 2\theta) \, d\theta$$

$$= a^3 \rho \left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\alpha}$$

$$= \rho \, a^3 \left(\alpha - \frac{\sin 2\alpha}{2}\right)$$

But $\quad \rho = \dfrac{M}{2a\alpha}$

$$\therefore \quad \text{M.I.} = \frac{M}{2a\alpha} a^3 \left(\alpha - \frac{\sin 2\alpha}{2}\right)$$

$$\boxed{\text{M.I.} = \frac{Ma^2}{2}\left(1 - \frac{\sin 2\alpha}{2\alpha}\right)} \qquad \text{... Ans}$$

Where ρ = density and A is area bounded by $x^2 = y$ and $y = x + 2$ as shown in the Fig. 15.61.

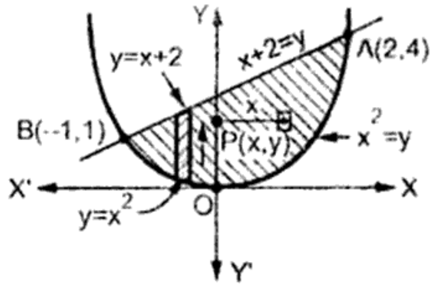


Fig. 15.61

$$\text{M.I.} = \rho \int_{-1}^{2} \int_{x^2}^{x+2} x^2 \, dx dy$$

$$(\because \rho = x)$$

$$= \rho \int_{-1}^{2} x^2 \, (y)_{x^2}^{x+2} \, dx$$

$$= \rho \int_{-1}^{2} x^2 \left[(x+2) - x^2\right] dx$$

$$= \rho \left(\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^5}{5}\right)_{-1}^{2}$$

$$\text{M.I.} = \rho \cdot \frac{63}{20}$$

Now, $\quad \text{Mass} = M = \iint \rho \, dx \, dy$

$$= \rho \int_{-1}^{2} \int_{x^2}^{x+2} dx \, dy$$

$$= \rho \int_{-1}^{2} \left[(x+2) - x^2\right] dx$$

$$= \rho \left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)_{-1}^{2}$$

$$= \rho \left(\frac{9}{2}\right)$$

or $\quad M = \frac{9}{2} \rho$

$\therefore \quad \rho = \frac{2M}{9}$

$\therefore \quad \text{M.I.} = \frac{63}{20} \cdot \frac{2M}{9}$

$$\boxed{\text{M.I.} = \frac{7M}{10}} \quad \text{... Ans}$$

Where A is area of the loop as shown in the Fig. 15.64.

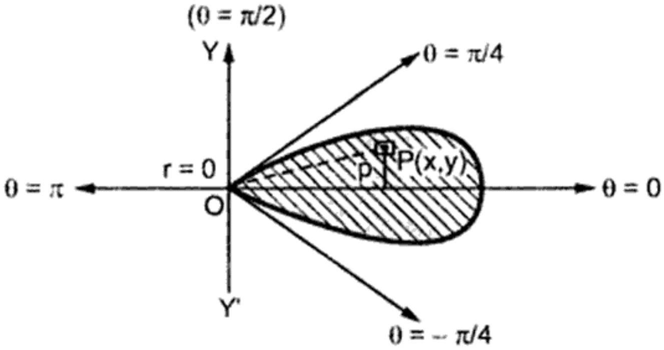


Fig. 15.64

From equation (1)

$$\text{M.I.} = \rho \int_{-\pi/4}^{\pi/4} \int_{0}^{a\sqrt{\cos 2\theta}} r^3 \sin^2\theta \; d\theta \; dr$$

$$= 2\rho \int_{0}^{\pi/4} \sin^2\theta \left(\int_{0}^{a\sqrt{\cos 2\theta}} r^3 \, dr \right) d\theta$$

$$= 2\rho \int_{0}^{\pi/4} \sin^2\theta \left(\frac{r^4}{4} \right)_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2\rho \frac{a^4}{4} \int_{0}^{\pi/4} \sin^2\theta \; \cos^2 2\theta \; d\theta$$

$$= \frac{\rho a^4}{2} \int_{0}^{\pi/4} \cos^2 2\theta \frac{(1 - \cos 2\theta)}{2} d\theta \qquad \text{Put } 2\theta = t$$

$$= \frac{\rho a^4}{4} \int_{0}^{\pi/2} \cos^2 t \,(1 - \cos t) \frac{dt}{2}$$

$$= \frac{\rho a^4}{4} \left[\frac{1}{2} \frac{\pi}{2} - \frac{2}{3} \right]$$

$$= \frac{\rho a^4}{96} (3\pi - 8)$$

Now, mass of the loop is

$$M = \iint \rho \; r \; d\theta \; dr$$

$$= \rho \int_{-\pi/4}^{\pi/4} \int_{0}^{a\sqrt{\cos 2\theta}} r \; d\theta \; dr$$

$$= 2\rho \int_0^{\pi/4} \left(\int_0^{a\sqrt{\cos 2\theta}} r\, dr \right) d\theta = 2\rho \int_0^{\pi/4} \left(\frac{r^2}{2} \right)_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \rho \int_0^{\pi/4} a^2 \cos 2\theta \; d\theta$$

$$= \rho\, a^2 \left(\frac{\sin 2\theta}{2} \right)_0^{\pi/4}$$

$$= \frac{\rho\, a^2}{2} (1 - 0)$$

$$M = \frac{\rho\, a^2}{2}$$

$$\therefore \quad \rho = \frac{2M}{a^2}$$

But

$$\text{M.I.} = \rho \frac{a^4}{96} (3\pi - 8)$$

$$= \frac{2M}{a^2} \frac{a^4}{96} (3\pi - 8)$$

$$\boxed{\text{M.I.} = \frac{(3\pi - 8)}{48} Ma^2} \qquad \text{... Ans}$$

Example 15.68 : *Find the M.I. of the area of upper half of the circle $x^2 + y^2 = a^2$ about the line $x + y = 2a$.* **(May-98)**

Solution :

$$\text{M.I.} = \iint_A p^2 \rho \; dx\, dy \qquad \text{... (1)}$$

where A is area of the circle $x^2 + y^2 = a^2$ as shown in the Fig. 15.65 and p = perpendicular distance of any point p(x, y) from the line x + y = 2a.

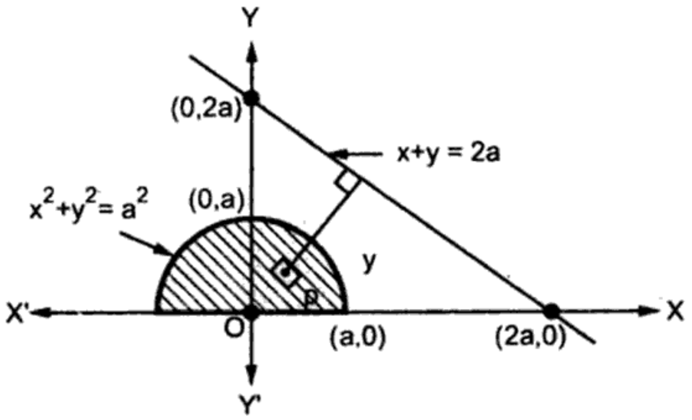


Fig. 15.65

$$\therefore \qquad p = \left| \frac{x + y - 2a}{\sqrt{2}} \right|$$

From equation (1),

$$\text{M.I.} = \rho \iint \frac{(x + y - 2a)^2}{2} \, dx\, dy$$

$$= \frac{\rho}{2} \iint \left[x^2 + y^2 + 2xy - 4a(x + y) + 4a^2\right] dxdy$$

Transforming into polar

$$= \frac{\rho}{2} \int_{\theta=0}^{\pi} \int_{r=0}^{a} \left[r^2 + 2r^2 \sin\theta \cos\theta - 4ar(\cos\theta + \sin\theta) + 4a^2\right] r \, d\theta \, dr$$

$$= \frac{\rho}{2} \int_0^{\pi} \left(\frac{a^4}{4} + \frac{a^4}{2} \sin\theta \cos\theta - \frac{4a^4}{3} (\cos\theta + \sin\theta) + 2a^4 \right) d\theta$$

$$= \frac{\rho}{2} \left[\frac{9}{4} a^4 (\pi) - \frac{4a^4}{3} (0 + 2) \right]$$

$$\boxed{\text{M.I.} = \rho\, a^4 \left(\frac{9\pi}{8} - \frac{4}{3} \right)} \qquad \text{... Ans}$$

Example 15.69 : *Find the M.I. of a semicircle about the line joining one end of the bounding diameter to the midpoint of the arc and show that radius of gyration* $k = a\sqrt{\left(\frac{3}{4} - \frac{3}{4\pi}\right)}$.

Solution : Let one end of semicircle be origin, the bounding diameter OA as the X-axis , the perpendicular to it through O be the Y-axis. The equation of circle of radius 'a' is

$$(x - a)^2 + y^2 = a^2$$

or $x^2 + y^2 = 2ax \Rightarrow r = 2a\cos\theta$

If B is midpoint of the arc then equation of OB is $y = x$.

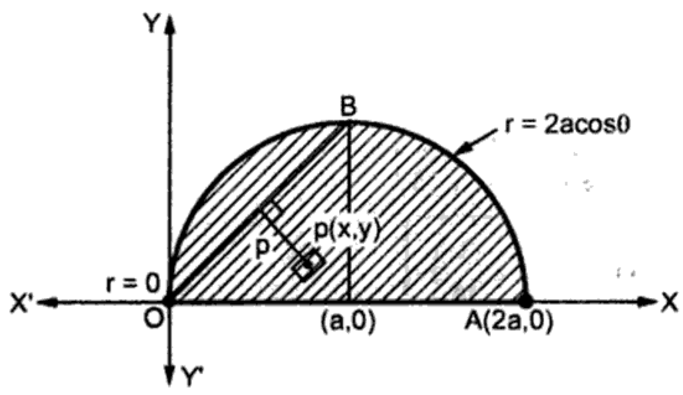


Fig. 15.66

Let p (x y) be any point of the semicircle then length of perpendicular from the point on the line OB is

$$p = \left|\frac{x-y}{\sqrt{2}}\right|$$

$$\therefore \quad \text{M.I.} = \iint \rho\, p^2\, dx\, dy$$

$$= \rho \iint \frac{(x-y)^2}{2}\, dx\, dy$$

$$= \frac{\rho}{2}\iint (x^2 + y^2 - 2xy)\, dx\, dy$$

Changing to polar co-ordinates

$$\text{M.I.} = \frac{\rho}{2}\int_0^{\pi/2}\int_0^{2a\cos\theta} \left(r^2 - 2r^2 \sin\theta\cos\theta\right) r\, d\theta\, dr$$

$$= \frac{\rho}{2}\int_0^{\pi/2} (1 - 2\sin\theta\cos\theta)\left(\frac{r^4}{4}\right)_0^{2a\cos\theta} d\theta$$

$$= \frac{\rho}{8}\int_0^{\pi/2} (1 - 2\sin\theta\cos\theta)\,(2a\cos\theta)^4\, d\theta$$

$$= 2a^4\rho \int_0^{\pi/2} \left(\cos^4\theta - 2\cos^5\theta\sin\theta\right) d\theta$$

$$= 2a^4\rho\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - 2\cdot\frac{1}{6}\right]$$

$$\text{M.I.} = a^4\rho\left[\frac{3\pi}{8} - \frac{2}{3}\right]$$

Since area of the semicircle is $\frac{\pi a^2}{2}$,

$$\therefore \quad \text{Mass} = \rho\frac{\pi a^2}{2} \qquad \therefore \rho = \frac{2M}{\pi a^2}$$

$$\therefore \quad \text{M.I.} = a^4\frac{2M}{\pi a^2}\left(\frac{3\pi}{8} - \frac{2}{3}\right)$$

$$\boxed{\text{M.I.} = Ma^4\left(\frac{3}{4} - \frac{3}{3\pi}\right)} \qquad \text{... Ans}$$

$$= \rho \int_0^c (y)_0^{c\cosh\left(\frac{x}{c}\right)} dx$$

$$= \rho \int_0^c c\cosh\left(\frac{x}{c}\right) dx$$

$$= \rho c^2 \left[\sinh\left(\frac{x}{c}\right)\right]_0^c$$

$$M = \frac{\rho c^2}{2}\left[e - \frac{1}{e}\right]$$

$$= \frac{\rho c^2}{2e}(e^2 - 1)$$

Now, the radius of gyration is

$$k^2 = \frac{\text{M.I.}}{\text{Mass}} = \rho \frac{c^4}{2e}(e^2 - 5) \frac{2e}{\rho c^2 (e^2 - 1)}$$

$$= \frac{c^2 (e^2 - 5)}{e^2 - 1}$$

or $$\boxed{k = \frac{c\sqrt{e^2 - 5}}{\sqrt{e^2 - 1}}}$$ ... Ans

Example 15.71 : *Find the M.I. about the X-axis of the area enclosed by the lines $x = 0, \frac{x}{a} + \frac{y}{b} = 1$.*

Solution :

$$\text{M.I.} = \iint_A \rho P^2 \, dx\, dy$$

$$\text{M.I.} = \iint_A \rho y^2 \, dx\, dy$$

$$(\because P = y) \quad \dots (1)$$

where A is area as shown in the Fig. 15.68.

Consider small area dxdy at a distance y from the X-axis.

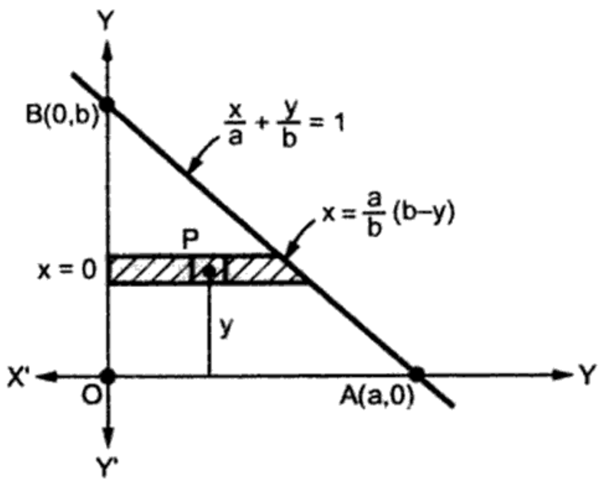


Fig. 15.68

From equation (1),

$$\text{M.I.} = \rho \int_{y=0}^{b} \int_0^{\frac{a}{b}(b-y)} y^2 \, dx\, dy$$

$$= \rho \int_0^b y^2 (x)_0^{\frac{a}{b}(b-y)} dy$$

$$= \rho \int_0^b y^2 \frac{a(b-y)}{b} dy$$

$$= \frac{\rho a}{b} \int_0^b (by^2 - y^3) dy$$

$$= \frac{\rho a}{b} \left(b\frac{y^3}{3} - \frac{y^4}{4} \right)_0^b$$

$$\text{M.I.} = \frac{\rho ab^3}{12}$$

Now, mass M of the area is

$$M = \rho \times \text{area of the } \Delta \text{ OAB}$$

$$M = \rho \frac{ab}{2}$$

$$\therefore \quad \rho = \frac{2M}{ab}$$

$$\therefore \quad \text{M.I.} = \frac{ab^3}{12} \times \frac{2M}{ab}$$

$$\boxed{\text{M.I.} = \frac{b^2 M}{6}} \quad \text{... Ans}$$

Example 15.72 : *The mass of a solid right circular cylinder of radius 'a' and height 'h' is M. Find the M.I. of the cylinder about (i) it's axis. ii) a line through it's C.G. perpendicular to it's axis. iii) any diameter through it's base.* (May-2000)

Solution : i) Consider a small disc of thickness dx, it's radius is 'a'. If ρ is density then mass M is

$$M = \rho \pi a^2 dx$$

Now, the M.I. of this elementary disc of mass M about the axis OX is by the result obtained as

$$\frac{Ma^2}{2} = \rho \frac{\pi a^4}{2} dx$$

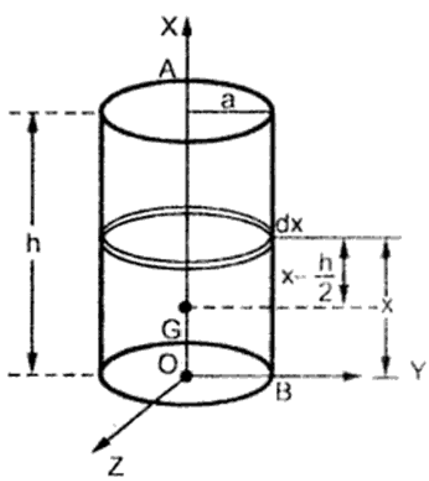


Fig. 15.69

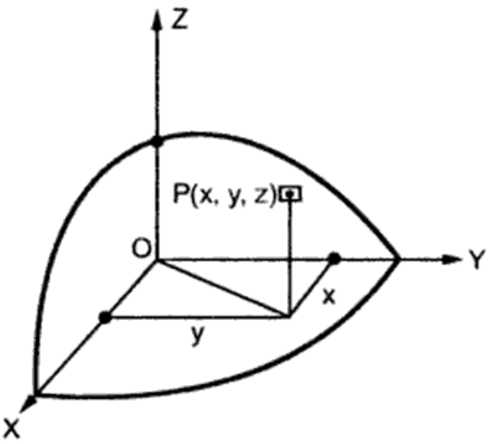


Fig. 15.71

Using spherical polar co-ordinates;

$x = a\ r \sin\theta \cos\phi$, $y = b\ r \sin\theta \sin\phi$, $z = c\ r \cos\theta$, $dx\ dy\ dz = a\ b\ c\ r^2 \sin\theta\ d\theta\ d\phi\ dr$

$$\text{M.I.} = \rho \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{1} \left(b^2\sin^2\theta \sin^2\phi + c^2\cos^2\theta\right) r^2\ abc\ r^2 \sin\theta\ d\theta\ d\phi\ dr$$

$$= \rho \int_0^{\pi/2} \int_0^{\pi/2} \left(b^2\sin^2\theta \sin^2\phi + c^2\cos^2\theta\right) abc \sin\theta\ d\theta \left(\frac{1}{5}\right)\ d\phi$$

$$= \frac{\rho\ abc}{5} \int_0^{\pi/2} \sin\theta \left[\int_0^{\pi/2} \left(b^2\sin^2\theta \sin^2\phi + c^2\cos^2\theta\right)\ d\phi\right] d\theta$$

$$= \frac{\rho\ abc}{5} \int_0^{\pi/2} \sin\theta \left\{b^2\sin^2\theta \frac{1}{2}\frac{\pi}{2} + c^2\cos^2\theta \cdot \frac{\pi}{2}\right\} d\phi$$

$$= \frac{\rho\ abc}{10} \int_0^{\pi/2} \left(\frac{b^2}{2} \sin^3\theta + c^2\cos^2\theta \sin\theta\right) d\theta$$

$$= \rho \frac{abc\pi}{10} \left[\frac{b^2}{2} \cdot \frac{2}{3} + \frac{c^2}{3}\right]$$

$$= \frac{\rho\ abc\pi}{10 \times 3}\left(b^2 + c^2\right)$$

$$\text{M.I.} = \frac{\rho\ abc\pi}{30}\left(b^2 + c^2\right)$$

As the volume of the ellipsoid is $\frac{4}{3}\pi abc$,

$\therefore$ Mass M of the octant is

$$M = \frac{1}{8}\frac{4}{3}\pi\, abc\, \rho$$

$$\therefore \quad \rho = \frac{6M}{\pi\, abc}$$

$$\therefore \quad M.I. = \frac{6M}{\pi\, abc}\cdot\frac{abc\pi}{30}\left(b^2 + c^2\right)$$

$$\boxed{M.I. = \frac{M\left(b^2 + c^2\right)}{5}} \quad \text{... Ans}$$

Example 15.76 : *Find the M.I. of a cone of base radius 'a' and height 'h' about it's axis.*

Solution : Consider a small disc of cone of thickness dy at a distance y from the vertex. Let radius of the disc be 'x'. If 'ρ' is density the mass m of the disc is

$$M = \rho\, \pi\, x^2\, dy$$

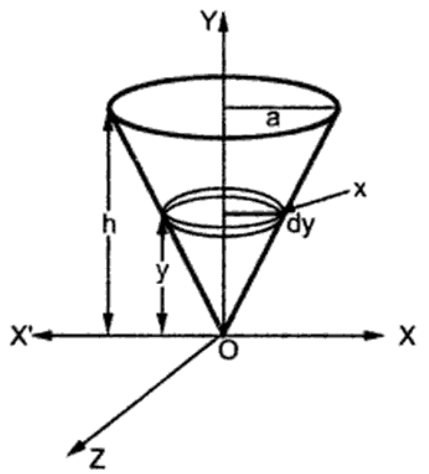


Fig. 15.72

The M.I. of this elementary disc of mass M about the axis OY is

$$\frac{Mx^2}{2} = \rho\, \pi x^2\, dy\cdot\frac{x^2}{2} \qquad \left(\because \text{M.I. of circular disc of radius } a = \frac{Ma^2}{2}\right)$$

$$= \rho\,\frac{\pi}{2}\, x^4\, dy$$

By considering similarity of triangles, we get $\frac{x}{a} = \frac{y}{h} \therefore x = \frac{a\,y}{h}$

Hence, M.I. of elementary disc is

$$\rho\,\frac{\pi}{2}\cdot\frac{a^4 y^4}{h^4}\, dy$$

$$= \frac{8\lambda}{3} a^{3/2} \left(\frac{x^{11/2}}{\frac{11}{2}} \right)_0^a$$

$$= \frac{16}{33} \lambda a^7$$

Now, $\quad \text{Mass} = M = \iint \rho \, dx\, dy$

$$= \int_0^a \int_0^{2\sqrt{ax}} \lambda x^3 \, dx\, dy$$

$$= \lambda \int_0^a x^3 \, (y)_0^{2\sqrt{ax}} \, dx$$

$$= 2\lambda \int_0^a \sqrt{a} \; x^{7/2} \, dx$$

$$= 2\lambda \sqrt{a} \cdot \left(\frac{x^{9/2}}{\frac{9}{2}} \right)_0^a$$

$$M = \frac{4}{9} \lambda a^5$$

$$\therefore \quad \lambda = \frac{9}{4} \frac{M}{a^5}$$

$$\therefore \quad \text{M.I.} = \frac{16}{33} \cdot \frac{9}{4} \cdot \frac{M}{a^5} a^7$$

$$\boxed{\text{M.I.} = \frac{12}{11} Ma^2} \quad \text{... Ans}$$

Example 15.78 : *Prove that the M.I. about an axis through the centre, perpendicular to the plane of a circular ring whose outer and inner radii are b and a is $\frac{1}{2} M(a^2 + b^2)$, where M is the mass of the ring.*

Solution : Consider a small area $r\, d\theta\, dr$ at a point $P(r, \theta)$.

The M.I. about the axis perpendicular to the plane through is

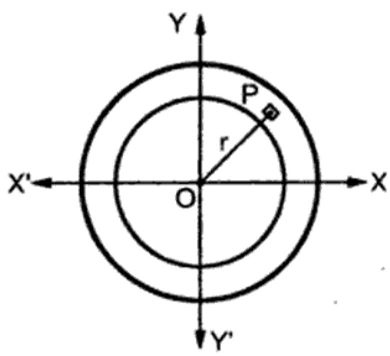


Fig. 15.74

$$\text{M.I.} = \iint \rho \, r^2 \cdot r \, d\theta \, dr$$

$$= \int_0^{2\pi} \int_a^b \rho \, r^3 \, d\theta \, dr$$

$$= \rho \int_0^{2\pi} \left(\frac{r^4}{4}\right)_a^b d\theta$$

$$= \frac{\rho}{4} \int_0^{2\pi} (b^4 - a^4)\, d\theta$$

$$= \frac{\rho\,(b^4 - a^4)}{4} \int_0^{2\pi} d\theta$$

$$= \frac{\rho}{4}(b^4 - a^4) \cdot 2\pi$$

$$\text{M.I.} = \frac{\rho}{2}(b^4 - a^4)\,\pi$$

But the mass of the ring is,

$$M = \pi\,(b^2 - a^2)\,\rho$$

$$\therefore \quad \rho = \frac{M}{\pi\,(b^2 - a^2)}$$

$$\therefore \quad \text{M.I.} = \frac{M}{\pi\,(b^2 - a^2)}(b^4 - a^4) \cdot \frac{\pi}{2}$$

$$\boxed{\text{M.I.} = \frac{M}{2}(a^2 + b^2)} \quad \text{... Ans}$$

Exercise 15.6

1) *Find the M.I. of a circular plate about a tangent.* **[Ans. $\frac{5}{4}Ma^2$]**

2) *Find the M.I. and the radius of gyration of the area bounded by $y = 4x(1-x)$ and the X-axis about X-axis.* **[Ans. : $\frac{8}{35}M, \sqrt{\frac{8}{35}}$]**

3) *Find the M.I. of the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi$.* **[Ans. : $\frac{4}{9}$]**

4) *Find the polar M.I. of the area bounded by the parabola $y^2 = 4ax$ and the line $y = x$.* **[Ans. : $\frac{768}{35}\rho a^4$]**

5) *Prove that the M.I. of the area included between the smaller arcs of $r = 2a\cos\theta$ and $r = 2a\sin\theta$ about an axis through the pole perpendicular to the plane of the curve is '$a^4\left(\frac{3\pi}{4} - 2\right)$'*

6) *Find the M.I. of a uniform Lamina bounded by the X-axis and one arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ from $\theta = 0$ to $\theta = 2\pi$ about X-axis.* **[Ans. : $\frac{35}{36}Ma^2$]**

7) *The surface density of a circular lamina varies as the square of the distance from a point O on the circumference. Find the M.I. of the area about an axis through O perpendicular to the plane of circle.* **[Ans. : $\frac{20}{9}$ Ma^2]**

8) *Find the M.I of one loop of $r^2 = a^2 \sin 2\theta$ about an axis perpendicular to its plane.* **[Ans. : $\frac{\rho \pi a^4}{16}$]**

9) *Find the M.I about the polar axis of the area enclosed by $r = a(1 + \cos\theta)$ in the upper half.* **[Ans. : $\frac{35\pi \rho a^4}{32}$]**

10) *A lamina in the first quadrant is bounded by $x = 0$, $y = 0$ and $x^2 + y^2 = a^2$ and $\rho = \frac{\lambda x^2 y}{a^3}$, ($\lambda$ is constant). Find the M.I. of the lamina about Z-axis.* **[Ans. : $\frac{5}{7}$ Ma^2]**

11) *Find the M.I. of the arc enclosed by $x^{2/3} + y^{2/3} = a^{2/3}$ about X-axis.* **[Ans. : $\frac{7}{64}$ Ma^2]**

12) *Find the M.I. of an arc in the form of cardiode $r = a(1 + \cos\theta)$ about the axis through the pole perpendicular to the plane of the arc.* **[Ans. : $\frac{256 a^3 \rho}{15}$]**

13) *Find the M.I. of a loop of the curve $r^2 = a^2 \cos 2\theta$ about*

   i) *a line through the pole perpendicular to its plane.* **[Ans. : $\frac{M}{8}\pi a^2$]**

   ii) *the tangent at the pole* **(May-02) [Ans. : $\frac{\pi a^4}{32}$]**

14) *A solid in the form of the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$ has its density given by $\rho = kxyz$, where k is constant. Find the M.I. about X-axis.* **[Ans. : $\frac{16 a}{35}$]**

15) *Find the M.I. of a circular disc of radius 'a' about an axis perpendicular to the disc and passing through the centre, the density being given by $\rho = \lambda r^2(1 + 3r)$.* **[Ans. : $2\pi \lambda a^6 \left(\frac{1}{6} + \frac{3}{7} a\right)$]**

16) *Find the M.I. of area bounded by the curves $r = a\cos\theta$ and $r = 2a\cos\theta$ about an axis through the origin perpendicular to XY plane if $\rho = kr$.* **[Ans. : $\frac{496}{76} \lambda a^5$]**

17) *Find the M.I. of the quadrant of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ of mass M about X-axis. If density at any point is proportional to xy.* **[Ans. : $\frac{Mb^2}{3}$]**

University Questions

Dec. - 2002

1) *Find the M.I. of one loop of the Laminscate $r^2 = a^2 \cos 2\theta$ above initial line.* **[8 Marks]**

May - 2006

1. *A rod of length a is divided into three parts. Find the Mean value of the product of these parts.* **[5 Marks]**
2. *Find the Moment of Inertia of the positive octant of the ellipsoid* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ *about the x-axis.* **[6 Marks]**
3. *Find the area inside the circle* $r = a \sin\theta$ *and outside the cardioide* $r = a(1 - \cos\theta)$. **[5 Marks]**
4. *Find the poistion of the centroid of the area under the parabola* $y = 4ax^2$ *from* $x = 0$ *to* $x = b$. **[5 Marks]**

Dec. - 2006

1. *Find the position of the centroid of the arc of the cardioide* $r = a(1+\cos\theta)$ *lying above the initial line.* **[5 Marks]**
2. *Find the area of the curve* $a^2x^2 = y^3(2a-y)$. **[5 Marks]**
3. *Find the volume bounded by the cylinder* $x^2 + y^2 = 4$ *and the planes* $y + z = 4$ *and* $z = 0$. **[5 Marks]**
4. *Find the mean value of* $e^{(x^2+y^2)}$ *over* $x^2 + y^2 = 1$. **[5 Marks]**
5. *Find the moment of inertia of the area in the XY plane bounded by* $y^2 = 2x$ *and* $y = x$, *about the axis through the origin perpendicular to the plane of the region. Assume constant density* ρ. **[6 Marks]**

May - 2007

1. *Find the M.V. and R.M.S. value of ordinates of cycloid* $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ *over the range* $\theta = -\pi$ *to* $\theta = \pi$. **[5 Marks]**
2. *Find area inside the circle* $r = a \sin\theta$ *and outside the cardiod* $r = a(1 - \cos\theta)$. **[5 Marks]**
3. *Find the moment of inertia of a sphere about a diameter.* **[6 Marks]**

Dec. - 2007

1. *Find the centre of gravity of the area bounded by* $y^2 = x$ *and* $x + y = 2$. **[5 Marks]**
2. *Find the area outside circle* $x^2 + y^2 = a^2$ *and inside cardiode* $r = a(1+\cos\theta)$. **[6 Marks]**
3. *Find the volume common to the cylinders* $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$ **[5 Marks]**
4. *Find the moment of inertia of one loop of Leminscate* $r^2 = a^2\cos 2\theta$ *about the initial line* $\theta = 0$. **[6 Marks]**

May - 2008

1. *Find the total area included between the cardioides* $r = a(1 + \cos\theta)$ *and* $r = a(1 - \cos\theta)$ **[5 Marks]**
2. *Find the volume bounded by the cylinders* $y^2 = x$, $x^2 = y$ *and the planes* $z = 0$ *and* $x + y + z = 2$. **[5 Marks]**
3. *The law of density* ρ *of a sphere of radius a is* $\rho = \rho_0 \dfrac{\sin(kr)}{nr}$ *where r is the distance from the centre,* ρ_0, *k and n are constants. Find the average density.* **[6 Marks]**
4. *Find the centroid of the loop of the curve* $r^2 = a^2\cos 2\theta$ **[5 Marks]**

Contents

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